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THE FUNDAMENTAL SU(2) HIGGS MODEL AT FINITE TEMPERATURE

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ABSTRACT

The fundamental SU(2) Higgs model at finite temperature is studied by means of Monte Carlo simulations, allowing for fluctuations in the radial degree of freedom of the Higgs fields. We find that the zero-temperature Higgs transition remains essentially unchanged at finite temperature. Monitoring the expectation value of the Polyakov line, as well as the contribution to the energy density from the gauge bosons and Higgs scalars, we find, when raising the temperature from the confined phase, a transition to a deconfined phase with a plasma of massless gauge bosons and massive Higgs scalars, quite analogous to fermion-gauge systems. On the other hand, starting from the Higgs phase and raising the temperature, we do not detect any signal for a transition into a "symmetry-restored" phase, which has been predicted in perturbative analyses.

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1. - INTRODUCTION

The fundamental SU(2) Higgs model is the SU(2) gauge theory coupled to a doublet of scalar fields in the fundamental representation of the gauge group. The classical effective potential of the scalar field can have a rather trivial structure (with just one minimum, situated at the origin) or have continuously degenerate global minima away from the origin, depending on the mass and coupling parameters chosen. Contrary to what its name indicates, such a model may therefore - depending on the parameter values - show both confinement behaviour and what is commonly known as a Higgs realization of the gauge symmetry.

When the radial part of the Higgs field is fixed at unit length, the difference between these two possible phases is only a question of quantitative changes in physical observables as one goes from one phase to the other. This is a consequence of a remarkable theorem¹⁾ which tells us that in this case the two phases are analytically connected.

If one introduces a Higgs potential of the form $\lambda(\phi(x)^2-1)^2$, and hence allows the magnitude of the Higgs field to vary, the phase diagram becomes a little more interesting. (The fixed-length case is recovered in the limit $\lambda \rightarrow \infty$). This is easily demonstrated in the lattice version of this model, where at $\beta = 4/g^2 = 0$ one can integrate out the gauge field exactly²⁾ and show the existence of a first-order phase transition for sufficiently small λ . Thus, for sufficiently small and fixed λ , the Higgs and confinement phases are no longer analytically connected in this model.

From Monte Carlo (MC) simulations, the full phase diagram of this model on the lattice is now known in quite great detail³⁾⁻⁵⁾, although the orders of the phase transition lines have not been conclusively identified in the whole parameter range. Preliminary mass determinations have also been performed⁵⁾.

Apart from answering several fundamental questions concerning gauge-matter systems, such investigations will ultimately also have phenomenological impact. Studies of the fundamental SU(2) Higgs model are the first steps towards a non-perturbative analysis of the standard model of weak and electromagnetic interactions. The simplifications may seem drastic: the whole fermion sector has been left out, as has the coupling to weak hypercharge. However, several important physical quantities, such as the Higgs boson mass for example, are not believed to depend strongly on the inclusion of the lepton sector. (This may, of course, be pure prejudice, but it is difficult to see an argument against it.) As for

the coupling to the extra $U(1)$ sector, MC simulations have already been performed with the full $SU(2) \times U(1)$ gauge group without encountering fundamental new difficulties⁶⁾.

We should strongly emphasize that, despite its enormous phenomenological success, the weak-coupling Weinberg-Salam model defined through its perturbation expansion will not provide us with all the information the theory contains. This not only pertains to such questions as whether a composite (i.e., "strongly-coupled") W and Z scenario is appropriate, what the mass of the Higgs boson is possibly predicted to be, etc., but may eventually put the very foundation of the theory into question by, for example, ruling out a meaningful non-trivial continuum limit in the scalar sector. In fact, in a recent preprint⁷⁾, A. and P. Hasenfratz argue that in the Higgs phase the model has no non-trivial continuum limit. However, they also argue that the cut-off may be chosen very large and that the model may serve very well as an effective theory, whose study would be of importance in its own right. Working on a lattice one has, of course, a finite cut-off, and investigations of the latticized model will hence be useful, regardless of the fate of a non-trivial continuum limit.

In this paper, we shall discuss the results of an MC simulation of this model at finite temperature. More than a decade has now passed since the prediction, based on weak-coupling perturbation theory⁸⁾, of a finite temperature phase transition from a Higgs phase to a "symmetry-restored" phase in the Weinberg-Salam model. Perhaps the above remarks should be sufficient to indicate why such a perturbative treatment may be insufficient, but in fact even more problems appear in this finite temperature case. The first problem is technical: the finite temperature perturbation series appears to be strongly gauge-dependent, and only the renormalizable gauge was found to give a meaningful expansion⁸⁾. Even then, the perturbation expansion breaks down in the neighbourhood of the phase transition due to long wavelength fluctuations.

Another problem seems more fundamental. Since at the phase transition the theory is supposed to go into a "symmetry-restored" phase of a confining gauge theory, it appears that non-perturbative effects may play an important rôle, at least close to the transition.

Closely related to this is the following puzzle. When described in the high-temperature (symmetry-restored) phase, the standard Weinberg-Salam model appears to be an ordinary confining gauge theory with confined leptons, vector bosons and scalars. Does this mean that when we heat up the system the electron

suddenly becomes "confined"? Furthermore, at finite temperature this confining gauge theory should, at a certain point, deconfine^{*)}. Should the temperature at which this occurs lie above or below the "symmetry restoration" phase transition? If the former were the case, one could naively expect a rather peculiar scenario in which the W's, the Z and all the leptons would first be in their ordinary "Higgs" realization (massive W's and Z), then would be confined in bound states, and finally, as one raised the temperature even further, would be liberated at the deconfinement temperature.

Some of these disturbing puzzles which appear to arise if one takes the notion of a symmetry-restored phase seriously have been mentioned previously in the literature⁹⁾, and their resolutions were there argued to lie in the so-called "complementarity principle" for gauge-Higgs theories with Higgs fields in the fundamental representation of the gauge group¹⁰⁾. Thus, it was argued in Ref. 9) that all regions in the coupling-temperature space are analytically connected. Whether true or not, this does not exclude the possibility of encountering a phase transition when moving on the "shortest" path from one region to another. In addition, one may encounter rapid changes in behaviour, such as in the deconfinement transition (see previous footnote). We shall here make a direct search for such phase transitions [in the loose sense^{**)}] in the finite temperature phase diagram.

Our paper is organized as follows. In the next section we introduce our notation for the fundamental SU(2) Higgs model, and give some details about our MC procedure. Section 3 is devoted to a brief discussion of gauge-Higgs thermodynamics. In Section 4 we present some of our results from one region of the finite temperature phase diagram which is particularly well understood and readily analyzed: the confinement-deconfinement region. Our results here compare nicely with both theoretical expectations and experience gained from earlier MC simulations for the SU(2) gauge theory with fermionic matter fields. The phase transitions between the confined-deconfined phases and the Higgs phase are analyzed in Section 5. Perhaps not surprisingly, we find that the zero temperature phase transition lines between the confinement and Higgs phases remain

*) Since matter fields are present, deconfinement may not necessarily be associated with a phase transition in the strict mathematical sense of a singularity in the infinite volume partition function. We use this term throughout the paper in the broad sense of a point at which the excess free energy of a static coloured source is substantially lowered.

***) Simulating a finite system numerically, we cannot make rigorous statements about true singularities in the infinite volume case anyway.

essentially unchanged, even in the finite temperature phase diagram. What is surprising, however, is that a line of phase transitions persists also between the deconfined phase and the Higgs phase. This has motivated us to perform an extensive search for a "symmetry-restored" region of the phase diagram (when starting from the Higgs phase), results of which are presented in Section 6. Our conclusion, at the same time intriguing and disturbing, is that with all the gauge-invariant observables that we are able to monitor, no sign whatsoever of a symmetry-restored phase is found. Possible pitfalls in our argumentation are discussed in Section 7. Finally, in an Appendix, we present some technical details related to the analysis of Section 4.

