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TENSOR GLUONIUM SPECTRUM IN QCD *

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ABSTRACT

We examine the $2^+$ gluonium spectrum in the framework of the Gauss-Weierstrass and Finite Energy QCD sum rules. The results of our analysis support the interpretation of the $\Theta(1710)$ as a tensor glueball, but they also suggest the existence of at least another state with a mass $M \sim 2$ GeV and a width of about 200 MeV.

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I. INTRODUCTION

The existence of colourless bound states of gluons \(^1\), in addition to ordinary quarkonium states, is a clear prediction of Quantum Chromodynamics (QCD). In the past there have been numerous attempts to elucidate the dynamical properties of the lowest \((gg)\) gluonia, particularly the mass spectrum, using various theoretical techniques \(^2\). Although often in quantitative disagreement, different theoretical calculations qualitatively agree in suggesting mass values less than 2-2.5 GeV for the lightest glueballs. On the experimental side \(^3\), signatures for such hadrons have been intensively searched for and at present there are several potential candidates, the \(2^{++}\) \(\theta(1710)\) being perhaps the most promising one.

In this paper we wish to study the tensor gluonium spectrum, with emphasis on the role of the existing candidates, in the framework of the Gauss-Weierstrass QCD sum rules \(^4\) and of the Finite Energy Sum Rules (FESR) which follow from the former. The method has already been described in detail in previous publications \(^4\),\(^5\). Let us just recall that in this formalism the notion of duality, which underlies the various versions of QCD sum rules, is implemented by the convolution of the hadronic spectral function \(\rho(s)\) with a gaussian, centered at an arbitrary point \(\hat{s}\) and having a finite-width resolution \(\sqrt{\Delta}\), i.e.

\[
G(\hat{s},\tau) = \frac{1}{\sqrt{4\pi \Delta}} \int_0^\infty ds \; e^{-\left(s - \hat{s}\right)^2/4\Delta} \rho(s).
\]  

(1)

This approach is particularly advantageous in cases where no beforehand information is available as \(e.g\). the case of gluonium. Some of the advantages are:

i) The possibility of sampling the hadronic spectral function at different duality intervals by tuning the resolution \(\sqrt{\Delta}\). Clearly, this can only be done up to some extent since the left-hand side of Eq.(1) is known in QCD just as a truncated series in \(1/\sqrt{\Delta}\), corresponding to the familiar expansion in a perturbative term plus a limited number of vacuum condensates.

ii) The one-dimensional heat equation satisfied by Eq.(1)

\[
\frac{\partial^2 G(\hat{s},\tau)}{\partial \hat{s}^2} = \frac{\partial G(\hat{s},\tau)}{\partial \tau},
\]

(2)
which allows the interpretation of \( G(\hat{s}, \tau) \) as the heat distribution in a semi-infinite rod \( 0 \leq \hat{s} \leq \infty \), evolving in "time" \( \tau \) from an initial distribution \( \rho(\hat{s}) \). For large enough \( \tau \) the hadronic heat distribution \( G(\hat{s}, \tau) \) should match the corresponding QCD distribution. In previous applications to the \( \rho \)-meson \(^4\), \(^5\) and \( \pi \)-meson \(^6\) channels, as well as to the scalar glueball \( 0^{++} \) channel \(^7\), this heat evolution test has proven a severe check of the consistency between a particular phenomenological parametrization of \( \rho(s) \) and a given set of QCD parameters.

iii) The possibility of deriving, unambiguously, FESR directly from Eq.(1) by using

\[
\int_0^\infty ds \ G(s, \tau) = \int_0^\infty ds \ \rho(s), \quad (\forall \ \tau)
\]

(3)

together with the fact that in QCD

\[
\lim_{\hat{s}/\sqrt{4\tau} \gg 1} G(\hat{s}, \tau) = \rho(\hat{s}).
\]

(4)

The above features make this formalism quite attractive to study the \( 2^{++} \) gluonium spectrum. Particularly, to examine the consistency between existing tensor glueball candidates and available QCD information; the latter being free from instanton effects which could vitiate the operator product expansion.

The paper is organized as follows. In Sec.II we derive the Gauss-Weierstrass and the finite-energy sum rules, and briefly discuss the values of the QCD parameters. In Sec.III we present the phenomenological finite-width parametrization of the hadronic spectral function. In Sec.IV we work out the eigenvalue solutions to the FESR, and in Sec.V we discuss stability tests and duality windows for those solutions. In Sec.VI we perform the heat evolution tests, and finally Sec.VII is devoted to a summary of our results and to concluding remarks.

