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REPRESENTATIONS OF THE ^{CI}
TWO-DIMENSIONAL CONFORMAL GROUP*

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I. INTRODUCTION TO THE CONFORMAL GROUP

The dynamics of a field A in D -dimensional Minkowski space-time can be invariant against a group of coordinate transformations, given infinitesimally by

$$\delta_f x^\mu = -f^\mu \quad (1.1)$$

which is larger than the Poincaré group of translations and Lorentz rotations. When the energy-momentum tensor $\theta^{\mu\nu}$ is not only conserved [as a consequence of translation invariance] and symmetric [as a consequence of Lorentz invariance] but is also traceless

$$\theta^\mu_\mu = 0 \quad (1.2)$$

then the following are conserved currents

$$j^\mu_f = \theta^{\mu\nu} f_\nu \quad (1.3)$$

provided f_μ is a conformal Killing vector.

$$\partial_\mu f_\nu + \partial_\nu f_\mu - \frac{2}{D} g_{\mu\nu} \partial_\alpha f^\alpha = 0 \quad (1.4)$$

For $D > 2$ the only solutions to this equation are the $(D+2)(D+1)/2$ vectors, at most quadratic in x^μ , which generate the $SO(D, 2)$ special conformal group.

A linear transformation law for the multicomponent field A is

$$\delta_f A = f^\mu \partial_\mu A + \partial_\mu f_\nu \left(g^{\mu\nu} \frac{d}{D} + \frac{1}{2} \Sigma^{\mu\nu} \right) A \quad (1.5)$$

Here, d is the scale dimension of A and $\Sigma^{\mu\nu}$ is its spin matrix. The conserved charges

$$Q_f = \int d\mathbf{r} \theta^{0\nu} f_\nu \quad (1.6)$$

generate this transformation by commutation.

$$i[Q_f, A] = \delta_f A \quad (1.7)$$

[Classically, the commutator is replaced by the Poisson bracket, divided by i .] Moreover, the charge algebra follows the Lie algebra of the group.

$$[\delta_f, \delta_g] = -\delta_{(f,g)} \quad (f, g)^\mu \equiv f^\alpha \partial_\alpha g^\mu - g^\alpha \partial_\alpha f^\mu \quad (1.8)$$

$$[Q_f, Q_g] = iQ_{(f,g)} \quad (1.9)$$

For further discussion, see Refs. [1] and [2].

At $D = 2$, the above continues to hold and describes an $SO(2, 2) = SO(2, 1) \otimes SO(2, 1)$ invariance group. However, there are additional conformal Killing vectors: any f^μ with + component depending on x^+ , $f^+(x^+)$, and - component depending on x^- , $f^-(x^-)$, satisfies (1.4). [Light-cone components are defined by $\pm \equiv \frac{1}{\sqrt{2}}(0 \pm 1)$.] These give rise to an infinite dimensional Lie group, whose action on the coordinates of two-dimensional Minkowski space-time [$x^\mu \equiv (t, x)$, metric $g_{\mu\nu} = \text{diag}(1, -1)$] takes light-cone components into arbitrary functions of themselves. The Lie algebra of these coordinate transformations is as in (1.8); the transformation of the fields is as in (1.5), except that in two dimensions $\Sigma^{\mu\nu}$ may be represented by $\epsilon^{\mu\nu} \Sigma$, hence (1.5) becomes

$$\delta_f A = f^\mu \partial_\mu A + \frac{1}{2} \partial_+ f^+ (d + \Sigma) A + \frac{1}{2} \partial_- f^- (d - \Sigma) A \quad (1.10)$$

Eq. (1.6) now gives rise to an infinite set of conserved charges. The energy-momentum tensor has only two independent components owing to its symmetry and tracelessness: $\theta^{01} = \theta^{10}$, $\theta^{00} = \theta^{11}$. Equivalently, the two components may be taken as θ_{++} and θ_{--} , since θ_{+-} vanishes. Moreover, conservation insures that θ_{++} is only a function of x^+ and θ_{--} only of x^- . Hence, (1.6) may be presented as

$$\begin{aligned} Q_f &= \frac{1}{\sqrt{2}} \int dx [\theta_{++}(x^+) f^+(x^+) + \theta_{--}(x^-) f^-(x^-)] \\ &= \int dz [\theta_{++}(z) f^+(z) + \theta_{--}(z) f^-(z)] \end{aligned} \quad (1.11)$$

where the t -independence is explicit in the second formula. These charges generate the transformation (1.7). (1.10); however, upon quantization their algebra does not

reproduce the group's Lie algebra (1.8). i.e. while (1.9) is a valid classical formula involving Poisson brackets, it is no longer true in the quantum theory when f^μ and g^μ range beyond the quadratic expressions appropriate to $SO(2,2)$.

To prove this, observe that (1.9) for arbitrary $f^\pm(x^\pm)$, $g^\pm(x^\pm)$ implies the following equal-time commutation relations.

$$[\theta_{\pm\pm}(t, x), \theta_{\pm\pm}(t, y)] = \pm 2i (\theta_{\pm\pm}(t, x) + \theta_{\pm\pm}(t, y)) \delta'(x - y) \quad (1.12)$$

$$[\theta_{\pm\pm}(t, x), \theta_{\mp\mp}(t, y)] = 0 \quad (1.13)$$

[In fact, $\theta_{\pm\pm}$ depends only the combination $\frac{1}{\sqrt{2}}(t \pm x)$.] By Lorentz invariance, a traceless and symmetric tensor has vanishing expected value in the vacuum $|0\rangle$, so the above requires $\langle 0 | [\theta_{\pm\pm}(t, x), \theta_{\pm\pm}(t, y)] | 0 \rangle$ to be zero. But one may show that in a theory with a non-negative energy spectrum, a unique ground [vacuum] state and positive norm in the Hilbert space, the commutator's vacuum expectation value cannot vanish. Consider the positive expectation of the Hamiltonian H in the states $|\alpha_\pm\rangle \equiv \int dx \alpha(x) \theta_{\pm\pm}(t, x) |0\rangle$, where α is an arbitrary real function with bounded support: $0 < \langle \alpha_\pm | H | \alpha_\pm \rangle = \int dx dy \alpha(x) \alpha(y) \langle 0 | \theta_{\pm\pm}(t, x) H \theta_{\pm\pm}(t, y) | 0 \rangle$. Two equivalent formulas for this expression are $\int dx dy \alpha(x) \alpha(y) \langle 0 | [\theta_{\pm\pm}(t, x), H] \theta_{\pm\pm}(t, y) | 0 \rangle$ and $\int dx dy \alpha(x) \alpha(y) \langle 0 | \theta_{\pm\pm}(t, x) [H, \theta_{\pm\pm}(t, y)] | 0 \rangle$. Since $i[H, \theta_{\pm\pm}] = \dot{\theta}_{\pm\pm} = \pm \theta'_{\pm\pm}$, where the dot [dash] indicates differentiation with respect to time [space], we find