2. - THE LATTICE ACTION

In a Euclidean space-time continuum, the action for the fundamental SU(2) Higgs model reads:

$$S_{\text{cont}} = - \int d^4x [|D_\mu \tilde{\phi}(x)|^2 + m^2 |\tilde{\phi}(x)|^2 + \tilde{\lambda} |\tilde{\phi}(x)|^4] \quad (2.1)$$

with $D_\mu = \partial_\mu + igA_\mu(x)$ being the covariant derivative and $\tilde{\lambda}$ being real and positive. As usual, m^2 is allowed to take negative values, which give rise to non-trivial minima of the classical $\tilde{\phi}$ -potential $V(|\tilde{\phi}|)$.

Adhering to a standard convention, we choose a latticized action of the form

$$S = S_g - \lambda \sum_x [\phi^\dagger(x) \phi(x) - 1]^2 - \sum_{x,y,\mu} \phi^\dagger(x) Q_\mu(x,y) \phi(y) \quad (2.2)$$

where S_g is the pure SU(2) gauge piece

$$S_g = \frac{1}{g^2} \sum_{\text{plaq.}} [\text{Tr } U_P + \text{Tr } U_P^\dagger] \quad (2.3)$$

As usual, we define $\beta = 4/g^2$. This should not be confused with the inverse physical temperature T^{-1} .

The coupling between the Higgs field $\phi(x)$ and the link variables $U_\mu(x) \in \text{SU}(2)$ occurs through the Q matrix:

$$Q_\mu(x,y) = \delta_{x,y} - \kappa [\delta_{x,y-\mu} U_\mu(x) + \delta_{x,y+\mu} U_\mu^\dagger(x)] \quad (2.4)$$

with a Higgs "hopping parameter" κ . The action is symmetric under $\kappa \rightarrow -\kappa^3$, and we need therefore only consider positive values of κ .

The lattice parametrization (2.2)-(2.4) is particularly useful because of a close analogy with the corresponding SU(2) lattice gauge theory coupled to fermions. The only price paid for this is a slightly awkward correspondence to the continuum parameters of (2.1):

$$\tilde{\Phi}(x) = \frac{\sqrt{\kappa}}{a} \phi(x), \quad \tilde{\lambda} = \frac{1}{\kappa^2} \lambda, \quad w^2 = \frac{1-2\lambda-8\kappa}{\kappa a^2} \quad (2.5)$$

Here a denotes the lattice spacing, which was set equal to unity in Eqs. (2.2) and (2.3).

For the theory at finite temperature T , we do not compute vacuum expectation values, but thermal averages. In our Euclidean formulation these are obtained in a standard manner by making the fields periodic in the Euclidean time direction. On the lattice we call the periodicity N_τ . Then $\phi(\vec{x}, t+N_\tau) = \phi(\vec{x}, t)$ and $U_\mu(\vec{x}, t+N_\tau) = U_\mu(\vec{x}, t)$ (for all μ), in an obvious notation. With a uniform lattice spacing a , the physical temperature is then $T = (aN_\tau)^{-1}$.

Analytic results can be obtained for various thermodynamic quantities in the high-temperature region, and in the case of a pure SU(N) gauge theory several of these have already been computed to high orders in weak-coupling perturbation theory¹¹⁾. The new contributions from the Higgs part of the action are readily computed as well, as will be shown in the next section.

MC simulations of the action (2.2) can easily be performed if one makes a change of variables from the Higgs fields $\phi(x)$, $\phi^\dagger(x)$ into a real and positive radial part $\rho(x)$, and the angular part $\Omega(x)$, which is an element of SU(2). The integration measure then becomes

$$d\phi d\phi^\dagger dU = g^3 dg d\Omega dU; \quad \Omega, U \in SU(2) \quad (2.6)$$

Viewing the $\prod_x \rho(x)^3$ part as a term in the action, Eq. (2.2) then takes the form

$$\begin{aligned} S = S_g - \lambda \sum_x [g(x)^2 - 1]^2 - \sum_x [g(x)^2 - \frac{3}{2} \ln g(x)^2] \\ + \kappa \sum_{x,\mu} g(x) g(x+\mu) \text{Tr}[\Omega^\dagger(x) U_\mu(x) \Omega(x+\mu)] \end{aligned} \quad (2.7)$$

which is to be integrated over the measure $dg d\Omega dU$.

For most of the MC data to be presented here we did not employ the full $SU(2)$ Haar measure, but only the 120-element icosahedral subgroup Y_{120} . This applies to both the gauge field $U_\mu(x)$ and the angular part of the Higgs field $\Omega(x)$. The standard Metropolis algorithm was used, with proposed changes of the group elements to one of the 12 nearest neighbours in the icosahedral subgroup. In a very large part of the phase diagram of the model, the restriction to this discrete subgroup is immaterial; it only introduces a small systematic deviation which will be well within our error bars. However, as was noted by Jersak et al. in Ref. 4), particular care is needed in the weak coupling region, where the line of "freezing" transitions associated with Y_{120} can cut significantly into the Higgs phase. For the few data points in question from that region of the phase diagram, we employed the full $SU(2)$ Haar measure. A Metropolis algorithm was also used for the radial part of the Higgs field $\rho(x)$. The proposed changes in $\rho(x)$ were chosen randomly in an interval $[\rho - \Delta\rho, \rho + \Delta\rho]$, with $\Delta\rho$ dynamically adjusted to obtain a 50% acceptance rate of the changes. This particular choice of $\Delta\rho$ minimizes sweep-sweep correlations.

One MC iteration then consisted of a sweep through the links of the lattice, with three hits per link, followed by a sweep through the sites of the lattice, with three hits per site for both the radial and angular parts of the Higgs field.

We have worked with lattices of sizes $8^3 \times 2$ and $8^3 \times 4$, supplemented by a few reference runs on a symmetric lattice of size 6^4 . Data were collected from random starts, unless otherwise noted. 1000 iterations were used for thermalization at each data point; measurements were taken from the following 24000 iterations on

the $8^3 \times 2$ lattice and 12000 iterations on the $8^3 \times 4$ lattice, unless otherwise noted.

Finally, as we shall make repeated references to the zero temperature phase diagram of this model, we display for the reader's convenience in Fig. 1 this phase diagram as obtained by Kühnelt, Lang and Vones [Ref. 3)]. [We agree completely with this zero temperature phase diagram, except for the phase transition line at $\lambda = 0$. At $\beta = 0$ it should start²⁾ at $\kappa = 1/8$ and, since it is a straight line at $\kappa = 1/8$ also for large β , it is presumably a horizontal line at $\kappa = 1/8$ all the way through the phase diagram.]

3. - GAUGE-HIGGS THERMODYNAMICS

At sufficiently high temperature, the gauge coupling constant will be so small that we can view the system as a free gauge boson gas interacting only weakly with itself and the Higgs sector. The internal energy of the pure gauge part of the action will then asymptotically show the classical Stefan-Boltzmann behaviour of an ideal Bose gas with $2(N^2-1) = 6$ degrees of freedom [a factor of 2 for the two transverse polarization states, the factor of 3 from the three different weak gauge bosons]. Thus, at an infinite volume and at zero couplings, one should have the internal energy density of the pure gauge part approach the Stefan-Boltzmann limit

$$\epsilon_{SB} = 2(N^2-1) \frac{\pi^2}{30} T^4 = \frac{\pi^2}{5} T^4 \quad (3.1)$$

in dimensionless units (i.e., $T = 1/N_\tau$).

On the lattice, the expression for the internal energy associated with the gauge fields is¹¹⁾

$$\epsilon_G = \frac{6N}{g^2} (\langle P_\sigma \rangle - \langle P_\tau \rangle) \quad (3.2)$$

ignoring terms arising from the implicit dependence of g^2 on the temperature. [These corrections will be small¹¹⁾.] Here $\langle P_\sigma \rangle$ and $\langle P_\tau \rangle$ are the expectation values of plaquettes in the space-space and space-time directions respectively. On a finite-size lattice ϵ_G can differ substantially from the infinite volume

prediction (3.1). Fortunately, ε_G of Eq. (3.2) has been computed perturbatively in g^2 (excluding zero momentum modes) for the lattice sizes of interest for our study [see the last of Ref. 11)]. Thus a direct comparison of our MC data with the Stefan-Boltzmann prediction is possible.

