REPRESENTING FUNCTIONALLY THE TWO-DIMENSIONAL CONFORMAL GROUP

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ABSTRACT

Projective representations of the two-dimensional conformal group are explicitly constructed in terms of propagation kernels. The representation functionals are then used to study the effect of conformal transformations on the states of a free massless scalar field theory.

I. INTRODUCTION

In two-dimensions, the conformal group is an infinite-dimensional Lie group. The study of its properties and those of the corresponding infinite-dimensional Lie algebra has always attracted a large interest. Many two-dimensional field theories are in fact invariant under this group of transformations,\(^1\) and also conformal symmetry plays a central role in the construction of the theory of strings\(^2\) as well as of several statistical models.\(^3\)

Here, we consider a massless scalar free field \(\Phi\), transforming homogeneously under the action of the conformal group, and study the transformation laws for the states of the corresponding field theory. This study requires the explicit construction of a representation for the finite elements of the group in terms of functional matrix elements and propagation kernels. This representation turns out to be projective and the corresponding 2-cocycle correctly reproduce the center of the conformal algebra.\(^4\) Representations for different numerical values of the center, although always larger than in the previous case, can be constructed as well by considering inhomogeneous transformations for \(\Phi\).

After briefly reviewing some characteristic properties of the two-dimensional conformal group, we discuss in detail how to construct the functional kernel which connects a state with the transformed one under conformal transformations. As we shall see, this is actually a part of the more general problem of finding a representation for the group of arbitrary linear canonical transformations on the dynamical variables of the theory. The transformation law of the states is subsequently discussed. Although the form of the transformed states can be explicitly computed, only rather formal expressions for them are available in general, showing that even for a free theory the conformal group acts in a non-trivial way on the states. The case of the ground state is of particular interest, as described at the end, its transformation law is directly related with the effect of particle creation in accelerated frames. Analogies with some quantum mechanical problems are briefly discussed in the Appendix.

* In a pure mathematical language, different representations of the two-dimensional conformal group are discussed in Refs. [4].
II. THE CONFORMAL GROUP IN TWO-DIMENSIONS

The following composition relations are well known and result from the invariance of the cross-ratio under the transformation 

\[ (x, y, z, w) \mapsto (x', y', z', w') \]

with 

\[ x' = \frac{x + y}{x + w}, \quad y' = \frac{y + z}{y + w}, \quad z' = \frac{z + x}{z + w}, \quad w' = \frac{w + y}{w + z}. \]

The cross-ratio is invariant under the automorphisms of the projective line, and these transformations form the group of conformal transformations in the complex plane. They are also known as M"obius transformations.

\[ \phi(x) = \frac{ax + b}{cx + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0. \]

The group of conformal transformations is isomorphic to the group of M"obius transformations.

\[ \text{Aut}(\mathbb{C}) \cong \text{PSL}(2, \mathbb{R}). \]

Due to this symmetry, there exists an infinite number of conformal curvatures.

\[ \phi' \circ \phi = \phi \circ \phi'. \]

The action of the group on the complex plane is given by

\[ \phi(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0. \]

The cross-ratio is invariant under these transformations.

\[ (x, y, z, w) \mapsto \left( \frac{ax + b}{cx + d}, \frac{ay + b}{cy + d}, \frac{az + b}{cz + d}, \frac{aw + b}{cw + d} \right). \]

This group is the group of conformal transformations in two dimensions.

\[ \frac{dz}{dz'} = \frac{a - c}{b - d}, \quad \text{for } a \neq c, b \neq d. \]

The cross-ratio is invariant under these transformations.

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This is true only at the classical level: at the quantum level the conformal algebra is realized with a non-number central extension, owing to the necessary presence of a Schwinger term in the commutators of the energy-momentum tensor.\footnote{The Rindler transformations, $x^+ = 1/2 e^{ax^+}$, $x^- = -1/2 e^{-ax^-}$, belong in fact to this subgroup.}

\[ [Q_f, Q_g] = iQ_{[f,g]} + c\Delta(f,g) \]  

(2.18)

with

\[ \Delta(f,g) = \frac{i}{48\pi} \int dx \left\{ \left[ f'^{-m} g^+ - g'^+ f^+ \right] - \left[ f'^{-m} g^- - g'^- f^- \right] \right\} \]  

(2.19)

and $c = 1$. The difference is a consequence of the fact that the products $(x^\pm)^2$ are not well-defined in the quantum theory. And, in fact, the charges (2.13) are usually defined using normal ordering with respect to the Fock vacuum of the theory. However, in the following different point of view is adopted: the algebra (2.18) will be recovered by explicitly constructing a representation for the Lie group, without any reference to the Fock vacuum of the theory.