$$\begin{aligned} 0 &< 2 \langle \alpha_\pm | H | \alpha_\pm \rangle \\ &= \int dx dy \alpha(x) \alpha(y) \{ \pm i \langle 0 | \theta'_{\pm\pm}(t, x) \theta_{\pm\pm}(t, y) | 0 \rangle \mp i \langle 0 | \theta_{\pm\pm}(t, x) \theta'_{\pm\pm}(t, y) | 0 \rangle \} \\ &= \mp i \int dx dy \alpha'(x) \alpha(y) \langle 0 | [\theta_{\pm\pm}(t, x), \theta_{\pm\pm}(t, y)] | 0 \rangle \end{aligned} \quad (1.14)$$

Thus, the vacuum expectation value of the commutator cannot vanish. Eq. (1.12), which is a valid classical formula when Poisson brackets are used, must acquire a quantum mechanical correction. Dimensional analysis suggests that the simplest modification of (1.12) is

$$[\theta_{\pm\pm}(t, x), \theta_{\pm\pm}(t, y)] = \pm 2i (\theta_{\pm\pm}(t, x) + \theta_{\pm\pm}(t, y)) \delta'(x - y) \mp \frac{ic_\pm}{6\pi} \delta'''(x - y) \quad (1.15)$$

where according to (1.14), $c_\pm > 0$. In a parity invariant theory, which we shall henceforth consider, $c_+ = c_- \equiv c$. Of course, more complicated modifications are also possible.

The addition to the commutator is frequently called a *Schwinger* term because the argument for its occurrence is an adaptation to the present context of Schwinger's proof that current commutators possess non-canonical contributions involving derivatives of delta functions. Also, it is a non-canonical commutator *anomaly*, giving rise to an *extension* or a *center* to the Lie algebra of generators.

$$\begin{aligned} [Q_f, Q_g] &= iQ_{(f,g)} \\ &\quad - \frac{ic}{24\pi} \int dx \left\{ \left(f^+ \frac{d^3}{dx^3} g^+ - g^+ \frac{d^3}{dx^3} f^+ \right) - \left(f^- \frac{d^3}{dx^3} g^- - g^- \frac{d^3}{dx^3} f^- \right) \right\} \end{aligned} \quad (1.16)$$

Notice that the extension vanishes for the $SO(2,2)$ subgroup when the conformal Killing vectors are at most quadratic. Also it vanishes for the subgroup of transformations for which $f^+(z) = -f^-(-z)$. We call these the *Rindler* transformations; they arise when $x+t$ and $x-t$ are transformed by the same function, which means that $t=0$ is left unchanged. The transformation to an accelerated coordinate system, frequently studied as an aid to understanding black holes, belongs to this class.³

The technical reason for the difference between the classical (1.12) and quantal (1.15) commutators is that the energy-momentum tensor is not well-defined in the quantum theory since it involves products of non-commuting operators. In the conventional resolution of this problem, one selects a Fock vacuum [not necessarily the true vacuum of the theory] and normal orders $\theta_{\mu\nu}$ with respect to this state. The commutator anomaly then emerges when the right-hand side of (1.12) is normal ordered. Alternatively, one may compute the true vacuum expectation value of the commutator in perturbation theory, using the Bjorken, Johnson, Low technique.

Here is presented another approach, which makes no reference to the vacuum state. Rather, we first explore, in Section II, the implication of a commutator anomaly for the representation theory of the conformal group: the representations are necessarily projective. Then, in Section III we construct the representation, determine the projective phase, and extract from it the extension (1.16) with $c=1$. In Section IV, it is shown how to construct representations which lead to arbitrary $c>1$. Finally, in Section V we discuss how the states of the theory transform under conformal transformations, with special attention to particle production in accelerated coordinates. References [4] and [5] comprise the original research papers on which these lectures are based.

II. COCYCLES AND EXTENSIONS IN REPRESENTATION THEORY

When one is dealing with a group of transformations g , which obey the composition law

$$g_1 g_2 = g_{1 \circ 2} \quad (2.1)$$

and transform variables q according to a definite rule,

$$q \xrightarrow{g} q^g \quad (2.2)$$

then a representation of this on functions of q , $\Psi(q)$, is gotten by associating with g an operator $U(g)$, which realizes (2.2) according to

$$U(g)qU^{-1}(g) = q^g \quad (2.3)$$

In the simplest case, $U(g)$ is defined to act on $\Psi(q)$ according to

$$U(g)\Psi(q) = \Psi(q^g) \quad (2.4)$$

and to satisfy a composition law which parallels (2.1).

$$U(g_1)U(g_2) = U(g_{1 \circ 2}) \quad (2.5)$$

The multiplication in (2.5) is associative.

$$\left(U(g_1)U(g_2) \right) U(g_3) = U(g_1) \left(U(g_2)U(g_3) \right) \quad (2.6)$$

However, it is possible to elaborate this simplest realization by introducing phases into formulas (2.4), (2.5) and (2.6). Such phases are called cocycles and the various conditions that they must satisfy have been elaborated in mathematics.⁶

It has been known for some time that cocycles arise in quantum mechanics of point particles. Moreover, recently it was appreciated that the anomaly phenomena of field theory makes use of them as well.

1-COCYCLE

In the simplest generalization, a phase factor is inserted in the action of the representation. Rather than (2.4) we have

$$U(g)\Psi(q) = e^{i2\pi\omega_1(q,g)}\Psi(q^g) \quad (2.7)$$

Consistency with (2.5) requires that ω_1 satisfy

$$\omega_1(q^{g_1}; g_2) - \omega_1(q; g_1 \circ g_2) + \omega_1(q; g_1) = 0 \pmod{\text{integer}} \quad (2.8)$$

A quantity $\omega_1(q, g)$ depending on one member of the transformation group, g , and possibly on the variable acted upon, q , is called a *1-cocycle*, if it obeys (2.8).