II. QCD SUM RULES

We begin by considering the two-point function

\[
\Pi_{\mu\nu;\alpha\beta}(q) = i \int d^4 x \ e^{iqx} \langle 0| T(\theta_{\mu\nu}(x)\theta_{\alpha\beta}(0))|0\rangle,
\]

(5)
where $\theta_{\mu \nu}$ is the gluon source

$$
\Theta_{\mu \nu} = - G_{\mu \alpha}^{a} G_{\nu \alpha}^{a} + \frac{i}{4} q_{\mu \nu}^{a} G_{\rho \beta}^{a} G_{\alpha \beta}^{a}, \quad (6)
$$

and quark contributions have been neglected. In this channel the unit operator (see Fig. 1a) starts at order $O(a_S^0)$, while the non-perturbative contributions, which start at dimension-eight $^8$, are given by Born diagrams with Wilson coefficients of order $O(a_S^{-1})$ (see Fig. 1b). Working at leading order in $a_S$ we can then ignore the trace anomaly: $\theta_{\mu} = O(a_S)$. As a result, the two-point function Eq. (5) is going to be traceless as well as transverse, i.e.

$$
\Pi_{\mu \nu, \alpha \beta} = \Pi_{\mu \nu; \alpha \beta} = 0, \quad (7)
$$

$$
q_{\mu} \cdot \Pi_{\mu \nu; \alpha \beta} = 0. \quad (8)
$$

Next, we define the function $T(Q^2)$, $Q^2 \equiv -q^2 > 0$, through

$$
\Pi_{\mu \nu; \alpha \beta} = \eta_{\mu \nu; \alpha \beta} T(Q^2), \quad (9)
$$

where $\eta_{\mu \nu; \alpha \beta}$ is the spin-2 projector

$$
\eta_{\mu \nu; \alpha \beta} = \eta_{\mu \alpha} \eta_{\nu \beta} + \eta_{\mu \beta} \eta_{\nu \alpha} - \frac{2}{3} \eta_{\mu \nu} \eta_{\alpha \beta}, \quad (10)
$$

with

$$
\eta_{\mu \nu} = - g_{\mu \nu} + \frac{q_{\mu} q_{\nu}}{q^2}. \quad (11)
$$

The QCD expression for $T(Q^2)$ is known to be given by $^8$

$$
T(Q^2) = C_0 (Q^2)^2 \ln \frac{\nu^2}{Q^2} + \frac{C_8 \langle O_8 \rangle}{(Q^2)^2} + \frac{C_{10} \langle O_{10} \rangle}{(Q^2)^3} + \cdots, \quad (12)
$$

where $\nu$ is a subtraction scale, and

$$
C_0 = \frac{1}{20 \pi^2}, \quad (13)
$$
\[ C_8 \langle O_8 \rangle = \frac{5\pi}{3} \alpha_s \langle 2O_1 - O_2 \rangle, \quad (14) \]

with
\[ O_1 = \left( \sum_{abc} \left( \sum_{\mu\nu\alpha} G_{\mu\alpha}^a G_{\nu\alpha}^b G_{\alpha\beta}^c \right)^2 \right), \quad (15) \]
\[ O_2 = \left( \sum_{abc} \left( \sum_{\mu\nu} G_{\mu\nu}^a G_{\alpha\beta}^b G_{\alpha\beta}^c \right)^2 \right). \quad (16) \]

Concerning the dimension-10 contribution in Eq.(12), both the Wilson coefficient and the vacuum matrix elements are unknown. Nevertheless, we have included it in Eq.(12) for reasons that will become clear later. Notice that from Eq.(12) the asymptotic behaviour of the spectral function is

\[ \rho(Q^2) \equiv \frac{1}{\pi} \text{Im} \frac{T(Q^2)}{Q^2} \sim C_0(Q^2)^2, \quad (17) \]

As suggested in Ref.4, in order to obtain the Gauss-Weierstrass transform \( G(\hat{s},\tau) \) of the function \( T(Q^2) \), Eq.(12), it is convenient to work out first the Laplace transform

\[ L(\sigma) = \int_0^{\infty} ds \, e^{-s\sigma} \rho(s), \quad (18) \]

noticing that due to the asymptotic behaviour (17) three subtractions must be performed. Applying familiar Borelization techniques to the function

\[ T_R(Q^2) = T(Q^2) - T(0) - Q^2 \frac{T}{T(0)} - \frac{1}{2} (Q^2)^2 T(0), \quad (19) \]

one easily finds the following Laplace transform:

\[ L(\sigma) = \frac{2C_0}{\sigma^3} + C_8 \langle O_8 \rangle \sigma - \frac{1}{2} C_{40} \langle O_{40} \rangle \sigma^2. \quad (20) \]

The Gaussian transform \( G(\hat{s},\tau) \) follows then from

\[ G(-\hat{s},\tau) = \frac{1}{2\pi} \mathcal{L} \left[ \frac{1}{\sigma} L(\sigma) \exp(-\hat{s}\sigma) \right], \quad (21) \]

where \( \mathcal{L} \) is the familiar operator.
\[ \hat{\lambda} = \lim_{\sigma^2 \to \infty \atop N \to \infty} \left( \frac{\sigma^2}{N} \right) \left( \frac{(-1)^N}{(N-1)!} \right) \left( \frac{\sigma^2}{N} \right)^N \frac{d^N}{(d\sigma^2)^N}, \]  (22)

and an analytic continuation in \( \hat{s} \) is understood in Eq.(21). Using the formulae given in Ref.4 we obtain