Finally, notice that there are subgroups of the conformal group for which the center is zero. One is the finite dimensional $SO(2,2) = SO(2,1) \times SO(2,1)$ restricted conformal group of translations, Lorentz transformations, special conformal transformations and dilations, for which $f^\pm(x)$ are at most quadratic in $x$. Another one is what we shall call the "Rindler group".\footnote{There is another conformally invariant theory in which the classical Lie algebra is realized with a central extension. This is the Liouville model, where the transformations (2.6) and (2.20) must be combined to achieve the symmetry transformation, with $1/4$ identified as the coupling constant of the theory.} It is characterized by the choice $X^-(x) = -X^+(x)$ for the transformation functions of Eq. (2.1). One can check that the infinitesimal form of these transformations satisfy the composition law (2.5), and moreover that $\Delta(f,g) = 0$. It is not difficult to convince oneself that these are the only subgroups of the two-dimensional conformal group with a Lie algebra realized without center, although one can easily find particular couples of Killing vectors $f$, $g$ for which $\Delta(f,g) = 0$.

The massless, free scalar theory also admits another symmetry: the field may be shifted by an arbitrary wave field

\[ \delta \Phi = \eta \quad \Box \eta = 0 \]  

(2.20)

A particular interesting choice for $\eta(x)$ is

\[ \eta = \lambda \partial_a f^a \]  

(2.21)

where $f^a$ is a Killing vector, and $\lambda$ an arbitrary constant. In fact, one can now combine Eq. (2.20) with the conformal transformation (2.6) to obtain an inhomogeneous symmetry transformation on $\Phi$

\[ \delta_{f,a} \Phi = f^a \partial_a \Phi + \lambda \partial_a f^a \]  

(2.22)

The corresponding conserved current $J^\mu_{f,a}$ can again be written as

\[ J^\mu_{f,a} = \Theta^\mu_{f,a} f_a \]  

(2.23)

with $\Theta^\mu_{f,a}$ an improved, traceless energy-momentum tensor, which differs from the canonical one (2.8) by a superpotential

\[ \Theta^\mu_{f,a} = \theta^\mu_{f,a} + 2\lambda (\delta^\mu \Box - \delta^\mu \partial \partial) \Phi 
\]

\[ = \partial^a \theta_{\partial_a \Phi} - \frac{1}{2} g^{\mu \nu} (\partial_a \Phi)^2 - 2\partial \partial \partial_a \Phi \]  

(2.24)

The corresponding charges

\[ Q^\mu_f = \frac{1}{2} \int dx \ f^\mu(x) (x^\mu(x))^2 \pm \sqrt{2} \lambda \int dx \ f^\mu(x)x^\mu(x) \]  

(2.25)

generate the infinitesimal transformation. Note that even at the classical level the charges $Q^\mu_f$ obey now the algebra (2.18), with $c = 48\pi \lambda^2$. As we shall see, at the quantum level $c$ becomes $1 + 48\pi \lambda^2$, again owing to the presence of the Schwinger term in the commutators of $\Theta^\mu_{f,a}$.

### III. The Functional Representations for the Conformal Group: \( c=1 \)

The operator which implements the conformal transformations on the states of the theory is $e^{i\Theta^\mu Q^\mu}$. It acts on the space spanned by the eigenstates $|\psi\rangle$ of $\Phi$

\[ \Phi(x)|\psi\rangle = \psi(x)|\psi\rangle \quad \langle \psi_1|\psi_2\rangle = \delta(\psi_1 - \psi_2) \]  

(3.1)

The states $|\psi\rangle$ are normalized with a functional $\delta$-function and functional integration is used to sum over $\psi$. In this basis, $x^\pm$ is represented by

\[ \langle \psi_1|x^\pm(x)|\psi_2\rangle = \left( -i \delta \delta \Phi / \delta \phi_1(x) \right) \delta(\psi_1 - \psi_2) \]  

(3.2)