Representations of Galilean boosts in quantum mechanics make use of a 1-cocycle. The Abelian group of Galileo transformations on the position vector $\mathbf{r}$ [q in the general discussion] is defined to act as $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{v}t$. Also, we have $\mathbf{p} = m\dot{\mathbf{r}} \rightarrow \mathbf{p} + m\mathbf{v}$ [m is the particle mass]. The transformation is labelled by $\mathbf{v}$ [corresponding to g in the general discussion] and is implemented on wave functions $\Psi(\mathbf{r})$ by the operator

$$\begin{aligned} U(\mathbf{v}) &= e^{i\mathbf{v} \cdot (\mathbf{p}t - m\mathbf{r})} \\ U(\mathbf{v})\mathbf{r}U^{-1}(\mathbf{v}) &= \mathbf{r} + \mathbf{v}t \\ U(\mathbf{v})\mathbf{p}U^{-1}(\mathbf{v}) &= \mathbf{p} + m\mathbf{v} \end{aligned} \quad (2.9)$$

as is established with the help of Heisenberg's algebra.

$$[r^i, r^j] = 0, \quad [r^i, p^j] = i\delta^{ij}, \quad [p^i, p^j] = 0 \quad (2.10)$$

One verifies that (2.5) is satisfied, but $U(\mathbf{v})$ acts on wavefunctions $\Psi(\mathbf{r})$ as in (2.7) with

$$2\pi\omega_1(\mathbf{r}; \mathbf{v}) = - \left(m\mathbf{v} \cdot \mathbf{r} + \frac{1}{2}m\mathbf{v}^2 t \right) \quad (2.11)$$

which satisfies (2.8).

2-COCYCLE

In the next generalization, a phase is introduced into the composition law (2.5).

$$U(g_1)U(g_2) = e^{i2\pi\omega_2(q;g_1,g_2)}U(g_{1\circ 2}) \quad (2.12)$$

A consistency condition on ω_2 follows from associativity (2.6).

$$\omega_2(q^{g_1}; g_2, g_3) - \omega_2(q; g_{1\circ 2}, g_3) + \omega_2(q; g_1, g_{2\circ 3}) - \omega_2(q; g_1, g_2) = 0 \pmod{\text{integer}} \quad (2.13)$$

When a quantity depends on two group elements and possibly on q , and also satisfies (2.13), it is called a *2-cocycle*. Representations that make use of 2-cocycles are called *projective* or *ray* representations and they too occur in quantum mechanics.

Translations on phase space: $\mathbf{r} \rightarrow \mathbf{r} + \mathbf{a}$, $\mathbf{p} \rightarrow \mathbf{p} + \mathbf{b}$, which arise when Galileo boosts are supplemented by spatial translations, are represented by the operator

$$\begin{aligned} U(\mathbf{a}, \mathbf{b}) &= e^{i(\mathbf{a} \cdot \mathbf{p} - \mathbf{b} \cdot \mathbf{r})} \\ U(\mathbf{a}, \mathbf{b})\mathbf{r}U^{-1}(\mathbf{a}, \mathbf{b}) &= \mathbf{r} + \mathbf{a} \\ U(\mathbf{a}, \mathbf{b})\mathbf{p}U^{-1}(\mathbf{a}, \mathbf{b}) &= \mathbf{p} + \mathbf{b} \end{aligned} \quad (2.14)$$

which composes according to (2.12) with

$$2\pi\omega_2(\mathbf{r}; \mathbf{a}_1\mathbf{b}_1, \mathbf{a}_2\mathbf{b}_2) = \frac{1}{2}(\mathbf{a}_2 \cdot \mathbf{b}_1 - \mathbf{a}_1 \cdot \mathbf{b}_2) \quad (2.15)$$

The above obeys the consistency condition (2.13).

When dealing with a continuous Lie group of transformations, the discussion may be carried out in infinitesimal terms. Corresponding to the finite group element g , there is an infinitesimal quantity θ , $g = e^\theta$, and the composition law (2.1) is reflected in a Lie algebra.

$$[\theta_1, \theta_2] = \theta_{(1,2)} \quad (2.16)$$

The infinitesimal action of the transformation on q is given by

$$q \rightarrow q + \delta_\theta q \quad (2.17)$$

and is represented by a generator G_θ .

$$i[G_\theta, q] = \delta_\theta q \quad (2.18)$$

The operator representing the finite transformation is the exponentiated generator, $U(g) = e^{iG_\theta}$, and the composition law without a cocycle (2.5) insures that G_θ satisfies the Lie algebra of the group.

$$i[G_{\theta_1}, G_{\theta_2}] = G_{\theta_{(1,2)}} \quad (2.19)$$

The presence of a 2-cocycle in the composition law, as in (2.12), means that the commutator algebra of the generators acquires an extension,

$$i[G_{\theta_1}, G_{\theta_2}] = G_{\theta_{(1,2)}} + \delta\omega_2 \quad (2.20)$$

where $\delta\omega_2$ is [proportional to] the infinitesimal portion of the 2-cocycle, whose consistency condition (2.13) insures that (2.20) does not contradict the Jacobi identity, which is the infinitesimal version of (2.6).

Thus, the Heisenberg algebra (2.10) is an extension of the Abelian algebra of translations, and we appreciate that the essence of quantum mechanics is a 2-cocycle. Also, we see that the extension in (1.16) signals that quantum mechanically the conformal transformation group is realized projectively.

HIGHER COCYCLES

I shall not pursue the subject of cocycles further, beyond mentioning that, not unexpectedly, a 3-cocycle involves abandoning associativity: Eq. (2.6) acquires a phase, while infinitesimally the Jacobi identity fails. This happens for gauge invariant velocity operators of a charged particle in the presence of an external magnetic field which is not source-free. The 3-cocycle then measures the magnetic flux enclosed by an arbitrary surface. The only physically conceivable magnetic sources are magnetic monopoles, and the magnetic flux they produce through closed surfaces is quantized in integer units, as a consequence of the Dirac quantization condition. Hence, the non-associativity is invisible since $e^{i2\pi n} = 1$. Indeed, one may view the demand of associativity as the origin of the Dirac quantization condition.