Defining the Higgs part of the action by $S_H = S - S_g$, we shall now derive the contribution to the internal energy from the Higgs part. We shall denote this contribution by ε_H . The parametrization (2.2) of the action is slightly inconvenient when one wishes to derive thermodynamic quantities from the action, due to the rescalings (2.5) which are necessary for obtaining the correct SU(2) gauge-Higgs action in the naive continuum limit. Instead, let us start with the more simple parametrization

$$S_H = \sum_{x,\mu} [\tilde{\Phi}^\dagger(x) U_\mu(x) \tilde{\Phi}(x+\mu) + \tilde{\Phi}^\dagger(x+\mu) U_\mu^\dagger(x) \tilde{\Phi}(x) - 2 \tilde{\Phi}^\dagger(x) \tilde{\Phi}(x)] \\ - \sum_x [m^2 \tilde{\Phi}^\dagger(x) \tilde{\Phi}(x) + \tilde{\lambda} (\tilde{\Phi}^\dagger(x) \tilde{\Phi}(x))^2] \quad (3.3)$$

which leads directly to the Higgs part of the action (2.1) in the naive continuum limit.

In order to derive ε_H from (3.3), we consider an asymmetric lattice of size $N_\sigma^3 N_\tau$ with asymmetric lattice spacings a_σ and a_τ . The Higgs action (3.3) then reads

$$S_H = a_\sigma^3 a_\tau \sum_x [a_\tau^{-2} (\tilde{\Phi}^\dagger(x) U_0(x) \tilde{\Phi}(x+\hat{0}) + \tilde{\Phi}^\dagger(x+\hat{0}) U_0^\dagger(x) \tilde{\Phi}(x) - 2 \tilde{\Phi}^\dagger(x) \tilde{\Phi}(x)) \\ + a_\sigma^{-2} \sum_{\mu \neq 0} (\tilde{\Phi}^\dagger(x) U_\mu(x) \tilde{\Phi}(x+\mu) + \tilde{\Phi}^\dagger(x+\mu) U_\mu^\dagger(x) \tilde{\Phi}(x) - 2 \tilde{\Phi}^\dagger(x) \tilde{\Phi}(x)) \\ + \sum_x [m^2 \tilde{\Phi}^\dagger(x) \tilde{\Phi}(x) + \tilde{\lambda} (\tilde{\Phi}^\dagger(x) \tilde{\Phi}(x))^2] \quad (3.4)$$

The physical temperature is $T = (N_\tau a_\tau)^{-1}$, and the volume equals $V_\sigma = N_\sigma^3 a_\sigma^3$. From the definition of the internal energy density we then have

$$\varepsilon = - \frac{1}{V_\sigma} \frac{\partial \ln Z}{\partial (1/T)} \Big|_{V_\sigma} = - \frac{1}{V_\sigma N_\tau} \frac{\partial \ln Z}{\partial a_\tau} \quad (3.5)$$

where

$$Z = \int \prod_x d\phi^\dagger(x) d\phi(x) \prod_{x,\mu} dU_\mu(x) \exp\{S_G + S_H\} \quad (3.6)$$

is the partition function.

As can be seen, the internal energy splits into two parts,

$$E = E_G + E_H \quad (3.7)$$

where E_G is as given in Eq. (3.2), and

$$\begin{aligned} E_H &= - \frac{1}{V_\sigma N_\tau} \left\langle \frac{\partial S_H}{\partial a_\tau} \right\rangle \\ &= \left\langle a_\tau^{-2} (\tilde{\phi}^\dagger(x) U_0(x) \tilde{\phi}(x+\hat{0}) + \tilde{\phi}^\dagger(x+\hat{0}) U_0^\dagger(x) \tilde{\phi}(x) - 2 \tilde{\phi}^\dagger(x) \tilde{\phi}(x)) \right. \\ &\quad \left. - a_\sigma^{-2} \sum_{\mu \neq 0} (\tilde{\phi}^\dagger(x) U_\mu(x) \tilde{\phi}(x+\hat{\mu}) + \tilde{\phi}^\dagger(x+\hat{\mu}) U_\mu^\dagger(x) \tilde{\phi}(x) - 2 \tilde{\phi}^\dagger(x) \tilde{\phi}(x)) \right. \\ &\quad \left. + m^2 \tilde{\phi}^\dagger(x) \tilde{\phi}(x) + \tilde{\lambda} (\tilde{\phi}^\dagger(x) \tilde{\phi}(x))^2 \right\rangle \end{aligned} \quad (3.8)$$

Having already taken the derivative with respect to a_τ , we can now set $a_\sigma = a_\tau = a$. After a rescaling $\tilde{\phi}(x) \rightarrow a^{-1} \phi(x)$, we then have

$$\begin{aligned} a^4 E_H &= \left\langle (\tilde{\phi}^\dagger(x) U_0(x) \tilde{\phi}(x+\hat{0}) + \text{h.c.}) - 3(\tilde{\phi}^\dagger(x) U_j(x) \tilde{\phi}(x+\hat{j}) + \text{h.c.}) \right. \\ &\quad \left. + [(ma)^2 + 4] \tilde{\phi}^\dagger(x) \tilde{\phi}(x) + \tilde{\lambda} (\tilde{\phi}^\dagger(x) \tilde{\phi}(x))^2 \right\rangle \end{aligned} \quad (3.9)$$

with $j = 1, 2$ or 3 , on account of the permutation symmetry among the three spatial axes.

Finally, performing the last rescalings of Eq. (2.5), $\tilde{\phi}(x) \rightarrow \sqrt{\kappa} \phi(x)$, with

$$\kappa = \frac{1 - 2\tilde{\lambda}}{(ma)^2 + 8}, \quad \lambda = \kappa^2 \tilde{\lambda} \quad (3.10)$$

we end up with

$$a^4 \epsilon_H = \langle \kappa (\phi^\dagger(x) U_0(x) \phi(x+\delta) + h.c.) - 3\kappa (\phi^\dagger(x) U_j(x) \phi(x+j) + h.c.) \\ + (1-2\lambda-4\kappa) \phi^\dagger(x) \phi(x) + \lambda (\phi^\dagger(x) \phi(x))^2 \rangle \quad (3.11)$$

From this should be subtracted the zero temperature contribution

$$a^4 \epsilon_H^{(0)} = \langle -2\kappa (\phi^\dagger(x) U_j(x) \phi(x+j) + h.c.) \\ + (1-2\lambda-4\kappa) \phi^\dagger(x) \phi(x) + \lambda (\phi^\dagger(x) \phi(x))^2 \rangle_0 \quad (3.12)$$

evaluated on a space-time symmetric lattice, preferably of extent N_σ^4 .

It should be noted that the expressions (3.11) and (3.12) are not unique, and that they can be rewritten using "Ward Identities" for the various expectation values. Such identities follow, for example, from the invariance of the partition function (3.6) under the change of variables $\phi(x) \rightarrow \gamma \phi(x)$, $\phi^\dagger(x) \rightarrow \gamma \phi^\dagger(x)$. In particular,

$$\frac{\partial}{\partial \gamma} \ln Z \Big|_{\gamma=1} = 0 \quad (3.13)$$

which immediately leads to

$$\langle \kappa (\phi^\dagger(x) U_0(x) \phi(x+\delta) + h.c.) + 3\kappa (\phi^\dagger(x) U_j(x) \phi(x+j) + h.c.) \\ - (1-2\lambda) \phi^\dagger(x) \phi(x) - 2\lambda (\phi^\dagger(x) \phi(x))^2 \rangle + 2 = 0 \quad (3.14)$$

for the finite temperature lattice, and similarly for the space-time symmetric lattice. Using this, one finds that the zero temperature contribution (3.12) reduces to

$$a^4 \epsilon_H^{(0)} = 1 + \left(\frac{1}{2} - \lambda - 4\kappa\right) \langle \phi^\dagger(x) \phi(x) \rangle_0 \quad (3.15)$$

which must be subtracted from the finite temperature result (3.11).