In order to avoid the ambiguities in the products $(x^\pm)^2$ and to build up well-defined generators, $f^\pm(x)$ are promoted to real, symmetric bilocal functions $F^\pm(x,y)$, which in the local limit become $f^\pm(x)\delta(x-y)$. The corresponding regulated generators

\[ Q^\pm = Q^\pm_F + Q^\pm_F 
\]

\[ Q^\pm_F - \frac{1}{2} \int dxdy \ x^\pm(x)F^\pm(x,y)\]  

(3.3)

where $Q^\pm_F$ and $Q^\pm_F$ are the Fock part of the generators and $(x^\pm)^2$ are the bilocal functions. This is the Liouville model, where the conformal transformations (2.6) and (2.20) must be combined to achieve the symmetry transformation, with $1/4$ identified as the coupling constant of the theory.
The information about the transformation of quantum field theory without...
\[ \Gamma_{\tau F} = -\frac{1}{2} \text{tr} \ln \left( 2\pi \cos \left( \frac{1}{2} \frac{\tau}{\psi} \right) \right) \quad \chi^\pm = \frac{1}{\sqrt{2}} (\psi_1 \pm \psi_2) \]

and

\[ \tilde{U}(\psi_1, \psi_2; \tau F^\pm) = \int D\varphi_1 e^{i \int dx dy \varphi_1(x) \varphi_2(x)} U(\psi_1, \psi_2; \tau F^\pm) \]
\[ = e^{i \frac{1}{2} \tau} e^{i \int dx dy \varphi_1(x) \varphi_2(x)} \times \exp \left\{ -\frac{1}{2} \int dx dy \left[ \psi_1(x) - \varphi_2(x) \right] A_{\tau F}^\pm(x, y) \right\} \]

\[ A_{\tau F}^\pm = \left( F^\pm \right)^{1/2} \left\{ -i \left( \frac{1}{2} \frac{\tau}{\psi} \right) \sin \left( \frac{1}{2} \frac{\tau}{\psi} \right) \right\} \left( F^\pm \right)^{-1/2} \] \hfill (3.17)

\[ \tilde{F}_{\tau F} = -\frac{1}{2} \text{tr} \ln \left( \left( F^\pm \right)^{-1/2} 2\pi i k^b \left( F^\pm \right)^{-1/2} \right) \] .

The functional representation \( U(\psi_1, \psi_2; \tau F) \) for the more general case \( F_1 = F_2 \equiv F^+ + F^- \), \( F_3 = F^+ - F^- \) (i.e., for a general conformal transformation) can now be obtained using

\[ U(\psi_1, \psi_2; \tau F) = \int D\varphi_1 U(\varphi_1, \psi_2; \tau F^+) U(\varphi_2, \varphi_2; \tau F^-) \] . \hfill (3.18)

It is again the Gaussian form (3.9) with

\[ A_{\tau F} = k + A_{\tau F}^+ - k \left( A_{\tau F}^+ + A_{\tau F}^- \right)^{-1} \left( A_{\tau F}^+ - k \right) \]
\[ B_{\tau F} = \left( A_{\tau F}^+ - k \right) \left( A_{\tau F}^+ + A_{\tau F}^- \right)^{-1} \left( A_{\tau F}^+ + k \right) \]
\[ C_{\tau F} = k + \left( A_{\tau F}^- - k \right) \left( A_{\tau F}^+ + A_{\tau F}^- \right)^{-1} \left( A_{\tau F}^- - k \right) \] \hfill (3.19)

\[ \Gamma_{\tau F} = \Gamma_{\tau F}^+ + \Gamma_{\tau F}^- = -\frac{1}{2} \text{tr} \ln \left\{ \frac{1}{2\pi} \left( A_{\tau F}^+ + A_{\tau F}^- \right) \right\} \] \hfill (3.20)

and it satisfies the composition law, as a consequence of (3.15).

By Eq. (3.18), the local limit \( F^\pm \to f^\pm \) of \( U(\psi_1, \psi_2; \tau F) \) can be discussed by examining the local limit of \( U(\psi_1, \psi_2; \tau F^\pm) \). No problems arise for the kernels \( \Gamma_{\tau F}^\pm \).