Finally, it must be stressed that cocycles are of interest only when they are non-trivial; i.e. when they cannot be removed by redefining phases. For example, a 2-cocycle of the form

$$\omega_2(q; g_1, g_2) = \alpha(q; g_1) + \alpha(q^{g_1}; g_2) - \alpha(q; g_{1 \circ 2}) \quad (2.21)$$

satisfies (2.13) with arbitrary α , but may be removed by redefining the phase of $U(g)$ by $2\pi\alpha(q; g)$. Hence it is trivial.

III. FUNCTIONAL REPRESENTATION OF CONFORMAL TRANSFORMATIONS, $c=1$

We now construct a representation of the conformal group for the simplest case: a two-dimensional spinless, dimensionless and massless free field Φ . The transformation law (1.10) reduces to the Lie derivative of Φ .

$$\delta_f \Phi = f^\alpha \partial_\alpha \Phi \quad (3.1)$$

We further restrict the discussion to one component of the group, transforming x^+ but not x^- . Since transformations belonging to different components act independently, we can always construct the full transformation sequentially. Thus we set $f^- = 0$ and define $f(x) \equiv \sqrt{2}f^+(x/\sqrt{2})$.

We adopt a canonical, fixed-time $[t = 0]$ description, where one deals with a pair of canonical variables Φ and $\Pi \equiv \dot{\Phi}$, which depend on x [but not on t] and satisfy the usual commutation relations. In terms of these, the transformation (3.1) is

$$\begin{aligned}\delta_f \Phi &= \frac{1}{2} f(\Pi + \Phi') \\ \delta_f \Pi &= \frac{1}{2} f(\Pi + \Phi')' + \frac{1}{2} f'(\Pi + \Phi')\end{aligned}\tag{3.2}$$

It is advantageous to introduce a self-dual field χ [it depends only on x^+]

$$\begin{aligned}\chi &\equiv \frac{1}{\sqrt{2}}(\Pi + \Phi') = \partial_+ \Phi \\ \dot{\chi} &= \chi' \\ [\chi(x), \chi(y)] &= i\delta'(x - y) \equiv k(x, y)\end{aligned}\tag{3.3}$$

Then, (3.2) reads

$$\delta_f \chi = (f\chi)'\tag{3.4}$$

The traceless energy momentum tensor of the theory

$$\theta_{\mu\nu} = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \Phi \partial_\alpha \Phi\tag{3.5a}$$

may also be written in the Sugawara-Sommerfield form in terms of a conserved current $J_\mu = \frac{1}{\sqrt{\pi}} \epsilon_{\mu\nu} \partial^\nu \Phi$.

$$\theta_{\mu\nu} = \pi \left(J_\mu J_\nu - \frac{1}{2} g_{\mu\nu} J^\alpha J_\alpha \right)\tag{3.5b}$$

[Similar formulas hold for $U(N)$ multiplets of free massless fermions, provided products of $U(N)$ currents are properly defined, as well as for bosons and fermions interacting with background gauge or gravitational fields.⁷] The generator of (3.4) involves θ_{++} .

$$Q_f = \frac{1}{2} \int dx f \chi^2\tag{3.6}$$

$$i[Q_f, \chi] = \delta_f \chi\tag{3.7}$$

and the finite transformation is given by

$$\begin{aligned}e^{i\tau Q_f} \chi(x) e^{-i\tau Q_f} &= \frac{f(X)}{f(x)} \chi(X) = \frac{\partial X(x)}{\partial x} \chi(X) \\ X(x) &= \mathcal{F}^{-1}(\tau + \mathcal{F}(x)) \quad , \quad \mathcal{F}(x) \equiv \int^x \frac{dz}{f(z)}\end{aligned}\tag{3.8}$$

[We assume that $f(x)$ has all the properties needed to justify the manipulations that we perform with it.]

Of course, Eqs. (3.6) – (3.8) are formal, since powers of the non-commuting operator $\chi(x)$ at the same point which occur in Q_f need to be defined. The difficulty with working with (3.6) is not apparent in (3.7) and (3.8), but when the commutator of two charges is evaluated formally, the necessary extension is not found. Rather than handling the problem conventionally by normal ordering χ^2 , we proceed by constructing a representation of the group of transformations (3.4). The Lie algebra may then be determined.

In order to build a well-defined generator, $f(x)$ is promoted to a real, symmetric bilocal function $F(x, y)$ which in the local limit becomes $f(x)\delta(x - y)$. The regulated generator

$$Q_F = \frac{1}{2} \int dx dy \chi(x) F(x, y) \chi(y) \quad (3.9)$$

is not beset by singularities; it satisfies the algebra without center.

$$[Q_F, Q_G] = iQ_{(F, G)} \quad , \quad i(F, G) \equiv FkG - GkF \quad (3.10)$$

but Q_F does not possess a local limit. [A functional matrix notation is being used: *e.g.* $(FkG)(x, y) \equiv \int dz dz' F(x, z) k(z, z') G(z', y) = i \int dz F(x, z) \frac{\partial}{\partial z} G(z, y)$.] Of course, Q_F no longer generates a symmetry, but it is a generator of a linear canonical transformation.⁸

As we shall see, a local limit can be taken if we add a *c*-number to Q_F , which is linear in F . So we define

$$: Q_F : \equiv Q_F - \frac{1}{4} \text{tr} \{ F^{1/2} \Omega F^{1/2} \} \quad (3.11)$$

and the formal definition (3.6) is replaced by $Q_f = \lim_{F \rightarrow f} : Q_F : \cdot \Omega(x, y)$ will be determined by the requirement that the representation of $e^{-i\tau : Q_F :}$ possesses a local limit. We emphasize that the colons do not signify normal ordering with respect to the Fock vacuum; they are defined by (3.11). Since the modification is a *c*-number, it does not affect the infinitesimal (3.7) nor finite (3.8) transformation law. However, non-linear relations like (3.10) are changed

$$[: Q_F :, : Q_G :] = i : Q_{(F, G)} : + \frac{i}{4} \text{tr} \{ (F, G) \Omega \} \quad (3.12)$$

To construct the representation, we consider eigenstates $|\varphi\rangle$ of Φ .