4. - THE DECONFINEMENT TRANSITION

At small κ the SU(2) gauge-Higgs system is (at zero temperature) in the confined phase. There we expect this system to behave in a very similar way to an SU(2) gauge theory coupled to dynamical fermions in the fundamental representation, the main difference being due to the absence of chiral symmetry in the gauge-Higgs system. In particular we expect, as the temperature is raised, a sharp crossover from confined behaviour at low temperature to a deconfined region at high temperature. The high-temperature phase should therefore correspond to a plasma of massless (and at very high temperature, due to asymptotic freedom, essentially non-interacting) gauge bosons and massive Higgs scalars.

In a previous letter¹²⁾ we have presented strong evidence for this rapid transition from the behaviour of the Polyakov line. Also Sexton¹³⁾ has reported similar findings. In the absence of matter fields in the fundamental representation, the expectation value $\langle W \rangle$ of the Polyakov line

$$W(x) = \frac{1}{2} \text{Tr} \left[\prod_{t=1}^{N_\tau} U_0(\vec{x}, t) \right] \quad (4.1)$$

is an order parameter for the deconfinement phase transition: it signals the breaking of the global Z(2) symmetry $U_0(\vec{x}, t^*) \rightarrow z U_0(\vec{x}, t^*)$, $z \in Z(2)$, for all $\vec{x}$ with t^* fixed, of the pure gauge part of the action (2.2)^{*}). Matter fields screen $\langle W \rangle$ even in the confinement region. Nevertheless, $\langle W \rangle$ still turns out to give a good signal for the deconfinement transition in the presence of matter fields, both in the form of dynamical fermions¹⁵⁾ and Higgs bosons¹²⁾. In fact, the same universal Z(2) (Ising) behaviour of the Polyakov line

$$\langle W \rangle = c (\beta - \beta_c)^{1/3} \quad (4.2)$$

near the transition point β_c has been observed^{12), 14), 16)}.

In the following subsection, we shall briefly outline a mean field treatment of the deconfinement transition, and compare the theoretical expectations with our MC data. We then present, in Section 4.2, our results for the internal energy densities in order to illustrate the behaviour of thermodynamic quantities around the deconfinement transition.

*) For a recent review, see, e.g., Ref. 14).

4.1 Deconfinement in a Mean Field Approximation

For pure SU(N) gauge theories, a mean field treatment of deconfinement transitions based on the effective Polyakov line model

$$Z = \int \prod_{\vec{x}} dW(\vec{x}) e^{S_{\text{eff}}}$$

$$S_{\text{eff}} = \frac{1}{2} J(\beta, N_\tau) \sum_{\vec{x}, k} [\text{Tr} W(\vec{x}) \text{Tr} W^\dagger(\vec{x}+k) + \text{h.c.}] \quad (4.3)$$

has produced rather good agreement between the predicted critical coupling β_c and MC data from lattices of temporal size $N_\tau = 1$ and 2^{17}). In Eq. (4.3), $W(\vec{x})$ is the untraced Polyakov line, Eq. (4.1) without the trace and without the factor $\frac{1}{2}$. The effective Polyakov line model (4.3) can be obtained at strong coupling (small β) with a character expansion, neglecting space-like plaquettes. For the effective coupling strength J one finds, for SU(2),

$$J(\beta, N_\tau) = \left[\frac{I_2(\beta)}{I_1(\beta)} \right]^{N_\tau} \quad (4.4)$$

with $I_n(x)$ being the modified Bessel function of order n .

Let us now consider the SU(2) gauge theory coupled to fundamental Higgs fields with zero (bare) self-coupling [$\lambda = 0$ in Eq. (2.2)]. The action is then quadratic in the Higgs fields, which can therefore be integrated out, giving a factor $[\det Q]^{-1} = \exp(-\text{Tr} \ln Q)$ in the partition function. At small κ we can make a hopping parameter expansion of $\text{Tr} \ln Q$. To leading order this gives a contribution (from Higgs scalars hopping around a plaquette) that just renormalizes the gauge coupling to

$$\beta_{\text{eff}} = \beta + 4\kappa^4 + \dots \quad (4.5)$$

It also gives an additional term to the effective action (4.3) from loops winding around the lattice in the temporal direction:

$$S_{\text{eff}} \rightarrow S_{\text{eff}} + h \sum_{\vec{x}} (\text{Tr} W(\vec{x}) + \text{Tr} W^\dagger(\vec{x})) \quad (4.6)$$

with $h = \kappa^{N_\tau}$. For $N_\tau < 4$ this term, analogous to an external magnetic field, is

the dominant effect from the Higgs fields. A term like (4.6) also results from a hopping parameter expansion of dynamical fermions¹⁷⁾.

A standard lowest-order mean-field treatment of the system (4.3) and (4.6) leads to the self-consistency equation

$$W = \frac{I_2(24JW + 4h)}{I_1(24JW + 4h)}, \quad W = \frac{1}{2} \langle \text{Tr } W(x) \rangle \quad (4.7)$$

For $h = 0$ (no matter fields), one finds a second-order (deconfining) phase transition at $J_c(\beta_c, N_\tau) = 1/6$. At $N_\tau = 2$ this leads, by Eq. (4.4), to $\beta_c = 1.8512$, in excellent agreement with MC results.

For non-zero h , Eq. (4.7) has always only one solution, and no phase transition appears. However, for small h the solution of Eq. (4.7), $W(J)$, shows a rapid rise around an almost critical point J_c . This is illustrated in Fig. 2, where we have plotted W against β for $N_\tau = 2$ and $\kappa = 0.15$, corresponding to $h = 0.0225$. For comparison we also show the results of our MC simulation¹²⁾ on an $8^3 \times 2$ lattice. The agreement is surprisingly good, even though in the MC simulation we used a (small) Higgs self-coupling of $\lambda = 0.1$.

As h increases, the rise in W becomes less pronounced since the magnetic field tends to align the Polyakov loops even in the confined region. This is shown in Fig. 3 for $\kappa = 0.32$ (i.e., $h \approx 0.1$), again for $N_\tau = 2$. For this κ -value we had to use a large self-coupling, $\lambda = 5.0$, in the MC simulation in order to stay (at zero temperature) in the confined phase. Nevertheless, the qualitative behaviour is still fairly well represented by the simple mean field analysis of the effective action (4.3)-(4.6).

Even in the absence of a real phase transition for non-zero h , we can still define an approximate critical J , $J_c(h)$, by the requirement

$$\left. \frac{d}{dJ} W(J) \right|_{J_c(h)} = \text{max.}, \quad h \text{ fixed} \quad (4.8)$$

That this gives quite a sensible procedure for defining $J_c(h)$ is illustrated in Fig. 4, where we plot the result of a numerical differentiation of the mean field prediction shown in Fig. 2. Although there is no real phase transition, a pronounced peak is nevertheless observed at an almost critical coupling $J_c(h)$.

The resulting $J_c(h)$ as a function of h is shown in Fig. 5. We have found numerically that it is an almost straight line with only a slight curvature at very small h , $h \lesssim 0.01$. At fixed N_τ , using the relations between β and J as well as between κ and h , this can be translated into a curve for $\beta_c(\kappa)$. For the case considered in Figs. 2 and 4, for example, we find $\beta_c(\kappa = 0.15) = 1.8192$.

In the large N limit, the effective Polyakov line model (4.3), (4.6) is exactly solvable¹⁸⁾. It is amusing to note that in this large N limit $J_c(h)$ is also predicted to be a straight line which, for three space dimensions, is given by

$$J_c(h) = \frac{1}{3} - \frac{2}{3}h, \quad N = \infty \quad (4.9)$$

with a third-order deconfinement transition along this line.

4.2 Behaviour of the Energy Density

Around the deconfinement transition thermodynamic observables should also change rapidly. In this section we shall discuss our MC results for the energy density.

In the confined phase all physical particles, "glueballs" (i.e., bound states of weak gauge bosons) and bound states of the Higgs scalars, are massive, and the energy density is expected to be small. In the deconfined phase the gauge bosons are massless and at high temperature only weakly interacting. Their contribution to the energy density ϵ_G of Eq. (3.2) should therefore approach the Stefan-Boltzmann law for massless non-interacting bosons, Eq. (3.1).