Explicitly one finds\(^*\) (for \( \tau > 0 \))

\[ A_{\tau F}^\pm(x, y) = -\tau \frac{1}{\tau f^\pm(x)} P \csc^2 \left\{ \frac{\pi}{2} \int_y^x \frac{dz}{\tau f^\pm(z)} \right\} \left( \frac{1}{\tau f^\pm(y)} \right) \] \hfill (3.21)

Here, \( P \) means principle value, and \( r \) must be understood as \( r = \text{int} \). On the other hand, the coefficients \( \Gamma_{\tau F}^\pm \) do not have a local limit. Nevertheless, if one chooses \( \Omega \) in (3.5) to coincide with \( \omega(x, y) = k(x, y) = -\frac{1}{4} \tau f^\pm(x, y) \), then \( e^{-i\tau X^\pm} \) have the representations\(^*\)*

\[ U: (\psi_1, \psi_2; \tau F^\pm) \equiv \left\{ \psi_1 e^{i\tau Q^\pm} \right\} \left( \psi_2 \right) \]
\[ = e^{i \frac{1}{2} \text{tr} (F^\pm \omega) U(\psi_1, \psi_2; \tau F^\pm)} \] \hfill (3.22)

which tend in the local limit to\(^**\)

\[ \hat{U}: (\psi_1, \psi_2; \tau f^\pm) \equiv \left\{ e^{i \frac{\pi}{2} \int \frac{dz}{\tau f^\pm(z)} \pm \int \frac{dz}{\tau f^\pm(z)} \varphi_2(x)} \right\} \]
\[ \times \exp \left\{ \frac{\pi}{2} \int dx dy [\varphi_1(x) - \varphi_2(x)] \frac{1}{\tau f^\pm(x)} P \csc^2 \left\{ \frac{\pi}{2} \int_y^x \frac{dz}{\tau f^\pm(z)} \right\} \right\} \] \hfill (3.23)

\[ \times \frac{1}{\tau f^\pm(y)} \left[ \varphi_1(y) - \varphi_2(y) \right] \} . \]

Similarly, \( \hat{U}: (\varphi_1, \varphi_2; \tau F^\pm) \equiv \left\{ \varphi_1 e^{i\tau Q^\pm} \right\} \varphi_2 \) and \( \hat{U}: (\psi_1, \psi_2; \tau F^\pm) \equiv \left\{ \psi_1 e^{i\tau Q^\pm} \right\} \) have the local limits

\[ \hat{U}: (\varphi_1, \varphi_2; \tau f^\pm) \equiv \left\{ \varphi_1 e^{-i\tau Q^\pm} \right\} \varphi_2 \) \hfill (3.24)

\[ \times \exp \left\{ -\frac{1}{2} \int dxdy \chi^\pm(x) \left\{ i ct h \frac{\pi}{2} \int_y^x \frac{dz}{\tau f^\pm(z)} \right\} (\chi^\pm(y)) \} \right\} \]

\* The combinations \( \Gamma_{\tau F}^+ + \frac{1}{2} \text{tr} (F^\pm \omega) \) really attain a finite local limit only after the subtraction of a \( F^- \) and \( r \)-independent logarithmic divergent constant in \( Z \), which can be reabsorbed in a rededinition of the functional measure: \( D\varphi \to Z^{-1} D\varphi \). This subtraction makes it rather difficult to check the composition law for the expression (3.23). Hence, explicit computations involving \( U: (\psi_1, \psi_2; \tau f^\pm) \) are always performed with in mind its bilocal version: \( U: (\psi_1, \psi_2; \tau F^\pm) \): the local limit it taken only at the end. Finally, note that \( U(\psi_1, \psi_2; \tau F^\pm) \) have only semigroup properties whenever the finite transformations \( X^\pm \) are not globally invertible.

\** Notice that for \( f^\pm = 1 \) the expressions (3.23) are the propagation kernels for the time evolution of the self-dual and antiself-dual fields \( X^\pm \) and \( \chi^\pm \); in fact, \( Q^\pm \) now exactly coincide with their corresponding Hamiltonians. Moreover, in the general case \( f^\pm(x) \) arbitrary functions, \( Q^\pm \) and \( U: (\psi_1, \psi_2; \tau F^\pm) \) may be viewed as Hamiltonians and propagation kernels for the self-dual and antiself-dual fields \( f^\pm X^\pm \) whose propagation is described with time intervals that vary in space as \( \Delta t = f^\pm(z) \).
CONFORMAL GROUP.