$$\Phi(x)|\varphi\rangle = \varphi(x)|\varphi\rangle \quad , \quad \langle\varphi_1|\varphi_2\rangle = \delta(\varphi_1(x) - \varphi_2(x)) \quad (3.13)$$

The scalar product involves a functional δ -function and functional integration is used to sum over φ . In this basis, χ is represented by

$$\langle\varphi_1|\chi(x)|\varphi_2\rangle = \left(\frac{1}{i} \frac{\delta}{\delta\varphi_1(x)} + \varphi_1'(x) \right) \delta(\varphi_1 - \varphi_2) \quad (3.14)$$

We seek the form of

$$U(\varphi_1, \varphi_2; \tau F) \equiv \langle \varphi_1 | e^{-i\tau Q_F} | \varphi_2 \rangle = U^*(\varphi_2, \varphi_1; -\tau F) , \quad (3.15)$$

which must be a Gaussian in φ_1 and φ_2 since Q_F is quadratic in χ . U may be determined by solving the Schrödinger-like equation obtained by differentiating (3.15) with respect to τ , evaluating the matrix element of Q_F from (3.14) and using the boundary condition (3.13). The result is

$$U(\varphi_1, \varphi_2; \tau F) = e^{\Gamma(\tau F)} e^{i \int dx \varphi_1'(x) \varphi_2(x)} \times \exp \frac{i}{2} \int dx dy [\varphi_1(x) - \varphi_2(x)] K_{\tau F}(x, y) [\varphi_1(y) - \varphi_2(y)] \quad (3.16)$$

where

$$K_{\tau F} = F^{-1/2} \left\{ k_F \operatorname{ctn} \left(\frac{\tau}{2} k_F \right) \right\} F^{-1/2} \quad (3.17a)$$

$$\Gamma(\tau F) = -\frac{1}{2} \operatorname{tr} \ln \left\{ F^{1/2} \frac{2\pi i \sin \left(\frac{\tau}{2} k_F \right)}{k_F} F^{1/2} \right\} \quad (3.17b)$$

$$k_F \equiv F^{1/2} k F^{1/2} . \quad (3.17c)$$

The functions of the kernels are defined either by power series, or by diagonalizing k_F . The composition law can be verified.

$$\int D\varphi_2 U(\varphi_1, \varphi_2; \tau F) U(\varphi_2, \varphi_3; \tau G) = U(\varphi_1, \varphi_3; \tau F \circ G) \quad (3.18)$$

Let us now examine the local limit $F \rightarrow f$. The spectrum of k_f is the real line and the normalized and complete eigenfunctions are

$$\int dy k_f(x, y) \psi_\lambda^f(y) = \lambda \psi_\lambda^f(x) \quad (3.19)$$

$$\psi_\lambda^f(x) = \frac{1}{\sqrt{2\pi f(x)}} \exp \left\{ -i\lambda \int_y^x \frac{dz}{f(z)} \right\} .$$

Consequently,

$$K_{\tau f}(x, y) = \frac{1}{f(x)} \int \frac{d\lambda}{2\pi} \lambda \operatorname{ctn} \left(\frac{\tau}{2} \lambda \right) \exp \left\{ -i\lambda \int_y^x \frac{dz}{f(z)} \right\} \frac{1}{f(y)} . \quad (3.20)$$

The λ integration requires a prescription for handling the singularities: continue to the imaginary axis $\tau \rightarrow -i\tau'$, evaluate the integral and continue back $\tau' \rightarrow i\tau + 0^+$. This gives [for $\tau > 0$]

$$K_{\tau f}(x, y) = -i\pi \frac{1}{\tau f(x)} P \csc^2 \left\{ \pi \int_y^x \frac{dz}{\tau f(z)} \right\} \frac{1}{\tau f(y)} \quad (3.21)$$

(P means principal value.) For the prefactor, Γ , we proceed as follows. After continuation, we have

$$\begin{aligned}
\Gamma(-i\tau'F) &= -\frac{1}{2} \text{tr} \ln \left\{ F^{1/2} \frac{2\pi \sinh\left(\frac{\tau'}{2} k_F\right)}{k_F} F^{1/2} \right\} \\
&= -\frac{1}{2} \text{tr} \ln \left\{ F^{1/2} \frac{2\pi \sinh\left(\frac{\tau'}{2} \omega_F\right)}{\omega_F} F^{1/2} \right\} \\
&= \frac{1}{2} \text{tr} \ln \left(\frac{1}{\pi} \omega \right) - \frac{1}{2} \text{tr} \ln \left\{ 1 - e^{-\tau' \omega_F} \right\} - \frac{\tau'}{4} \text{tr} \omega_F
\end{aligned} \tag{3.22}$$

where $\omega_F(x, y) \equiv |k_F|(x, y)$ and $\omega(x, y) \equiv |k|(x, y) = \int \frac{d\lambda}{2\pi} e^{-i\lambda(x-y)} |\lambda| = -P \frac{1}{\pi(x-y)^2}$. The first term is an infinite τ' -independent constant, $\ln Z \equiv \frac{1}{2} \text{tr} \ln \left(\frac{1}{\pi} \omega \right) \sim \frac{1}{2} \int dx \int \frac{d\lambda}{2\pi} \ln \frac{1}{\pi} |\lambda|$, which may be removed by redefining the functional integration measure $\mathcal{D}\varphi \rightarrow Z^{-1} \mathcal{D}\varphi$. The second attains a finite local limit.

$$\gamma(-i\tau'f) = -\frac{1}{2} \int \frac{dx}{f(x)} \int \frac{d\lambda}{2\pi} \ln(1 - e^{-\tau'|\lambda|}) = \frac{\pi}{12} \int \frac{dx}{\tau'f(x)} \tag{3.23}$$

The third has no local limit. However, we observe that

$$\begin{aligned}
\omega_f(x, y) &= \frac{1}{f^{1/2}(x)} \int \frac{d\lambda}{2\pi} |\lambda| \exp \left\{ -i\lambda \int_y^x \frac{dz}{f(z)} \right\} \frac{1}{f^{1/2}(y)} \\
&= -\frac{1}{f^{1/2}(x)} P \frac{1}{\pi \left(\int_y^x \frac{dz}{f(z)} \right)^2} \frac{1}{f^{1/2}(y)} \\
&= \frac{1}{f^{1/2}(x)} \frac{(x-y)^2}{\left(\int_y^x \frac{dz}{f(z)} \right)^2} \frac{1}{f^{1/2}(y)} \omega(x, y)
\end{aligned} \tag{3.24}$$