\[ G(\hat{s}, \tau) = C \sqrt{\frac{4}{\pi}} \left\{ \frac{4}{\sqrt{\pi}} H_4(\hat{\chi}) e^{-\hat{\chi}^2} + \frac{1}{2} \text{Erfc}(\hat{\chi}) [4 H_0(\hat{\chi}) + \right. \]
\[ \left. + H_2(\hat{\chi}) \right] \right\} - \frac{C_8 \langle 0^8 \rangle}{4 \tau \sqrt{\pi}} H_4(\hat{\chi}) e^{-\hat{\chi}^2} + \]
\[ + \frac{C_{10} \langle 0_{10} \rangle}{16 \sqrt{\pi} \tau^{3/2}} H_2(\hat{\chi}) e^{-\hat{\chi}^2}, \]  (23)

where \( H_n(\hat{\chi}) \) are Hermite polynomials

\[ \hat{\chi} = \frac{\hat{s}}{2 \sqrt{\tau}}, \]  (24)

and

\[ \text{Erfc}(\hat{\chi}) = \frac{2}{\sqrt{\pi}} \int_{\hat{\chi}}^\infty e^{-y^2} dy = 1 - \text{erf}(\hat{\chi}). \]  (25)

In the derivation of the FESR one also needs the transforms

\[ U^{(+)}(\hat{s}, \tau) = G(\hat{s}, \tau) \pm G(-\hat{s}, \tau), \]  (26)

which correspond to solutions of Eq.(2) with the following alternative choices of boundary conditions:

\[ U(0, \tau) = 0, \text{ for } \tau > 0, \text{ U}^{(-)}\text{-solution} \]  (27)

\[ \frac{\partial U(0, \tau)}{\partial \hat{s}} = 0, \text{ for } \tau > 0, \text{ U}^{(+)}\text{-solution} \]  (28)

and the initial condition

\[ U(\hat{s}, 0) = \rho(\hat{s}). \]  (29)
After a straightforward computation we find

\[ U^{(+)}(\hat{s}, \tau) = C_0 \tau \left[ 4 H_0(\hat{x}) + H_2(\hat{x}) \right] + \]
\[ + \frac{C_{10} \langle O_{10} \rangle}{8 \sqrt{\pi} \tau^{3/2}} H_2(\hat{x}) e^{-\hat{x}^2}, \tag{30} \]

\[ U^{(-)}(\hat{s}, \tau) = C_0 \tau \left\{ \frac{2}{\sqrt{\pi}} H_1(\hat{x}) e^{-\hat{x}^2} + e_{f}^{(\hat{p})(\hat{x})} \right\} \left[ 4 H_0(\hat{x}) + \right. \]
\[ + \left. H_2(\hat{x}) \right] - \frac{C_8 \langle O_8 \rangle}{2 \tau \sqrt{\pi}} H_4(\hat{x}) e^{-\hat{x}^2}. \tag{31} \]

Notice that from Eqs. (30) and (31) one finds

\[ \lim_{\hat{x} \to \infty} U^{(\pm)}(\hat{s}, \tau) = C_0 \hat{S}_z^2, \tag{32} \]

in agreement with Eqs. (4) and (17). After taking suitable Hermite moments of \( U^{(\pm)}(\hat{s}, \tau) \) one easily obtains the FESR, as discussed in Ref. 4 (for alternative derivations see Refs. 6 and 9). Retaining non-perturbative contributions up to \( C_{10} \langle O_{10} \rangle \) we obtain the following three FESR:

\[ C_0 \frac{S_0^3}{3} = \int_0^{S_0} ds \rho(s), \tag{33} \]

\[ C_0 \frac{S_0^4}{4} - C_8 \langle O_8 \rangle = \int_0^{S_0} ds \ s \rho(s), \tag{34} \]

\[ C_0 \frac{S_0^5}{5} + C_{10} \langle O_{10} \rangle = \int_0^{S_0} ds \ s^2 \rho(s), \tag{35} \]

where \( s_0 \) is the threshold of asymptotic freedom, i.e. for \( s > s_0 \) the hadronic spectral function \( \rho(s) \) coincides with the QCD asymptotic expression (17). We wish to stress that the actual value of \( s_0 \) is determined (in this framework), together with the glueball mass and width, by solving the eigenvalue problem posed by Eqs. (33)-(35).

To complete this section we need an estimate of \( C_8 \langle O_8 \rangle \) given in Eqs. (14)-(16). Assuming dominance of the vacuum intermediate state (vacuum factorization) one obtains \(^8\)
\[ C_8 \langle 0_8 \rangle \bigg|_{\text{FACT.}} = -\frac{45}{16 \pi^2} \frac{1}{(\alpha_s / \pi)} \left( \frac{\pi}{3} \alpha_s G_{\mu \nu}^2 \right)^2, \]  

(36)

where \( \langle \alpha_s G_{\mu \nu}^2 \rangle \) is the familiar gluon condensate. The above factorization hypothesis has been tested in the charmonium system \(^{10}\) and is expected to provide a reasonable estimate of \( C_8 \langle 0_8 \rangle \) within a factor of two-three. Using the standard value of the gluon condensate \(^{11}\), \( \langle \frac{\pi}{3} \alpha_s G_{\mu \nu}^2 \rangle \approx 0.04 \text{ GeV}^4 \), in Eq.(36) gives

\[ C_8 \langle 0_8 \rangle \bigg|_{\text{FACT.}} \approx -0.0046 \text{ GeV}^8. \]