2.1. The Functional Representations for the Conformal Group. Let \( \Omega \) be a domain in \( \mathbb{C} \). For each \( \Omega \), consider the conformal group \( \mathcal{C}(\Omega) \) of \( \Omega \), which consists of all orientation-preserving biholomorphic mappings of \( \Omega \) onto itself.

\( \mathcal{C}(\Omega) \) is a Lie group, and its Lie algebra \( \mathfrak{c}(\Omega) \) consists of all first-order differential operators on \( \Omega \) that are symmetric and positive definite on \( \Omega \).

The Lie algebra \( \mathfrak{c}(\Omega) \) is isomorphic to the Lie algebra \( \mathfrak{sl}(2,\mathbb{R}) \) of the Lorentz group, which is the group of all orientation-preserving transformations of \( \mathbb{R}^2 \) that preserve the Lorentz inner product.

The group of all orientation-preserving conformal transformations of \( \mathbb{R}^2 \) is isomorphic to the group \( \mathfrak{sl}(2,\mathbb{R}) \), and its Lie algebra \( \mathfrak{c}(\mathbb{R}^2) \) consists of all first-order differential operators on \( \mathbb{R}^2 \) that are symmetric and positive definite on \( \mathbb{R}^2 \).

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Note that $\hat{\chi}$ still satisfies the commutation relations (2.15), since it differs from $\chi^+$ only for a $c$-number.

A functional matrix representation

$$:U:(p_1, p_2; \tau F^+, \lambda) = \langle \psi_1 | e^{-irQ_{2,1}} | \psi_2 \rangle$$

(4.4)

for $e^{-irQ_{2,1}}$ can now be constructed as in the previous case. Here again we have introduced the regularized charge

$$Q_{2,1}^+ = \frac{1}{2} \int dx dy |\hat{\chi}^+(x) F^+(x, y) \hat{\chi}^+(y)| \lambda^2 \int dx |f^+(x)|^2 (\ln f^+(x))^\prime$$

(4.5)

$F^+(x, y)$ being a bilocalization of $f^+$: $Q_{2,1}^+$ differs from $Q_{2,1}^+$ by the addition of a $c$-number, to be determined by the requirement that $:U:(p_1, p_2; \tau F^+, \lambda)$ possesses a well-defined local limit as $F^+(x, y) \rightarrow f^+(x) \delta(x - y)$. It turns out that the representation (4.4) is projective, and the corresponding 2-cocycle becomes again non-trivial in the local limit. In this limit one explicitly finds:

$$:U:(p_1, p_2; \tau f^+) \times \exp i \lambda \left\{ \lambda \int dx \int dy |\hat{\chi}^+(x) (\ln f^+(x))^\prime - 2 \int dx |f^+(x)|^2 (\ln f^+(x))^\prime (\psi_1(x) - \psi_2(x)) \right\}.$$

(4.6)

Moreover, the expression of the two-cocycle differs from (3.28) by the additional term

$$r \lambda^2 \left\{ \int dx \left[ f^+ (\ln f^+)^\prime + g^+ (\ln g^+)^\prime - (f^+ \circ g^+)^\prime (\ln f^+ \circ g^+) \right] + 2i \tau \int dxdy \left( \frac{f^+(x)}{g^+(x)} \right) (A^+_f(x, y) + A^+_g(x, y))^{-1} \left( \frac{f^+(y)}{g^+(y)} \right) \right\}$$

(4.7)

At the lowest order in $\tau$, this produces a new contribution to the center of the Lie algebra of the generators: $Q_{2,1}^+$: this algebra is still of the conformal form (2.18), but with $c = 1 + 48 \lambda^2$.

**V. THE TRANSFORMATION LAW FOR THE STATES**

The representation functional $:U:(p_1, p_2; \tau f)$ can now be used to exhibit the effect of a conformal transformation on the functional states of the theory. An arbitrary Gaussian

$$\Psi(\varphi) = e^{-\frac{1}{2} \int dxdy \varphi(x) \Omega(x, y) \varphi(y)}$$

(5.1)

changes under a general conformal transformation into a new Gaussian state given by

$$\Psi_{\tau F}(\varphi) = \int D\varphi_1 :U:(\varphi_1, p_1; \tau F) \Psi(\varphi_1)$$

$$= e^{\sigma_{\tau F} - \frac{1}{2} \int dxdy \varphi(x) \Omega_{\tau F}(x, y) \varphi(y)}$$