Since the quantity multiplying ω in the last formula of (3.25) becomes $f(x)$ for $x = y$ we see that

$$\text{tr} \omega_F = \text{tr}(F^{1/2} \omega F^{1/2}) + \left\{ \begin{array}{l} \text{terms that vanish} \\ \text{in the local limit} \end{array} \right\} . \tag{3.25}$$

Thus, if we chose Ω in (3.11) to coincide with ω , $e^{-i\tau:Q_F:}$ has the representation

$$Z^{-1/2} \langle \varphi_1 | e^{-i\tau:Q_F:} | \varphi_2 \rangle Z^{-1/2} = Z^{-1} e^{\frac{i\tau}{4} \text{tr}(F\omega)} U(\varphi_1, \varphi_2; \tau F) \tag{3.26}$$

which tends in the local limit to

$$\begin{aligned}
:U: (\varphi_1, \varphi_2; \tau f) &= \exp -i \left\{ \frac{\pi}{12} \int \frac{dx}{\tau f(x)} - \int dx \varphi_1'(x) \varphi_2(x) \right\} \times \\
&\exp \frac{\pi}{2} \int dx dy [\varphi_1(x) - \varphi_2(x)] \frac{1}{\tau f(x)} P \csc^2 \left(\pi \int_y^x \frac{dz}{\tau f(z)} \right) \frac{1}{\tau f(y)} [\varphi_1(y) - \varphi_2(y)]
\end{aligned} \tag{3.27}$$

But the composition law changes. From (3.18) it follows that instead of that formula a projective rule with a 2-cocycle is satisfied.

$$\int D\varphi_2 : U : (\varphi_1, \varphi_2; \tau F) : U : (\varphi_2, \varphi_3; \tau G) = e^{i\omega_2(F, G)} : U : (\varphi_1, \varphi_3; \tau F \circ G) \quad (3.28)$$

The 2-cocycle in the local limit is given by

$$\omega_2(f, g) = \lim_{\substack{F \rightarrow f \\ G \rightarrow g}} \frac{\tau}{4} \text{tr} \{ (F + G - F \circ G) \omega \} . \quad (3.29)$$

Before the limit is taken, the cocycle is trivial in the sense (2.21); it arises because the phase of U has been redefined, and ω_2 can similarly be removed. However, after the local limit, a non-trivial expression remains which to $O(\tau^2)$ produces the extension in (1.16). This is evaluated as follows. The $O(\tau^2)$ contribution to (3.29) is $-\frac{\tau^2}{8} \text{tr} \{ (F, G) \omega \}$. In terms of a Fourier representation $F(x, y) = \int \frac{dp}{2\pi} \frac{dq}{2\pi} e^{-i\frac{p}{2}(x+y)} e^{-iq(z-y)} \tilde{F}(p, q)$. $\tilde{F}(p, q) = \tilde{F}(p, -q) \xrightarrow{F \rightarrow f} \tilde{f}(p) = \int dx e^{ipx} f(x)$, and similarly for G , this reads

$$\begin{aligned} -\frac{\tau^2}{8} \text{tr} \{ (F, G) \omega \} &= \frac{i\tau^2}{8} \text{tr} \{ (FkG - GkF) \omega \} \\ &= \frac{i\tau^2}{8} \left\{ \int \frac{dp}{2\pi} \frac{dq}{2\pi} \epsilon \left(q + \frac{p}{2} \right) \left(q^2 - \frac{p^2}{4} \right) \tilde{F}(p, q) \tilde{G}(-p, q) - (F \leftrightarrow G) \right\} \\ &= \frac{i\tau^2}{8} \left\{ \int \frac{dp}{2\pi} \int_{-p/2}^{p/2} \frac{dq}{2\pi} \left(q^2 - \frac{p^2}{4} \right) \tilde{F}(p, q) \tilde{G}(-p, q) - (F \leftrightarrow G) \right\} \end{aligned}$$

where the last expression follows from the previous owing to the the symmetry of $\tilde{F}$ and $\tilde{G}$ in q . Next, we pass to the local limit, perform the q integral and are left with

$$-\frac{i\tau^2}{96\pi} \int \frac{dp}{2\pi} p^3 \left\{ \tilde{f}(p) \tilde{g}(-p) - \tilde{g}(p) \tilde{f}(-p) \right\} = -\frac{\tau^2}{2} \frac{1}{48\pi} \int dx (f'''g - g'''f)$$

This corresponds to the charge algebra

$$[Q_f, Q_g] = iQ_{(f,g)} + \frac{i}{48\pi} \int dx (f'''g - g'''f) \quad (3.30)$$

i.e. in (1.16), $c = 1$.

Note the following properties of the representation functional (3.27). $: U :$ is invariant against the replacement of $\varphi_i(x)$ by $\varphi_i(X(x))$. This is a consequence of the identity $\frac{dX}{f(X)} = \frac{dx}{f(x)}$ and the coordinate invariance of the symplectic structure $\int dx \varphi'_1 \varphi_2 = -\int \varphi_1 d\varphi_2$. The occurrence of this symplectic form is natural for self-dual scalar fields. A Lagrangian for a self-dual field is

$$L_X = \frac{1}{4} \int dx dy X(t, x) \epsilon(x - y) \dot{X}(t, y) - \frac{1}{2} \int dx X^2(t, x) \quad (3.31)$$

which leads to the Hamiltonian

$$H_X = \frac{1}{2} \int dx \chi^2 = Q_1 \quad (3.32)$$

and canonical quantization produces precisely the commutator (3.3). Moreover, L_X is invariant under (3.4), and Q_f of (3.6) is the associated Noether constant of motion. So we recognize that $:U: (\varphi_1, \varphi_2; t)$ is the propagation kernel for time evolution of a self-dual field. Moreover, Q_f and $:U: (\varphi_1, \varphi_2; tf)$ may be viewed as the Hamiltonian and propagation kernel for a self-dual field $f\chi$ whose propagation is described with time intervals that vary in space as $\Delta t = \frac{1}{f(x)}$.