(37)

Since there are indications \(^{5,12}\) that the standard value of the gluon condensate is an underestimate by up to a factor of three, we shall also use in our analysis the alternative choice

\[ C_8 \langle 0_8 \rangle \bigg|_{\text{FACT.}} \approx -0.041 \text{ GeV}^8. \]

(38)

In Eqs.(37) and (38) we have used \( \alpha_s / \pi \approx 0.1 \), corresponding to a scale of 1 GeV\(^2\). At the level of accuracy we are working the effects of the running of \( \alpha_s \) are negligible and thus we shall keep \( \alpha_s \) constant. Finally, as mentioned before, \( C_{10} \langle 0_{10} \rangle \) is unknown and, therefore, we shall treat it as a free parameter.

III. HADRONIC SPECTRAL FUNCTION

The lowest lying intermediate state contributing to the hadronic spectral function is clearly the two-pion state. We need then to consider the matrix element

\[ \tau_{\mu \nu} = \langle \pi(p_1), \pi(p_2) | \theta_{\mu \nu} | 0 \rangle = A r_{\mu} r_{\nu} + \]

\[ + B q_{\mu} q_{\nu} + C g_{\mu \nu} + D(r_{\mu} q_{\nu} + r_{\nu} q_{\mu}), \]

(39)

where \( r = p_1 - p_2 \) and \( q = p_1 + p_2 \). The functions \( A, B, C \) and \( D \) can be related to each other by working in the chiral limit \( p_1^2 = p_2^2 = m_\pi^2 = 0 \) and imposing on Eq.(39) the following constraints:
a) transversality: \[ q^\nu \langle \pi \pi | \theta_{\mu \nu} | 0 \rangle = q^\nu \langle \pi \pi | \theta_{\mu \nu} | 0 \rangle = 0 \]

b) traceless-ness
\[ \langle \pi \pi | \theta_{\mu \mu} | 0 \rangle = 0, \quad \text{to order } O(\alpha_s) \]

c) crossed channel behaviour:
\[ \langle \pi (p_1) | \theta_{\mu \nu} | \pi (p_2) \rangle \rightarrow \frac{2 p_\mu p_\nu p_G}{p_1 = p_2 = p} \]

where \( p_G \) has been interpreted as the share of the pion momentum by the gluons (\( p_G \leq 1 \)). After using the above constraints we obtain
\[ T_{\mu \nu} = \frac{p_G}{640 \pi^2} \left( 3 r_{\mu \nu} - q_{\mu} q_{\nu} + q^2 q_{\mu \nu} \right), \tag{40} \]

and \( p_G \) will be considered as a free multiplicative parameter to be determined from the FESR, subject to the constraint \( p_G \leq 1 \). In addition, Eq.(40) should have a complex pole at \( \sqrt{q^2} = M + i \Gamma \), with \( M \) and \( \Gamma \) the mass and width of the tensor gluonium resonance, respectively. After working out the phase space and spin projections we obtain the following spectral function:
\[ \rho(s) = \frac{p_G^2}{640 \pi^2} s^2 \frac{M^2 (M^2 + \Gamma^2)}{(s-M^2)^2 + M^2 \Gamma^2} \Theta(s). \tag{41} \]

In order to be able to explore energy regions above 1 GeV we should also include in \( \rho(s) \) the \( K \bar{K} \) and \( \eta \eta \) intermediate states. From the flavour independence of the source \( \theta_{\mu \nu} \), the simplest generalization of Eq.(41) can be written as
\[ \rho(s) = \frac{p_G^2}{640 \pi^2} s^2 \frac{M^2 (M^2 + \Gamma^2)}{(s-M^2)^2 + M^2 \Gamma^2} \left[ \Theta(s) + \frac{5}{3} \left( 1 - \frac{4 \mu_\eta^2}{s} \right)^{5/2} \Theta(s-4 \mu_\eta^2) \right], \tag{42} \]

where we have taken into account kinematical corrections to the chiral \( SU(3) \times SU(3) \) limit, but kept \( \mu_\eta^2 = 0 \). For the \( K \bar{K} \) and \( \eta \eta \) threshold we shall use a common average value, viz.
\[ 4 \mu_\eta^2 = 1.09 \text{ GeV}^2 \tag{43} \]
IV. SOLUTIONS TO THE FESR

To begin this section we wish to look for the implications of a smooth, non-resonant spectral function saturation of the FESR (33)-(35). A simple, albeit not unique, model for such a $\rho(s)$ which satisfies all the low-energy constraints may be derived from Eq.(41) by formally taking the limit $M \to \infty$. This amounts to an extrapolation up to $s = s_0$ of the low energy, incoherent, two-pion intermediate state contribution to the spectral function. With such a choice of $\rho(s)$ the FESR (33)-(35) reduce to

\[
C_0 = \frac{\rho_G^2}{640 \pi^2},
\]