Since the Stefan-Boltzmann law strongly depends on the energy-momentum dispersion relation, the continuum Stefan-Boltzmann law is modified on a finite lattice^{11),19)}. In the last of Ref. 11), the energy density for a pure SU(2) gauge theory has been computed on finite lattices including the first correction from the weak interactions between the gauge bosons, with the result:

$$\begin{aligned} a^4 \epsilon_G^{(asym.)} &= 0.21277 + 0.00806 g^2, \quad 8^3 \times 2 \text{ lattice} \\ a^4 \epsilon_G^{(asym.)} &= 0.010786 + 3.37 \cdot 10^{-4} g^2, \quad 8^3 \times 4 \text{ lattice} \end{aligned} \quad (4.10)$$

As in Eq. (3.2), terms arising from the implicit dependence of g^2 on T have been ignored in Eq. (4.10). They contribute at order g^2 only, and in MC calculations require accurate data on symmetric lattices, which we did not have available. Fortunately these corrections are small¹¹⁾.

In Fig. 6 we show this gauge boson contribution ϵ_G to the energy density for fixed $\lambda = 0.1$ and $\kappa = 0.15$ as a function of β from Monte Carlo runs on $8^3 \times 2$ and $8^3 \times 4$ lattices. They show the expected sharp rise around the deconfinement transition to values in agreement with the high temperature predictions of the pure gauge theory, Eq. (4.10). We note that in perturbation theory the Higgs fields contribute to ϵ_G only at order g^2 through their contribution to the vacuum polarization, but we have not computed this contribution. Figure 6 gives good evidence for the existence of a plasma of massless, weakly interacting gauge bosons at high temperature.

We note that we change the temperature by varying the gauge coupling β . This would be correct if, with λ and κ fixed, the only coupling renormalization needed when changing the cut-off (the lattice spacing a) is the one associated with the gauge coupling β . Then β and T are related by the one-loop formula

$$T = \frac{\Delta_L}{N_\tau} \exp\left[\frac{\beta}{8\gamma_0}\right] \quad (4.11)$$

with $\gamma_0 = 43/96\pi^2$ being the one-loop β -function coefficient of this theory. In the confinement region the scaling relation (4.11) is quite well obeyed when going from a lattice of size $8^3 \times 2$ to one of size $8^3 \times 4$ ¹²⁾. As the coupling constants mix under renormalization, the above scaling law should not be expected to hold with too high accuracy. We shall return to this issue in Section 6, where we shall discuss the high-temperature limit of the Higgs phase.

The full physical energy density has, in addition, contributions from the Higgs fields, Eq. (3.11), (3.12) and (3.15):

$$a^4 \epsilon_H^{\text{phys.}} = a^4 \epsilon_H(N_\sigma^3 \times N_\tau) - a^4 \epsilon_H(N_\sigma^4) \quad (4.12)$$

Contrary to the gauge boson contribution ϵ_G of Eq. (3.2), which vanishes on a symmetric lattice (i.e., at "zero" temperature), ϵ_H^{phys} needs a subtraction of the zero temperature contribution, Eqs. (3.12) or (3.15). Therefore, MC measurements on symmetric lattices, preferably of size N_σ^4 , are also required for the determi-

nation of ϵ_H^{phys} on an asymmetric lattice of size $N_\sigma^3 \times N_\tau$. Due to limits on available computer time we do not have any data from 8^4 lattices. We did a few relatively short runs, 2000 iterations after thermalization, on a 6^4 lattice. The zero temperature contribution to the Higgs energy density on this symmetric lattice is, as expected, only slightly dependent on β . We have used linear interpolation to get the plot of the physical Higgs energy density for the $8^3 \times 2$ lattice at $\lambda = 0.1$ and $\kappa = 0.15$ shown in Fig. 7. Again we see a rise of ϵ_H^{phys} around the deconfinement temperature.

For free massless bosons the Stefan-Boltzmann law on an $8^3 \times 2$ lattice predicts¹⁹⁾:

$$a^4 \epsilon_{SB}^{\text{phys.}} = n_{df} \times 3.56 \cdot 10^{-2} \quad (4.13)$$

with the number of bosons indicated by n_{df} . In the deconfined phase we have $n_{df} = 4$. We see that even at fairly high temperatures, where ϵ_G has long reached the Stefan-Boltzmann limit, this is not the case for $\epsilon_H^{\text{phys.}}$.

We take the above facts as an indication that the four Higgs scalars in this regime are rather massive. In the continuum, the energy density of massive bosons decreases from the Stefan-Boltzmann law for small m/T as

$$\frac{\epsilon}{T^4} = \frac{\pi^2}{30} - \frac{1}{24} \left(\frac{m}{T} \right)^2 + \dots \quad (4.14)$$

per degree of freedom (see the Appendix for details). For $m/T \gtrsim 1$ it drops rapidly and goes to zero for large m/T as $\exp(-m/T)$. In Fig. 8 we plot ϵ/T^4 as a function of m/T (in the continuum). We refer the reader to the Appendix for technical details.

On the $8^3 \times 4$ lattice we were unable to get a signal for the Higgs contribution to the energy density in the confinement-deconfinement part of the phase diagram. We can therefore only say that for all ρ values considered on this lattice size it must be considerably smaller than the Stefan-Boltzmann limit for massless bosons.

It would be interesting to see what happens to the deconfinement transition when the Higgs scalars are as light as possible, since then their screening effect should be biggest. We would also expect to see there the largest differ-

ence (if any) between Higgs scalars and dynamical fermions, since the latter might never become really light due to chiral symmetry breaking. Denoting the critical κ -value for the (de)confined-Higgs transition by $\kappa_c(\beta)$, this would require simulating the system at various β -values with κ just below $\kappa_c(\beta)$. Unfortunately we know from the last of Refs. 4) and 5) that the masses change quite rapidly in a very narrow κ -interval around the Higgs transition. Therefore $\kappa_c(\beta)$ would have to be determined with a high accuracy for each β value considered, an undertaking that far exceeds our computer time resources.

5. - CONFINEMENT-HIGGS TRANSITIONS AT FINITE TEMPERATURE

Although finite temperature introduces a new regime of the phase diagram, the deconfinement region, we have found that the main features of the zero temperature phase diagram remain unchanged. In particular, we shall here give numerical evidence for the persistence of the confinement-Higgs phase transition lines at parameter values indistinguishable from those at zero temperature. We have already reported first evidence of this fact in Ref. 12), and Sexton¹³⁾ has reported similar findings.

As a first indication, consider the behaviour of the Polyakov loop $\langle W \rangle$ when we, for fixed β and λ , vary the Higgs hopping parameter from the confinement to the Higgs phase. For $\beta < \beta_c$ and $\kappa = 0$ we know that $W(x)$ is protected from acquiring an expectation value due to the unbroken $Z(2)$ symmetry. On the other hand, when κ takes values in the Higgs phase, we should certainly expect

$$\langle W \rangle = e^{-(E - E_{\text{vac}})/T} \neq 0 \quad (5.1)$$

due to screening. Crossing a first-order phase transition line we should expect $\langle W \rangle$ to show a finite discontinuity, or, because of finite size effects, at least show a rapid variation.

These expectations are indeed confirmed by our MC data. In Fig. 9 we show $\langle W \rangle$ as a function of κ for fixed $\beta = 1.25$ and $\lambda = 0.10$. There is a clear change of regime around $\kappa_c = 0.25$ -0.26, nicely overlapping with the singular κ -value in the zero temperature phase diagram³⁾. The rise in $\langle W \rangle$ is much sharper on the $8^3 \times 4$ lattice than on the $8^3 \times 2$ lattice¹²⁾, again indicating that a temporal extent of just $N_\tau = 2$ on a lattice of spatial volume 8^3 introduces substantial finite size effects. Even on the $8^3 \times 4$ lattice we do not see any sign of a discontinuity in $\langle W \rangle$, despite the transition presumably being (weakly) of first order^{4),5)}.