(5.2)

where

$$\Omega_{\tau F} = A_{\tau F} - B_{\tau F} (\Omega + C_{\tau F})^{-1} B^T_{\tau F}$$

$$\sigma_{\tau F} = \gamma_{\tau F} - \frac{1}{2} \tau \ln \left\{ \frac{1}{2\pi} (\Omega + C_{\tau F}) \right\}$$

(5.3)

and $\gamma_{\tau F}$ is the part of $\Gamma_{\tau F}$ which attains a finite limit as $F^+ \rightarrow f^+$. In this limit, $\Psi_{\tau F}$ becomes $\Psi_{\tau F}(\varphi) = e^{\gamma_{\tau F} - \frac{1}{2} \int dxdy \varphi(x) \Omega_{\tau F}(x, y) \varphi(y)}$, and $\Omega_{\tau F}, \sigma_{\tau F}$ are simply the local limit version of (5.3) and (5.4).

By repeated action of (5.2), one can verify that $\Omega_{\tau F}$ gives a representation of the conformal Lie algebra, but without center:

$$[\delta_{\tau F}, k_G] \Omega = -\delta_{k_G} \Omega$$

(5.5)

$$\delta_{\tau F} \Omega \equiv \Omega_{\tau F} - \Omega = \frac{-r}{2} \left\{ (\Omega + k) F^+ (\Omega + k) + (\Omega - k) F^- (\Omega + k) \right\}.$$ 

(5.6)

The center comes from the composition law for the $\sigma$'s:

$$[\delta_{\tau F}, k_G] \sigma = -k_G (\Omega - \omega)$$

(5.7)

$$[\delta_{\tau F}, k_G] \sigma = \frac{-i}{4} \text{tr} \left\{ (F^+ + F^-) (\Omega - \omega) \right\}.$$ 

(5.8)

$\delta_{\omega G}(F, G)$ is the $O(r^2)$ contribution to the 2-cocycle of Eq. (3.27), which in the local limit, as already observed in section III, exactly reproduce the center in (2.18) (with $c = 1$). Note that $\Omega_{\tau F}$ is also a solution of the following differential equation

$$2i \frac{d}{dr} \Omega_{\tau F} = \frac{1}{2} \left\{ (\Omega + k) F^+ (\Omega + k) + (\Omega - k) F^- (\Omega + k) \right\}$$

(5.9)

with the initial condition $\Omega_{\tau F}|_{r=0} = \Omega$.

Let us consider now the ground state functional

$$\Psi(\varphi) = e^{-\frac{1}{2} \int dxdy \varphi(x) \Omega(x, y) \varphi(x)}$$

(5.10)

** In terms of $\sigma$'s, the finite 2-cocycle can be written as

$$\epsilon_{\omega G}(F, G) = \lim_{\sigma \rightarrow 0} \left\{ \rho_{\sigma F} + \sigma_{\delta F} - \sigma_{\rho F} \right\} - \frac{1}{2} \text{tr} \ln \left\{ (\Omega + C_{\rho F})^{-1} (\Omega_{\rho F} + C_{\rho F}) \right\}.$$ 

** The corresponding equation for $\sigma_{\tau F}$ is: $i\frac{d}{dr} \sigma_{\tau F} = i \frac{1}{2} \text{tr} \left\{ (F^+ + F^-) (\Omega_{\tau F} - \omega) \right\}$, with $\sigma_{\tau F}|_{r=0} = 0$. Both these equations are a consequence of: $i\frac{d}{dr} \Psi_{\tau F} = Q_{\tau F} \Psi_{\tau F}$. 
In the electromagnetic vacuum, the electromagnetic field is a vector field that is a solution to Maxwell's equations. The electromagnetic field is described by the electric field $E$ and the magnetic field $B$. The electromagnetic field is a fundamental concept in physics, particularly in electromagnetism. It is the field that surrounds objects that carry electric charge or current.