(44)

\[
C_0 \frac{S_0^4}{4} - C_8 \langle O_8 \rangle = \frac{\rho_G^2}{640 \pi^2} \frac{S_0^4}{4},
\]

(45)

\[
C_0 \frac{S_0^5}{5} + C_{10} \langle O_{10} \rangle = \frac{\rho_G^2}{640 \pi^2} \frac{S_0^5}{5},
\]

(46)

which implies $\rho_G^2 = 32$ and the vanishing of all the non-perturbative contributions; $C_8 \langle O_8 \rangle = C_{10} \langle O_{10} \rangle = \ldots = 0$. Since $\rho_G \leq 1$, and the vacuum condensates are in principle non-zero, such a solution looks quite unreasonable. With due reservations, on account of the model choice of $\rho(s)$, this may be taken as an indication of the existence of a gluonium bound state.

In order to predict the mass and width of tensor gluonium we have at our disposal the three FESR (33)-(35). However, $s_0$, $\rho_G$ and $C_{10} \langle O_{10} \rangle$ are also part of the unknowns. Therefore, we choose the following strategy: we tentatively set $C_{10} \langle O_{10} \rangle = 0$, input a particular value for the width $\Gamma$, and solve the eigenvalue problem to find $M$, $s_0$ and $\rho_G$. Since $\rho_G$ increases with increasing $\Gamma$, and $\rho_G \leq 1$, we stop the procedure whenever the width makes $\rho_G$ exceed its bound. The eigenvalues so obtained are presented in Tables I and II corresponding, respectively, to the two alternative choices of $C_8 \langle O_8 \rangle$ in Eqs.(37) and (38), and using just the $\pi\pi$ contribution to the spectral function, Eq.(41). Use of the complete spectral function (42) leads to essentially the same solutions for the mass, and slightly smaller values of $\rho_G$. However, in this case the approximation $C_{10} \langle O_{10} \rangle = 0$ is not always consistent, i.e. for some values of the width there is no eigenvalue solution unless $C_{10} \langle O_{10} \rangle \neq 0$. A more refined analysis to be performed in the next section will clarify this issue.
An inspection of the eigenvalue solutions in Tables I and II shows that, depending on the choice of $C_8 \langle 0_8 \rangle$, the present method can easily accommodate either the $\Theta(1710)$ or the $g_T(2050)$, two of the candidates in this channel. In particular, if we use as an input to the spectral function the measured parameters of the $\Theta(1710)$, i.e. $M = 1.71$ GeV, $\Gamma \approx 150$ MeV, and solve the FESR for $s_0$, $\rho_G$ and $C_8 \langle 0_8 \rangle$ (with $C_{10} \langle 0_{10} \rangle = 0$) we find

$$S_0 = \begin{cases} 3.89 \text{ GeV}^2 & (a) \\ 3.93 \text{ GeV}^2 & (b) \end{cases}$$

$$\rho_G = \begin{cases} 0.90 & (a) \\ 0.74 & (b) \end{cases}$$

$$C_8 \langle 0_8 \rangle = \begin{cases} -0.0071 \text{ GeV}^8 & (a) \\ -0.0075 \text{ GeV}^8 & (b) \end{cases}$$

where the solutions labelled (a) and (b) correspond, respectively, to a $\pi\pi$ and a $\pi\pi + K\bar{K} + \eta\eta$ saturation of the hadronic spectral function. The above eigenvalue solutions are indeed very reasonable. In particular, the values of $C_8 \langle 0_8 \rangle$ deviate from the standard determination (37) by less than a factor of two. This is quite reassuring, given the approximate nature of the vacuum saturation hypothesis. However, in spite of all this a single resonance saturation of $\rho(s)$ leads to eigenvalue solutions which are not very stable against changes in $s_0$, as will be discussed in the next section. From being a handicap, our next analysis of the stability problem will show that when both the $\Theta(1710)$ and the $g_T(2050)$ are incorporated into the spectral function the eigenvalue solutions become remarkably stable.

V. STABILITY TESTS AND DUALITY WINDOWS

The solution to the FESR eigenvalue problem carried out in Sec.IV should be regarded as only the first step in a more thorough analysis. In fact, in order for the eigenvalue solutions to be meaningful they should exhibit some stability against changes in the asymptotic freedom threshold $s_0$. Parameters such as the resonance mass and width should remain reasonably constant as $s_0$ is varied within a (hopefully) wide interval, the so-called duality window. In other words, the hadronic and the QCD sides of the
FESR should match inside the duality window, the wider this window the more accurate the eigenvalue predictions.