Other expectation values show the confinement-Higgs phase transition in an equally clear manner. Experience has shown that $\langle \phi^\dagger \phi \rangle$, although not exactly an order parameter, is a good indicator of confinement and Higgs phases in the zero temperature phase diagram^{3),4)}. As can be seen from Fig. 10, this holds in the finite temperature phase diagram as well. The black squares in Fig. 10 indicate data from the same parameter choices as in Fig. 9. There is again a clear change of behaviour around $\kappa_c \approx 0.25-0.26$. In the confinement region $0 \leq \kappa < \kappa_c$, $\langle \phi^\dagger \phi \rangle$ is small and close to being constant; it rises quickly at $\kappa = \kappa_c$ and grows almost exactly as a linear function of κ for $\kappa > \kappa_c$. For $\kappa \rightarrow \infty$ one can argue³⁾ that since the Ω field of Eq. (2.7) will freeze into ± 1 , and the Higgs field will therefore be aligned, a semi-classical estimate for $\langle \phi^\dagger \phi \rangle = \langle \rho^2 \rangle$ determined from the minimum of the ρ potential will be reliable. This leads to

$$\langle \phi^\dagger \phi \rangle \rightarrow \frac{4\kappa}{\lambda} + \text{const.}, \quad \kappa \rightarrow \infty \quad (5.2)$$

a prediction which is shown as the straight line in Fig. 10. This simple $\kappa \rightarrow \infty$ estimate is satisfied to a remarkable degree of accuracy even for $\kappa \leq 1$.

As in the zero temperature case, the string bit $\langle \phi^\dagger(x) U_\mu(x) \phi(x+\mu) \rangle$ can also be used to determine the phase transition. This is shown with black squares for the same values of β and λ in Fig. 11a. In Fig. 11b we have plotted the normalized expectation value $\langle \phi^\dagger(x) U_\mu(x) \phi(x+\mu) \rangle / \langle \phi^\dagger(x) \phi(x) \rangle$, which shows the transition even more clearly.

Surprisingly, even for $\beta > \beta_c$ we find exactly the same signals for a phase transition, now between a deconfined phase and the Higgs phase. A first indication of this is seen already in $\langle W \rangle$. Of course, $\langle W \rangle$ will also be large in the deconfined phase, but for β values not too far from β_c one can still distinguish different behaviour in the deconfinement region and the Higgs phase (see Fig. 9). In the whole deconfinement phase $\langle W \rangle$ for fixed β is very close to being constant, whereas it grows slowly as a function of κ in the Higgs phase. It is clear, however, that for $\beta \rightarrow \infty$ we cannot use $\langle W \rangle$ to distinguish between the two phases, since $\langle W \rangle \rightarrow 1$ in both regions.

Instead, both expectation values $\langle \phi^\dagger \phi \rangle$, $\langle \phi^\dagger(x) U_\mu(x) \phi(x+\mu) \rangle$ and their ratio $\langle \phi^\dagger(x) U_\mu(x) \phi(x+\mu) \rangle / \langle \phi^\dagger \phi \rangle$ can be used to distinguish between deconfinement and Higgs phases. This can be seen from Figs. 10 and 11, where MC data for $\beta = 2.75$ and $\lambda = 0.10$ are shown as filled circles. Recall that for this lattice size ($8^3 \times 4$) the deconfinement transition occurs at $\beta_c \approx 2.25$. In comparing with the

corresponding data for $\beta = 1.25$, the transition deconfinement-Higgs appears to be of the same strength as the one between confinement and Higgs. Furthermore, at $\beta = 2.75$ and $\lambda = 0.10$, the transition occurs at $\kappa_c \approx 0.18$. This coincides exactly with the confinement-Higgs phase transition point in the zero temperature theory³⁾.

6. - THE HIGGS PHASE AT HIGH TEMPERATURE

Surprised by the results presented in the previous section, which showed the Higgs-confinement phase transitions to persist even into the high-temperature (deconfined) region, we have made a careful search for phase transitions between the Higgs phase and a "symmetry-restored" phase at high temperature. On the theoretical side, we have already mentioned in the Introduction some of the conceptual difficulties with the notion of a symmetry-restored phase (local gauge invariance is never broken, even in the Higgs phase), and part of the criticism one can raise against the prediction of a phase transition from the Higgs phase into this new phase. Some of these difficulties were actually discussed in some detail in the original paper of Weinberg⁸⁾, but they seem to have been largely ignored by subsequent workers in the field.

To analyze the question of finite temperature symmetry restoration, we have performed several series of MC runs starting from the Higgs phase at fixed λ and κ , while varying β . One technical difficulty arises from the fact that Y_{120} , our discrete SU(2) subgroup, introduces a spurious phase transition, which for $\kappa = 0$ occurs around $\beta \approx 6.0$. This "freezing transition" remains at roughly the same β value throughout the (de)confinement phase, but as noted by Jersak et al. in Ref. 4), it bends sharply upon crossing the (de)confinement-Higgs phase transition line, and cuts significantly into the Higgs part of the phase diagram. In order to probe the phase diagram beyond the freezing transitions, we therefore supplemented our calculation with MC simulations using the full SU(2) gauge group when needed.

Let us begin by presenting some results for the Polyakov line $\langle W \rangle$. As we showed in Figs. 9-11, on an $8^3 \times 4$ lattice with $\beta = 2.75$ there is a clear phase transition between the deconfined phase and the Higgs phase. In Fig. 12 we show (as filled circles) results from starting at $\kappa = 0.20$ (in the Higgs phase, just above the deconfinement-Higgs phase transition) at $\beta = 2.75$ and then increasing β up to $\beta = 4.75$. As can be seen, $\langle W \rangle$ shows no structure at all in this β interval. These data were obtained by using the icosahedral subgroup of SU(2), and there-

fore we could not search at higher β -values. Instead, we made a search at a very large λ -value ($\lambda = 5.0$), and deeper into the Higgs phase ($\kappa = 0.60$). These results are shown in the same figure as black squares. Again, no signal for a phase transition is found.

Remaining for a moment at the same parameter values as above, we show in Fig. 13 results for $\langle \phi^2 \rangle$ and $\langle \phi^\dagger U \phi \rangle$. Also these observables are completely smooth through the same β -interval. This β -interval (from $\beta = 1.5$ to $\beta = 4.5$) may seem rather short. However, it should be kept in mind that if the scaling relation (4.11) is valid in the Higgs phase (as we verified that it was to a good approximation in the confinement phase), then going from $\beta = 1.5$ to $\beta = 4.5$ corresponds to raising the temperature by a factor of ~ 4000 .

No sign of a phase transition is found in the internal energy density of the gauge bosons either. In Fig. 14 we show how ϵ_G approaches the Stefan-Boltzmann limit for massless bosons in the same β -interval. The fact that ϵ_G is so close to the massless Stefan-Boltzmann limit already at $\beta \approx 2.0$ makes it very plausible that we are indeed probing the Higgs phase at physical temperatures T where $M_W/T \ll 1$, M_W = gauge boson masses. Still in order to make sure that we are not missing crucial information from the large- β part of the phase diagram, we have extended the above searches up to $\beta = 50.0$. These computations of course employed the full $SU(2)$ gauge group.

In Fig. 15a we show the results of the above searches for β up to 50.0 at fixed $\kappa = 0.35$, $\lambda = 0.10$, i.e., deep in the Higgs phase. As can be seen, no structure whatsoever is observed in any of the observables ϵ_G , $\langle \phi^\dagger \phi \rangle$ or $\langle \phi^\dagger U \phi \rangle$. As all three quantities change most for $0 < \beta < 5.0$, we display in Fig. 15b an enlarged plot of this interval; again, no structure is found.