The self-dual field χ , with its non-vanishing commutation relation (3.3), is somewhat analogous to the velocity operator of a charged particle in an external magnetic field. That operator also does not commute with itself; the commutator gives rise to the external magnetic field. So we may view a self-dual field as defined on a [functional] configuration space with a [functional] $U(1)$ connection that gives rise to a constant [functional] curvature given by (3.3).⁹ Indeed, the propagator for a charged particle in a constant magnetic field bears close analogy to our functional propagators.⁵

Finally, let us observe that the kernel which connects an eigenstate of Π with that of Φ is given by an elegant expression. We record the formula for the unregulated bilocal transformation function.

$$\begin{aligned} \tilde{U}(\pi_1, \varphi_2; \tau F) &\equiv \langle \pi_1 | e^{-i\tau Q_F} | \varphi_2 \rangle / \langle \pi_1 | \varphi_2 \rangle \\ \int D\varphi_1 e^{-i \int dx \pi_1 (\varphi_1 - \varphi_2)} U(\varphi_1, \varphi_2; \tau F) &= e^{\tilde{\Gamma}(\tau F)} \exp -\frac{i}{2} \int dx \chi(x) K_{\tau F}^{-1}(x, y) \chi(y) \\ \chi &\equiv \pi_1 + \varphi_2' \\ K_{\tau F}^{-1} &= F^{1/2} \left\{ \frac{\tan(\frac{\tau}{2} k_F)}{k_F} \right\} F^{1/2} \\ \tilde{\Gamma}(\tau F) &= -\frac{1}{2} \text{tr} \ln \left\{ \cos \frac{\tau}{2} k_F \right\} \end{aligned} \quad (3.33)$$

As mentioned earlier, the representation functional for a full transformation acting both on x^+ and x^- may be obtained by composing the individual ones. In general, only formal expressions emerge which involve separate kernels K corresponding to the two separate transformations. A simple formula arises in the Rindler case, $-f^-(-x) = f^+(x) = \frac{1}{\sqrt{2}} f(\sqrt{2}x)$.

$$:U: (\varphi_1, \varphi_2; \tau f) = \delta(\varphi_1(X^{-1}(x)) - \varphi_2(x)) \quad (3.34)$$

where $X(x)$ is given in (3.8). As discussed in the Introduction, no center is present in the Lie algebra of the Rindler subgroup of transformations. Correspondingly, the representation (3.34) is faithful; it composes without a cocycle.

A closed formula for the representation functional may also be obtained when $f^+ = f^-$.⁵

IV. FUNCTIONAL REPRESENTATION OF CONFORMAL TRANSFORMATIONS, $c > 1$

It is also possible to find representations which lead to an extension in (1.16) with $c > 1$. We consider again the scalar field Φ but postulate a formula for the energy-momentum tensor which modifies and "improves" upon (3.5).

$$\begin{aligned}\theta_{\mu\nu}^\lambda &= \theta_{\mu\nu} + 2\lambda (g_{\mu\nu}\square - \partial_\mu\partial_\nu)\Phi \\ &= \theta_{\mu\nu} - 2\lambda\partial_\mu\partial_\nu\Phi\end{aligned}\quad (4.1)$$

$\theta_{\mu\nu}^\lambda$ remains conserved, symmetric and traceless, and we may construct the conformal charges from $\theta_{\mu\nu}^\lambda$, rather than from $\theta_{\mu\nu}$.

$$Q_f^\lambda = Q_f - 2\lambda \int dx f^\mu \partial_\mu \Phi \quad (4.2)$$

The modified transformation generated by Q_f^λ is inhomogeneous.

$$i[Q_f^\lambda, \Phi] = f^\mu \partial_\mu \Phi + \lambda \partial_\mu f^\mu \quad (4.3)$$

This is a symmetry operation because $\lambda \partial_\mu f^\mu$ shifts Φ by a harmonic function [$\square \partial_\mu f^\mu = 0$, when f^μ is a conformal Killing vector] which is an invariance of a free, massless field.

In terms of the previously defined canonical variables we have

$$Q_f^\lambda = Q_f + \sqrt{2}\lambda \int dx f' \chi \quad (4.4)$$

It is now easy to compute the modified Lie algebra from (3.7), (3.30). Eq. (3.30) is again true for Q_f^λ except the center acquires the factor $1 + 48\pi\lambda^2$; i.e. $c = 1 + 48\pi\lambda^2$. Also, the representation functional for the finite transformation acquires a further phase.

$$:U^\lambda : (\varphi_1, \varphi_2; \tau f) = :U : (\varphi_1, \varphi_2; \tau f) \times \exp -i2\lambda \int dx (\ln \tau f)' \left(\varphi_1 - \varphi_2 - \frac{\tau\lambda}{2} f' \right) \quad (4.5)$$

The extension is already present classically: evaluation of the Lie algebra with Poisson brackets produces $c = 48\pi\lambda^2$, while the quantum correction adds the additional 1. It should be recalled that while the free theory allows any value of λ , there exists a 2-dimensional interacting theory, the Liouville model, with conformal invariance realized inhomogeneously as in (4.3), where λ is identified with the Liouville coupling constant.¹⁰

Are there models with $c < 1$? It is known that unitarity restricts values of c below 1.¹¹ The first permitted is $c = \frac{1}{2}$ and this is realized for free Majorana or Weyl fermions. [Dirac fermions, even when interacting as in the Thirring model,² possess $c = 1$.] For $\frac{1}{2} < c < 1$, there are known statistical models at discrete values of c , but

apparently there are no field theories. However, I know of no proof that for quantum field theory $c = 1/2$ or $c \geq 1$. Perhaps a spectral sum rule of the Källen-Lehmann sort is called for.

V. CONFORMALLY TRANSFORMED STATES AND PARTICLE PRODUCTION IN ACCELERATED FRAMES

I shall now discuss how the states of our massless free field theory transform under conformal transformations. Of course, the general result is clear: for any state, $\Psi(\varphi)$, which in our representation is given by a functional of φ , the transformed state $\tilde{\Psi}(\varphi)$ is

$$\tilde{\Psi}(\varphi) = \int \mathcal{D}\tilde{\varphi} : U : (\varphi, \tilde{\varphi}; \tau f) \Psi(\tilde{\varphi}) \quad (5.1a)$$

In order to deal with well-behaved expressions, it is better to view (5.1a) as the local limit of

$$\Psi_{\tau F}(\varphi) = \int \mathcal{D}\tilde{\varphi} : U : (\varphi, \tilde{\varphi}; \tau F) \Psi(\tilde{\varphi}) \quad (5.1b)$$

We present some explicit formulas.