Following the criterion adopted in previous applications\(^5\)-\(^7\) we study the stability of the eigenvalue solutions by considering the ratio of the first two FESR, Eqs. (33) and (34), i.e.

\[
1 - \frac{4 C_8 \langle O_8 \rangle}{C_0 S_0^4} = \left( \frac{4}{3 S_0} \right) \frac{\mathcal{I}_3 (S_0, M, \Gamma)}{\mathcal{I}_2 (S_0, M, \Gamma)} \equiv \left( \frac{4}{3 S_0} \right) \frac{\int_0^{S_0} ds \ s \ \rho(s)}{\int_0^{S_0} ds \ \rho(s)} .
\] (50)

In Fig. 2 we compare the QCD-left-hand side of Eq. (50) (solid line) with the hadronic-right-hand side (dashed line), as a function of \(S_0\), for a mass and width corresponding to the \(\Theta(1710)\) and using the value of \(C_8 \langle O_8 \rangle\) obtained in Eq. (49). Notice that the normalization of the spectral function, \(\rho_G\), drops out in the ratio (50). As seen from Fig. 2 there is essentially no duality window, as it reduces to just a crossover point corresponding to the eigenvalue solution for \(S_0\), Eq. (47). A similar situation of instability holds for the other eigenvalue solutions listed in Tables I and II. As a result of this we conclude that a single resonance parametrization of the hadronic spectral function does not saturate the FESR. Additional hadronic information, e.g. another resonance, is clearly needed. The occurrence of such a situation somehow depends upon the channel under consideration. For instance, in a previous application to the \(0^{++}\) gluonium\(^7\), a single resonance saturation of the spectral function was enough to obtain reasonably wide duality windows. However, in the case of the \(\rho\)-meson the situation resembles the present one, i.e. a single resonance is not enough to saturate the spectral function in the \(1^{-+}\) channel as it leads to unstable eigenvalue solutions to the FESR\(^5\).

An obvious solution to the above instability problem may be to add a second resonance to \(\rho(s)\), Eq. (42). A simple, albeit not unique, generalization of Eq. (42) is

\[
\rho(s) = \frac{\rho_G}{640 \pi^2} s^2 \left[ \Theta(s) + \frac{5}{3} \left( 1 - \frac{4 \mu^2}{5} \right)^{5/2} \Theta(s - 4 \mu^2) \right] \times
\times \left[ \frac{M_4^2 (M_4^2 + T_4^2)}{(s - M_4^2)^2 + M_4^2 T_4^2} + \lambda \frac{M_2^2 (M_2^2 + T_2^2)}{(s - M_2^2)^2 + M_2^2 T_2^2} \right] ,
\] (51)
where $\lambda$ is a free parameter. In spite of Eq.(51) being the most economical generalization of Eq.(42), the number of unknown hadronic parameters has become too large. With only three FESR at our disposal it is clear that the analysis is bound to be only semiquantitative. Nevertheless, we can study the consistency between a given set of input values for the mass and width of the two resonances and the resulting values of $\rho_G$, $C_8\langle 0_8 \rangle$, and $C_{10}\langle 0_{10} \rangle$ obtained by solving the FESR. By adjusting the parameter $\lambda$ we can see whether a reasonably wide duality window now exists for the ratio (50) as well as for $\rho_G$, $C_8\langle 0_8 \rangle$ and $C_{10}\langle 0_{10} \rangle$. These parameters should not only be stable against changes in $s_0$ but also their values should not differ much from those of the single resonance analysis. Following this strategy we have found that using the $\Theta(1710)$ and $g_T(2050)$ parameters it is indeed possible to achieve a remarkable overall stability, to wit. In Fig.3 we show the ratio (50) as a function of $s_0$ for $\lambda = 0.5$. The triangles correspond to the hadronic right-hand side of Eq.(50) with $M_1$ = 1.71 GeV, $\Gamma_1 = 150$ MeV, $M_2 = 2.15$ GeV and $\Gamma_2 = 200$ MeV, while the closed circles correspond to the same $M_1$, $\Gamma_1$ but $M_2 = 2$ GeV, and $\Gamma_2 = 200$ MeV. The solid curve corresponds to the QCD-left-hand side of Eq.(50) with $C_8\langle 0_8 \rangle = -0.0115$ GeV$^8$ and the broken curves (a) and (b) are calculated with the two extreme choices $C_8\langle 0_8 \rangle = -0.041$ GeV$^8$ and $C_8\langle 0_8 \rangle = -0.0046$ GeV$^8$, respectively. Figs.4, 5 and 6 show the predictions for $\rho_G$, $C_8\langle 0_8 \rangle$ and $C_{10}\langle 0_{10} \rangle$, respectively, obtained by solving the FESR (33)-(35) with $\rho(s)$ given by Eq.(51) with $\lambda = 0.5$ and the $\Theta(1710)$ and $g_T(2050)$ parameters. For comparison we also show in Figs.4-6 the corresponding predictions using the single resonance spectral function (42) with the $\Theta(1710)$ mass and width.