Since at $\beta = 50.0$, the energy density ϵ_G has approached the asymptotic massless Stefan-Boltzmann limit, indicating again that this indeed corresponds to a temperature where the gauge bosons appear effectively massless: $M_W/T \ll 1$, one might think that at these large β -values the distinction between the Higgs phase and the deconfined phase has disappeared. This is not the case. Shown in Fig. 16 is a plot of ϵ_G , $\langle \phi^\dagger \phi \rangle$ and $\langle \phi^\dagger U \phi \rangle$ versus κ for fixed $\lambda = 0.1$ and $\beta = 50.0$. As expected, ϵ_G shows no variation while crossing from the small- κ (deconfined) to the large- κ (Higgs) region, since at this large temperature the gauge bosons are effectively massless in both phases. Nevertheless, $\langle \phi^\dagger \phi \rangle$ and $\langle \phi^\dagger U \phi \rangle$ both show the characteristic behaviour of a deconfinement-Higgs phase transition at $\kappa_c \approx 0.17$. This κ_c coincides nicely with the singular κ -value for the zero-temperature

confinement-Higgs phase transition³⁾. Finally, for comparison we show a similar plot in Fig. 17 for the same λ -value, but now with $\beta = 2.5$. As can be seen, the behaviour in $\langle \phi^\dagger \phi \rangle$ and $\langle \phi^\dagger U \phi \rangle$ is identical to the behaviour at $\beta = 50.0$. Only ε_G shows a different behaviour: it is essentially constant in the deconfined phase (close to the Stefan-Boltzmann limit) and then falls towards zero as one crosses the deconfinement Higgs phase transition. This, again, is as expected. In the Higgs phase the gauge bosons appear effectively massive^{*)}, thus lowering ε_G from the massless limit as explained in the previous section. As κ is increased and we move deeper into the Higgs phase, the masses increase and ε_G should therefore approach zero, in complete agreement with the behaviour seen in Fig. 17.

Although we believe that the results discussed above give substantial evidence to the claim that the large- β region indeed corresponds to high temperatures even in the Higgs phase, we should add some words of caution. In the confined-deconfined phases we have seen that the two Higgs couplings κ and λ do not play an important rôle, and approximate renormalization group scaling is obtained for just $\beta = \beta(a)$, i.e., the one-loop relation (4.11) between T and β is obeyed to a reasonable degree of accuracy. We have no such direct evidence from the Higgs phase.

To illustrate some of the difficulties that occur in the presence of several coupling constants, we now analyze the renormalization group equations of the SU(2) Higgs system using the notation of Ref. 7) (i.e., $u = \tilde{\lambda}/6$, and r instead of κ). Any physical quantity with the dimension of a mass, e.g., the temperature T , the Λ_L parameter and all masses, have to satisfy the Callan-Symanzik equation

$$\left(a \frac{\partial}{\partial a} + a \frac{dg^2(a)}{da} \frac{\partial}{\partial g^2} + a \frac{du(a)}{da} \frac{\partial}{\partial u} + a \frac{dr(a)}{da} \frac{\partial}{\partial r} \right) M = 0 \quad (6.1)$$

where

$$a \frac{d}{da} = \frac{d}{dt}$$

in the notation of Ref. 7). The renormalization group equations for the three coupling constants can be found there, too.

*) Strictly speaking: the gauge-invariant states which, after gauge fixing, can be identified with the gauge bosons.

Assuming for this illustration that we can neglect the dependence on r , we find the solution to Eq. (6.1) at one-loop level to be:

$$M(a, g^2(a), u(a)) = \frac{1}{a} \left\{ c_1 \exp\left(-\frac{1}{2\gamma_0 g^2(a)}\right) + c_2 \exp\left(-\frac{1}{2\gamma_0 g^2(a)} \left[1 - \frac{1}{f\left(\frac{2u(a)}{g^2(a)}\right)}\right]\right) \right\} \quad (6.2)$$

with constants c_1 and c_2 . The function f is given by

$$\begin{aligned} f(x) &= c_3 \exp\left\{\frac{b}{\alpha} \delta(x)\right\} \\ \delta(x) &= \arctan\left(\frac{x + \frac{4}{3}}{\alpha}\right) \\ b &= 16\pi^2 \gamma_0, \quad \alpha = \sqrt{\frac{27}{4} - \frac{16}{9}} \end{aligned} \quad (6.3)$$

with $c_3 \neq 0$ being an arbitrary constant.

Taking for M in Eq. (6.2) the Λ_L parameter itself, we see that, contrary to the case of a pure gauge theory, not even its functional form is completely determined by the renormalization group equation (6.1). Additional input is needed to fix the constants c_3 and the ratio c_1/c_2 . Examples might be the requirement that the gauge-to-Higgs boson mass ratio M_W/M_{H_7} stay constant and/or that one stays on the critical surface. The latter leads to⁷⁾:

$$c_1 = 0, \quad c_3 = \exp\left\{-\frac{b}{\alpha} \arctan\left(\frac{4}{3\alpha}\right)\right\} \quad (6.4)$$

However, since $0 < \delta(x) < \pi/2$ for positive x , we conclude from Eqs. (6.2) and (6.3) that the relation between T and $\beta [= 4/g^2]$ is always dominated by a term of the form

$$T = (N_T a)^{-1} \sim \exp(\text{const.} \cdot \beta) \quad (6.5)$$

This suggests that changing β from 2 to 50 should vary T over several orders of magnitude.

Let us finally point out that Montvay's observation of lines of "constant physics" (at fixed λ) which for large β seem only weakly dependent on κ ⁵⁾ is also suggestive of a scaling relation between T and β as in Eq. (6.5).

7. - DISCUSSION

The finite temperature effects on the confinement part of the phase diagram nicely confirmed our expectations. What corresponds to the confinement part in the zero-temperature phase diagram splits at finite temperature into two regions: confinement and deconfinement regimes. They are separated by a line of rapid crossover at gauge couplings $\beta_c(\kappa)$ which are almost constant as a function of κ .

Interestingly, we have found that even for $\beta > \beta_c(\kappa)$ a line of phase transitions persists between the Higgs and the now deconfined phase. For all observables we have checked, the Higgs phase appears to extend all the way to $\beta = \infty$ without crossing any new phase transition lines. In particular, we have found no evidence for the existence of a "symmetry-restored" phase at large β , separated from the Higgs phase by a phase transition.

In our search for a "symmetry-restored" phase in the high-temperature Higgs region, we have not used any of the highly non-local order parameters which have been proposed for the zero-temperature theory²⁰⁾. However, in the zero-temperature theory none of these order parameters are needed to establish the phase diagram, and, as we have shown here, this holds for all known phases in the finite temperature case as well. Whatever the circumstances, it appears highly unlikely that any of the proposed order parameters (which, if blindly applied to our finite temperature case, would probe only the spatial large-distance behaviour) can be used to localize a "symmetry-restored" phase in the high-temperature Higgs region²¹⁾. The very notion of such a "symmetry-restored" phase, if it can be made meaningful at all, appears to be in need of a clear definition^{*)}.

If it can be confirmed by other approaches, a lack of phase transitions into a "symmetry-restored" phase could have serious consequences for cosmological models describing developments in the early Universe. If not present in the fundamental SU(2) Higgs model, it is difficult to see why it should be present in the conceptually essentially identical models such as SU(2) $\times$ U(1) or, for that

*) It is even intuitively difficult to see any reason for "symmetry restoration". Since the Higgs phase should properly be defined as the highly-screened phase, one would expect, if anything, this screening to increase as the temperature is raised, since now thermal excitations can screen as well. This is indeed one way to interpret our results in Section 6.

matter, any other unified gauge group. Clearly, more work should be done to clarify this issue.