The electromagnetic field is described by the electric field $E$ and the magnetic field $B$. The electromagnetic field is a solution to Maxwell's equations, which relate the electric and magnetic fields to the sources of the fields. Maxwell's equations are:

\[ \nabla \times E = -\frac{\partial B}{\partial t} \]

\[ \nabla \times B = \mu_0 J + \frac{\partial E}{\partial t} \]

\[ \nabla \cdot B = \mu_0 J \]

\[ \nabla \cdot E = \rho \]

where $\nabla$ is the gradient operator, $E$ is the electric field, $B$ is the magnetic field, $\mu_0$ is the permeability of free space, $J$ is the current density, and $\rho$ is the charge density.

The electromagnetic field is a vector field that varies in space and time. It is characterized by its strength and direction. The electromagnetic field is a fundamental concept in physics, particularly in electromagnetism. It is the field that surrounds objects that carry electric charge or current.
consequence of the non-trivial change of the states under the action of the conformal group.

The operator which implements the transformations (2.1) between the two observers is, in fact, $e^{-iQf_\gamma}$, thus

$$a_{f\gamma}(p)\Psi^{(0)}(\phi) = e^{-iQf_\gamma}a(p)e^{iQf_\gamma}\Psi^{(0)}(\phi)$$

$$= \frac{1}{\sqrt{4\pi|p|}} \int dx \ e^{-i\phi (x-x_0)} \Psi^{(0)}(\phi) .$$

(Eq. 5.18)

Evaluation of the functional integral (5.17) gives then

$$N(p,p') = \frac{1}{4\pi} \frac{1}{\sqrt{|p||p'|}} \int dxdy \left\{ \theta(p)\theta(p') e^{-i\phi X^-} e^{i\phi Y^-} [\omega(x,y) + k(x,y)] + \theta(-p)\theta(-p') e^{-i\phi X^-} e^{i\phi Y^-} [\omega(x,y) - k(x,y)] \right\} .$$

(Eq. 5.19)

However, one can also note that

$$N(p,p') = \int D\phi \Psi^{(0)}_{-\gamma}(\phi) a^+(p')a(p)\Psi^{(0)}_{\gamma}(\phi) .$$

(Eq. 5.20)

$\Psi^{(0)}_{-\gamma}$ has been already evaluated, so that

$$a(p)\Psi^{(0)}_{-\gamma}(\phi) = \frac{1}{\sqrt{4\pi|p|}} \int dx \ e^{-i\phi (x-x_0)} [p]\phi(x) - \int dx \omega_{-\gamma}(x,y)\phi(x)] \Psi^{(0)}_{\gamma}(\phi) .$$

(Eq. 5.21)

Inserting this in (5.20) and performing the functional integral, one finally gets

$$N(p,p') = \frac{1}{4\pi} \frac{1}{\sqrt{|p||p'|}} \int dxdy e^{-ips} e^{i\phi_\gamma}$$

$$\times \left( \theta(p)\theta(p') [\omega_{-\gamma}(x,y) + k(x,y)] + \theta(-p)\theta(-p') [\omega_{-\gamma}(x,y) - k(x,y)] \right)$$

(Eq. 5.23)

where now $\omega_{-\gamma}(x,y)$ is given by Eq. (5.15). The nature of this spectrum is determined, in general, by the analytic properties of the function $X(x)$, in particular, if $X^{-1}(x) = \frac{x}{x^2}$, i.e. in the case of the uniform accelerating observer with acceleration $a$, $N(p,p')$ becomes a purely thermal spectrum

$$N(p,p') = \delta(p-p') \frac{1}{\epsilon^2|p| - 1}$$

(Eq. 5.24)

with temperature proportional to the acceleration.

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REFERENCES

The boundary conditions are a consequence of the no-moving condition (11) in a different context. The model was originally studied in Ref. [15].

\[
\begin{aligned}
\left\{ \frac{\partial}{\partial t} - \frac{\partial}{\partial s} \right\}^2 &+ \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial s} \right)^2 = \frac{\partial}{\partial t} - \frac{\partial}{\partial s} \\
\int_0^1 &\left( \frac{\partial}{\partial x} + \frac{\partial}{\partial s} \right) \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial s} \right) \frac{\partial}{\partial x} \\
&= \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial s} \right) \\
\end{aligned}
\]