An arbitrary Gaussian

$$\Psi_{\Omega}(\varphi) = e^{-\frac{1}{2} \int dx dy \varphi(x) \Omega(x, y) \varphi(y)} \quad (5.2)$$

transforms according to (5.1) into

$$\Psi_{\Omega_{\tau F}}(\varphi) = e^{\sigma(\tau F)} e^{-\frac{1}{2} \int dx dy \varphi(x) \Omega_{\tau F}(x, y) \varphi(y)} \quad (5.3)$$

where [see (3.17) and (3.23)].

$$\Omega_{\tau F} = \Omega - (\Omega + k)(\Omega - iK_{\tau F})^{-1}(\Omega - k) \quad (5.4a)$$

$$\sigma(\tau F) = \gamma(\tau F) - \frac{1}{2} \text{tr} \ln \frac{1}{2\pi} (\Omega - iK_{\tau F}) \quad (5.4b)$$

Infinitesimally, these formulas read

$$\delta_F \Omega = -\frac{i\tau}{2} (\Omega + k) F (\Omega - k) \quad (5.5a)$$

$$\delta_F \sigma = -\frac{i\tau}{4} \text{tr} F (\Omega - \omega) \quad (5.5b)$$

and one verifies that (5.5a) gives rise to the Lie algebra without center, which resides in (5.5b).

Of particular interest, of course, is the vacuum state for the Hamiltonian of our theory.

$$H = \frac{1}{2} \int dx (\Pi^2 + \Phi'^2) \quad (5.6)$$

The ground state functional is Gaussian

$$\begin{aligned}\Psi_\omega(\varphi) &= e^{-\frac{1}{2} \int dx dy \varphi(x) \omega(x,y) \varphi(y)} \\ \omega(x,y) &= -\frac{1}{\pi} P \frac{1}{(x-y)^2}\end{aligned}\tag{5.7}$$

The occurrence of precisely the kernel ω in the ground state shows that our group theoretically motivated subtraction in Q_F coincides with the conventional subtraction. From (3.24) we see that ω_f may also be written as $\omega_f(x,y) = f^{1/2}(x) \left\{ \frac{1}{f(x)} \omega \left(\int^x \frac{dz}{f(z)}, \int^y \frac{dz}{f(z)} \right) \frac{1}{f(y)} \right\} f^{1/2}(y)$. A Gaussian with the kernel as in the curly brackets is an eigenstate of the unsubtracted Q_f , Eq. (3.6), with eigenvalue $\frac{1}{4} \text{tr} \omega_f$. Hence, we may also view our regularization of Q_f as normal ordering with respect to the “vacuum” of Q_f .

The conformally transformed vacuum is given by Eqs. (5.3) and (5.4) with $\Omega = \omega$. By iteration, one can obtain the relevant expressions for the case when both x^+ and x^- are transformed.

Evidently, the ground state is not invariant under arbitrary conformal transformations. This, of course, is no surprise: when a transformation is represented projectively there can be no invariant states; equivalently, a central extension in the Lie algebra of generators prevents the generators from annihilating states. On the other hand, since the $SO(2,2)$ subgroup possess no center, one expects invariance of the ground state. Indeed, one can confirm this from (5.4): with $\Omega = \omega$ and $F \rightarrow f = 1, x, x^2$, the second member of the right-hand side of (5.4a) vanishes.¹²

The Rindler subgroup also has no center in the Lie algebra and is represented faithfully by (3.34). But the ground state is not invariant, as follows from (3.34) and (5.1). The transformed vacuum is a Gaussian with kernel $\tilde{\omega}$.

$$\begin{aligned}\tilde{\omega}(x,y) &= \frac{\partial X(x)}{\partial x} \omega(X(x), Y(y)) \frac{\partial Y(y)}{\partial y} \\ \frac{\partial X(x)}{\partial x} &= \frac{f(X)}{f(x)}, \quad \frac{\partial Y(y)}{\partial y} = \frac{f(Y)}{f(y)}\end{aligned}\tag{5.8}$$

Thus for the conformal group, with the exception of the $SO(2,2)$ subgroup, we are facing a kind of “spontaneous symmetry breaking”, which is triggered by the central extension in general, but remains mysterious for the Rindler subgroup.¹² Moreover, this breaking can be related to known physical effects in an accelerated reference frame, which is what a conformally transformed coordinate system describes. We define the number operator

$$N(p, p') = a^\dagger(p) a(p')\tag{5.9}$$

where

$$a(p) = \int \frac{dx}{\sqrt{4\pi|p|}} e^{-ipx} [|p| \Phi(x) + i\Pi(x)]\tag{5.10}$$

and compute its expectation in the transformed vacuum $\Psi_{\tilde{\omega}}$. An alternative and equivalent computation is the expectation of the transformed number operator in the standard vacuum (5.2). An explicit formula can be given,⁵ which for the Rindler subgroup is

$$\begin{aligned} \langle N(p, p') \rangle &= \frac{1}{4\pi\sqrt{|pp'|}} \int dx dy e^{+ipx} e^{-ipy} \times \\ &\times \left(\frac{\partial X^{-1}(x)}{\partial x} \omega(X^{-1}(x), Y^{-1}(y)) \frac{\partial Y^{-1}(y)}{\partial y} \mp k(x, y) \right) \end{aligned} \quad (5.11)$$

where the upper sign holds for p and p' positive, the lower for p and p' negative, and $\langle N \rangle$ vanishes otherwise. The nature of the spectrum is determined by the analytic properties of the new coordinates in terms of the old.¹³ In particular,

$$X^{-1}(x) = \frac{1}{\mu} e^{\mu x} \quad (5.12)$$

i.e. the transformation is to a system with uniform acceleration μ , Eq. (5.11) reduces to the well-known thermal spectrum.¹⁴

$$\langle N(p, p') \rangle = \delta(p - p') \frac{1}{e^{\frac{2\pi}{\mu}|p|} - 1} \quad (5.13)$$

[It should be emphasized that (5.12) is not properly a member of a group, rather it belongs to a semi-group.] We thus arrive at an interpretation for (5.11), (5.13) and for particle production in an accelerated frame as spontaneous breaking of conformal symmetry.

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