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APPENDIX

In this Appendix we shall briefly derive some thermodynamic properties of free massive bosons in the continuum. The partition function is given by

$$Z = \text{Tr } e^{-H/T} \quad (\text{A.1})$$

with the (normal-ordered) Hamiltonian

$$H = \sum_{\vec{k}} \sqrt{\vec{k}^2 + m^2} a_{\vec{k}}^\dagger a_{\vec{k}} \quad (\text{A.2})$$

Evaluating Z in the occupation number basis, we find

$$\begin{aligned} Z &= \sum_{\{n_{\vec{k}}\}} \langle \{n_{\vec{k}}\} | e^{-\frac{1}{T} \sum_{\vec{k}} \sqrt{\vec{k}^2 + m^2} a_{\vec{k}}^\dagger a_{\vec{k}}} | \{n_{\vec{k}}\} \rangle \\ &= \prod_{\vec{k}} \sum_{n_{\vec{k}}} e^{-\frac{1}{T} \sqrt{\vec{k}^2 + m^2} n_{\vec{k}}} \\ &= \prod_{\vec{k}} [1 - e^{-\frac{1}{T} \sqrt{\vec{k}^2 + m^2}}]^{-1} \\ &= \exp \left\{ -V \int \frac{d^3 k}{(2\pi)^3} \ln(1 - e^{-\frac{1}{T} \sqrt{\vec{k}^2 + m^2}}) \right\} \end{aligned} \quad (\text{A.3})$$

Defining

$$G(\mu^2) = \frac{1}{2\pi^2} \int_0^\infty dx \, x^2 \ln(1 - e^{-\sqrt{x^2 + \mu^2}}) \quad (\text{A.4})$$

we hence obtain the partition function

$$Z = \exp[-VT^3 G((m/T)^2)]$$

From this we find the energy density to be

$$\epsilon = - \frac{1}{V} \frac{\partial \ln Z}{\partial (1/T)} \Big|_V = T^4 \left\{ -3G((m/T)^2) + 2 \left(\frac{m}{T} \right)^2 \frac{d}{d\mu^2} G(\mu^2) \Big|_{\mu = \frac{m}{T}} \right\} \quad (\text{A.5})$$

The integral (A.4) has been evaluated by Dolan and Jackiw⁸⁾ for small μ^2 with the result:

$$G(\mu^2) = - \frac{\pi^2}{90} + \frac{1}{24} \mu^2 - \frac{1}{12\pi} \mu^3 + \mathcal{O}(\mu^4 \ln \mu^2) \quad (\text{A.6})$$

Inserting this into (A.5) we find for small (m/T) :

$$\frac{\epsilon}{T^4} = \frac{\pi^2}{30} - \frac{1}{24} \left(\frac{m}{T} \right)^2 + \mathcal{O} \left(\left(\frac{m}{T} \right)^4 \ln \left(\frac{m}{T} \right)^2 \right) \quad (\text{A.7})$$

the expression quoted in Eq. (4.13).

For $\mu^2 \gg 1$, we can expand the logarithm in Eq. (A.4):

$$\begin{aligned} G(\mu^2) &= - \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} dx \, x^2 e^{-n\sqrt{x^2+\mu^2}} \\ &= - \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{3} \int_0^{\infty} dx \, x^4 (x^2+\mu^2)^{-1/2} e^{-n\sqrt{x^2+\mu^2}} \\ &= - \frac{\mu^2}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(n\mu) \end{aligned} \quad (\text{A.8})$$

The second line was obtained by an integration by parts, and the third line follows from I.S. Gradshteyn and I.M. Ryzhik, formula 8.432.9.

For large μ^2 , this becomes

$$G(\mu^2) = -\frac{\mu^2}{2\pi^2} \sqrt{\frac{\pi}{2\mu}} e^{-\mu} (1 + \mathcal{O}(\mu^{-2})) \quad (\text{A.9})$$

and we therefore find for $(m/T) \gg 1$:

$$\frac{\epsilon}{T^4} = \frac{1}{2\pi^2} \sqrt{\frac{\pi}{2}} \left(\frac{m}{T}\right)^{5/2} e^{-m/T} (1 + \mathcal{O}((T/m)^2)) \quad (\text{A.10})$$

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FIGURE CAPTIONS

- Fig. 1 : The zero temperature phase diagram, from Ref. 3).
- Fig. 2 : The mean field prediction for the Polyakov line expectation value, plotted as a function of β for $N_\tau = 2$ and $\kappa = 0.15$, i.e. $h = 0.0225$. The full squares are Monte Carlo data from an $8^3 \times 2$ lattice¹²⁾.
- Fig. 3 : Same as Fig. 2, but for $\kappa = 0.32$, i.e., $h \approx 0.1$. The full squares are Monte Carlo results from an $8^3 \times 2$ lattice with a large Higgs self-coupling of $\lambda = 5.0$.
- Fig. 4 : The derivative of the mean field prediction of the Polyakov line expectation value with respect to the coupling of the effective Polyakov line action (4.3)-(4.6) at constant external magnetic field h . An illustration for the determination of the "critical coupling" J_c of Eq. (4.8).
- Fig. 5 : J_c as a function of the external magnetic field h , determined from the criterion Eq. (4.8).
- Fig. 6 : The gauge boson contribution ε_G to the energy density for fixed $\kappa = 0.15$ and $\lambda = 0.1$ on an $8^3 \times 2$ lattice (a) and an $8^3 \times 4$ lattice (b). Also shown are the (lattice) Stefan-Boltzmann limit (dashed-dotted line), and the lowest-order corrections due to the weak interactions for a pure SU(2) lattice gauge theory (dashed line).
- Fig. 7 : The physical contributions from the Higgs scalars to the energy density on an $8^3 \times 2$ lattice at fixed $\kappa = 0.15$ and $\lambda = 0.1$. The zero temperature subtractions were taken from a 6^4 lattice at the same couplings.
- Fig. 8 : The energy density for free massive bosons as a function of the "effective dimensionless mass" m/T .
- Fig. 9 : The Polyakov line expectation value as a function of κ for fixed $\lambda = 0.1$ on an $8^3 \times 4$. The full dots are results from fixed $\beta = 1.25$, i.e., from the confined phase (for small κ). The full squares are results from fixed $\beta = 2.75$, i.e., from the deconfined phase (for small κ).

- Fig. 10 : $\langle \phi^\dagger \phi \rangle$ as a function of κ . Symbols and parameters are as in Fig. 9. Straight lines give the large- κ prediction (5.2).
- Fig. 11 : The "string bit" (in spatial directions) $\langle \phi^\dagger(x) U_j(x) \phi(x+j) \rangle$ (a), and the normalized string bit $\langle \phi^\dagger(x) U_j(x) \phi(x+j) \rangle / \langle \phi^\dagger \phi \rangle$ (b) as a function of κ . Symbols and parameters are as in Fig. 9.
- Fig. 12 : Expectation value of the Polyakov line for fixed λ and fixed κ in the Higgs phase as a function of β . Full squares denote results on an $8^3 \times 2$ lattice at $\lambda = 5.0$ and $\kappa = 0.6$. Full circles are data from an $8^3 \times 4$ lattice at $\lambda = 0.1$ and $\kappa = 0.2$.
- Fig. 13 : $\langle \phi^\dagger \phi \rangle$ (full squares and full circles) and the string bit $\langle \phi^\dagger(x) U_j(x) \phi(x+j) \rangle$ (open squares and open circles) as functions of β . Squares refer to the $8^3 \times 2$ lattice, and circles refer to the $8^3 \times 4$ lattice. Parameters are as in Fig. 12.
- Fig. 14 : The gauge boson contribution ϵ_G to the energy density for fixed $\lambda = 5.0$ and $\kappa = 0.6$ (i.e., in the Higgs phase) as a function of β on an $8^3 \times 2$ lattice. Also shown is the (lowest-order) lattice Stefan-Boltzmann limit. The freezing transition of the icosahedral $SU(2)$ subgroup occurs somewhere between $\beta = 4.0$ and $\beta = 5.0$.
- Fig. 15 : Results from runs on an $8^3 \times 2$ lattice, using the full $U(2)$ group, in the Higgs phase at fixed $\lambda = 0.1$ and $\kappa = 0.35$. Shown are the gauge boson contribution ϵ_G to the energy density (full circles), $\langle \phi^\dagger \phi \rangle$ (full squares) and the string bit in spatial directions $\langle \phi^\dagger(x) U_j(x) \phi(x+j) \rangle$ (crosses). Figure 15a shows the full range of β explored, while Fig. 15b is a blow-up of the "small" β region $0 < \beta < 5.0$.
- Fig. 16 : Results for fixed $\beta = 50.0$ and $\lambda = 0.1$ on an $8^3 \times 2$ lattice with varying κ . Observables and symbols are as in Fig. 15.
- Fig. 17 : Same as in Fig. 16, but for fixed $\beta = 2.50$.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6a

Fig. 6b

Fig. 7

Fig. 8

Fig. 9

Fig. 10

Fig. 11a

Fig. 11b

Fig. 12

Fig. 13

Fig. 14

Fig. 15a

Fig. 15b

Fig. 16

Fig. 17