A comparison of Fig.3 with Fig.2 shows quite clearly how the addition of a second resonance to $\rho(s)$ solves the stability problem. In fact, the two-resonance saturation of the spectral function leads to a wide duality window extending from $s_0 \simeq 3$ GeV$^2$ to $s_0 \simeq 5.5$ GeV$^2$. Furthermore, as seen from Figs.4-6 the parameters $\rho_G$, $C_8\langle 0_8 \rangle$ and $C_{10}\langle 0_{10} \rangle$ become remarkably more stable after a second resonance is added to $\rho(s)$. We wish to reiterate, however, that given the approximations contained in Eq.(51) and given the number of unknown parameters, we cannot claim to have predicted the mass and width of the two glueballs. Our analysis only shows that if the $\Theta(1710)$ and the $g_T(2050)$ were to be established experimentally as boudine tensor glueballs, then their masses and widths would be compatible with QCD in a stable sense. However, our analysis of the stability of the eigenvalue solutions to the FESR has provided a strong hint that in addition to a ground state glueball with $M \simeq 1.6-1.7$ GeV, $\Gamma \simeq 100-200$ MeV, there should exist at least another resonance with $M \gtrsim 2$ GeV, $\Gamma \simeq 200$ MeV. In spite of being qualitative, we find this conclusion quite interesting by itself.
VI. HEAT EVOLUTION TESTS

As anticipated in Sec.I the heat evolution test following from Eq.(2) provides an adequate quantitative check of the consistency between a given ansatz for the hadronic spectral function and the corresponding QCD expression. In the present instance the phenomenological expressions of the Gauss-Weierstrass transforms \( \mathcal{U}^{(\pm)}(\hat{s}, \tau) \) defined in Eq.(26) are

\[
\mathcal{U}^{(\pm)}(\hat{s}, \tau) \bigg|_{\text{PHEN.}} = \frac{1}{\sqrt{4\pi \tau}} \int_{0}^{\infty} ds \rho(s) \left[ e^{-\left(s-\hat{s}\right)^{2}/4\tau} \mp e^{-\left(s+\hat{s}\right)^{2}/4\tau} \right] + \\
+ C_{0} \left\{ \sqrt{\frac{\tau}{\pi}} \left[ (s_{\pm} + \hat{s}) e^{-\left(s_{\pm} + \hat{s}\right)^{2}/4\tau} \mp (s_{\pm} - \hat{s}) e^{-\left(s_{\pm} - \hat{s}\right)^{2}/4\tau} \right] \\
+ \tau \left[ \mathcal{E}_{\chi}^{\pm} \left( \frac{s_{\pm} - \hat{s}}{2\sqrt{\tau}} \right) \mp \mathcal{E}_{\chi}^{\pm} \left( \frac{s_{\pm} + \hat{s}}{2\sqrt{\tau}} \right) \right] \left( 1 + \frac{\hat{s}^{2}}{2\tau} \right) \right\}, \tag{52}
\]

where the first term corresponds to the resonance contribution and the second one is the result of an analytical integration of the QCD continuum contribution. Notice that Eq.(52) has the correct asymptotic behaviour

\[
\mathcal{U}^{(\pm)}(\hat{s}, \tau) \bigg|_{\text{PHEN.}} \sim C_{0} \hat{s}^{2}, \tag{53}
\]

in agreement with Eqs.(4) and (32).

According to the heat evolution tests the Gaussian transforms \( \mathcal{U}^{(\pm)}(s, \tau) \), Eq.(52), should match their QCD counterparts, Eqs.(30) and (31), after a "time" \( \tau \lesssim 1 \text{ GeV}^{4} \), if there is duality between the set of hadronic parameters contained in \( \rho(s) \) and the set of QCD parameters. In Figs.7 and 8 we show the results of such a comparison for \( \mathcal{U}^{(+)}(s, \tau) \) at \( \tau = 0.5 \text{ GeV}^{4} \) and \( \tau = 1.5 \text{ GeV}^{4} \), respectively, using the two-resonance spectral function (51) with the \( \Theta(1710) \) and \( g_{T}(2050) \) masses and widths. The rest of the parameters have been fixed at the values corresponding to the duality window of stability found in Sec.V. It should be clear from these figures that the agreement between phenomenology and QCD, already satisfactory at \( \tau = 0.5 \text{ GeV}^{4} \), becomes almost exact at \( \tau = 1.5 \text{ GeV}^{4} \) and beyond. A similar behaviour is exhibited by \( \mathcal{U}^{(-)}(s, \tau) \) and \( \mathcal{G}(s, \tau) \). We wish to stress in closing the non-trivial nature of this test. In fact, the quantitative
definition of duality through the heat equation analogy provides a rather stringent constraint on the allowed values of hadronic parameters consistent with a given QCD information.

VII. SUMMARY

In this paper we have analyzed the $2^{++}$ gluonium spectrum using the Gauss-Weierstrass QCD sum rules and the FESR which follow from them. The fact that the gaussian transforms obey the heat equation leads to a quantitative formulation of the notion of duality which in turn provides a rather stringent consistency check between a given set of QCD parameters and the hadronic information contained in the spectral function. In addition to solving the FESR in order to predict the mass and width of the tensor gluonium, we have studied the stability of these eigenvalue solutions against changes in the location of the asymptotic freedom threshold. This analysis may provide indications as to whether a single resonance parametrization of the spectral function leads to saturation or some additional structure, e.g. a second resonance, is needed in order to achieve stability. The results of our analysis may be summarized as follows.