Until the corresponding commutation version (3.10) and (3.11), these equations can be

\[
\begin{aligned}
G^\prime(t,x) &= \frac{\partial G}{\partial t} \\
G^\prime(t,x) &= \frac{\partial G}{\partial x} \\
G^\prime(t,x) &= \frac{\partial G}{\partial x} \\
\end{aligned}
\]

equations for the coefficients \( A, B, C \), and 1. By taking the derivatives with respect to \( t \) of (A.1) one gets a set of differential

\[
\begin{aligned}
\left[ \frac{\partial}{\partial x} + \frac{\partial}{\partial s} \right] \frac{\partial}{\partial x} &+ \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial s} \right) \frac{\partial}{\partial x} = \frac{\partial}{\partial t} - \frac{\partial}{\partial s} \\
\end{aligned}
\]

which may be written in \( x_1 \) and \( \xi \), where the function \( C \) is quadratic in \( \xi \),

\[
\begin{aligned}
\left[ \frac{\partial}{\partial x} + \frac{\partial}{\partial s} \right] \frac{\partial}{\partial x} &+ \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial s} \right) \frac{\partial}{\partial x} = C \\
\end{aligned}
\]

and \( \xi \)

\[
\begin{aligned}
\left[ \frac{\partial}{\partial x} + \frac{\partial}{\partial s} \right] \frac{\partial}{\partial x} &+ \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial s} \right) \frac{\partial}{\partial x} = D \\
\end{aligned}
\]

We seek the form of \( \xi = \xi(x, t) \) is then represented by the coordinates \( x, \xi \) and \( x, \tau \) are compatible canonical coordinates. In the space spanned by the coordinates \( x, \xi \) and \( x, \tau \), the coordinates \( \xi, \tau \) are related to the SO(2) group. The study of this representation in section 4.2 uses a quantum mechanical context, the study of the representation of the conformal group discussed

APPENDIX
where \( \omega = \sqrt{q_1 q_3 - q_2^2} \). The composition law for \( U(x_1, x_2; ra) \) may be verified** and then (A.6) gives an explicit representation for the starting \( SO(2,1) \) group.

Another amusing similarity with a quantum mechanical problem can be found by comparing the functional representation (3.13) with the time propagation kernel for the motion of a particle of mass \( m \) and electric charge \( e \) on a plane, in a uniform magnetic field \( B \) orthogonal to the plane.

Starting with the appropriate Lagrangian

\[
L = \frac{m}{2} v^2 + ev \cdot A
\]

\[
A_i = -\frac{B}{2} \epsilon^{ij} x^j = \frac{B}{2} \epsilon^{ij} x^j
\]

one can quantize the system using the corresponding Hamiltonian

\[
H = \frac{1}{2} \dot{r}^2 + kr^2 x^i x^j
\]

(A.8)

and the canonical commutation relations

\[
[v^i, v^j] = 2ik \epsilon^{ij}
\]

(A.9)

we have chosen units such that \( m = 1 \) and put \( k = \frac{eB}{2} \). The propagation kernel

\[
U(x_1, x_2; t) = \langle x_2 | e^{-iHt} | x_1 \rangle
\]

is given by

\[
U(x_1, x_2; t) = e^{it} e^{-\frac{1}{2} \{x_1 - x_2\}^2} \langle x_2 | e^{-iHt} | x_1 \rangle
\]

(A.10)

with

\[
A = -i k \cot \left( \frac{kt}{2} \right)
\]

\[
\gamma = -\frac{1}{2} \ln \left( \frac{2\pi^2 \sin (kt/2)}{k} \right)
\]

(A.11)

The analogy between \( v^i \) and the self-dual field \( \chi^+(x) = \frac{1}{2\pi} (\pi(x) + \psi'(x)) \). and between (A.10)–(A.11) and (3.13)–(3.12) is evident. Actually, this is more than analogy. In fact, the term \( \psi'(x) \) which appear in the definition of \( \chi^+(x) \) can be interpreted as a functional \( U(1) \) connection in the space of the variables \( \psi(x) \).** It corresponds to a uniform functional magnetic field \( B = 2\psi'(x - y) \).

---

* Hyperbolic functions occur in \( U(x_1, x_2; ra) \) when \( \omega^2 < 0 \).

** Here again, the general composition law \( \int dz_3 U(x_1, x_2; ra) U(x_3, x_2; ra) = U(x_1, x_3; ra \circ a') \) can be checked in practice only for the first terms in a \( r \)-power series, since a \( \circ a' \) is given by such a series.

*** \( U(1) \)-connections in field space arise also in Yang-Mills theories; for a discussion, see Ref. [16].