First of all we found that a flat, non-resonant parametrization of the spectral function leads to inconsistencies as it implies the vanishing of the non-perturbative contributions to all dimensions, and a normalization of $\rho(s)$ exceeding its bound by roughly an order of magnitude. With due reservations this result may be taken as evidence for confinement in this channel. Next, starting from a general finite-width single resonance parametrization of $\rho(s)$, plus the standard asymptotic freedom QCD continuum, we found mass eigenvalues close to the $\Theta(1710)$ or to the $a_T(2050)$, depending on the choice of $C_8 \left< 0_8 \right>$. In both cases the width was found in the range $\Gamma \approx 100-250$ MeV. However, our subsequent analysis of the stability of these eigenvalues showed quite clearly that additional hadronic information was needed in the spectral function in order to obtain reasonably wide duality windows. Adding a second resonance to $\rho(s)$, even if done in the most economical fashion, reduces the quantitative predictive power of the method. Nevertheless, we have found that by assuming the lowest tensor glueball to have a mass close to the $\Theta(1710)$, then the second resonance mass turned out to be close to that of $a_T(2050)$.

Our prediction of the tensor glueball mass, in the single resonance saturation of the finite-width spectral function, is in reasonable agreement with earlier estimates $8,14$) based on Laplace transform QCD sum rules using a zero-width approximation. However, the Laplace sum rules being less
sensitive to $s_0$, and with their natural exponential suppression, are not capable of uncovering the additional structure at $M \gtrsim 2$ GeV revealed by the stability analysis of the PESR.

ACKNOWLEDGMENTS

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REFERENCES


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5) The original paper, Ref.4, contains a detailed description of the method. For further details see e.g. R.A. Bertlmann, C.A. Dominguez, M. Loewe, M. Petrottet and E. de Rafael, Univ. Wien Report (to appear).


### Table I

Eigenvalue solutions to the FESR corresponding to the choice of $\langle 0_8 \rangle$ in Eq. (37) by using the spectral function in Eq. (41).

<table>
<thead>
<tr>
<th>$\Gamma$(MeV)</th>
<th>M(MeV)</th>
<th>$s_0$(GeV$^2$)</th>
<th>$\rho_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1610</td>
<td>3.422</td>
<td>0.733</td>
</tr>
<tr>
<td>150</td>
<td>1624</td>
<td>3.510</td>
<td>0.927</td>
</tr>
<tr>
<td>200</td>
<td>1636</td>
<td>3.592</td>
<td>1.10</td>
</tr>
</tbody>
</table>

### Table II

Eigenvalue solutions to the FESR corresponding to the choice of $\langle 0_8 \rangle$ in Eq. (38) and using the spectral function in Eq. (41).

<table>
<thead>
<tr>
<th>$\Gamma$(MeV)</th>
<th>M(MeV)</th>
<th>$s_0$(GeV$^2$)</th>
<th>$\rho_G$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>2109</td>
<td>5.842</td>
<td>0.629</td>
</tr>
<tr>
<td>150</td>
<td>2122</td>
<td>5.956</td>
<td>0.791</td>
</tr>
<tr>
<td>200</td>
<td>2135</td>
<td>6.070</td>
<td>0.935</td>
</tr>
<tr>
<td>250</td>
<td>2148</td>
<td>6.179</td>
<td>1.070</td>
</tr>
</tbody>
</table>
FIGURE CAPTIONS

Fig.1a
Lowest order perturbative contribution to $\Pi_{\mu\nu;\alpha\beta}$ in Eq.(5).

Fig.1b
First non-zero non-perturbative contribution to $\Pi_{\mu\nu;\alpha\beta}$
in Eq.(5).

Fig.2
Ratio of the first two FESR as defined in Eq.(50).
Dashed line: hadronic right-hand side of Eq.(50)
corresponding to the $\Theta(1710)$ parameters and using
$\rho(s)$ of Eq.(41). Solid line: QCD left-hand side of
Eq.(50) for $C_{8}\langle O_{8} \rangle$ in Eq.(49).

Fig.3
The ratio Eq.(50) using $\rho(s)$ in Eq.(51) with $\lambda = 0.5$.
Triangles: hadronic right-hand side with $M_1 = 1.71 \text{ GeV}$,
$\Gamma_1 = 150 \text{ MeV}$, $M_2 = 2.15 \text{ GeV}$, $\Gamma_2 = 200 \text{ MeV}$. Closed circles:
same $M_1$, $\Gamma_1$ but $M_2 = 2 \text{ GeV}$, $\Gamma_2 = 200 \text{ MeV}$. Solid curve:
QCD left-hand side with $C_{8}\langle O_{8} \rangle = -0.0115 \text{ GeV}^8$. Broken
curves (a) and (b): same with $C_{8}\langle O_{8} \rangle = -0.041 \text{ GeV}^8$
and $C_{8}\langle O_{8} \rangle = -0.0046 \text{ GeV}^8$, respectively.

Fig.4
Curve (a): The normalization $\rho_G$ obtained from the
FESR using $\rho(s)$ of Eq.(51) with $\lambda = 0.5$ and the $\Theta(1710)$
and $g_{\tau}(2050)$ parameters. Curve (b): corresponding
predictions using $\rho(s)$ of Eq.(42) with the $\Theta(1710)$
parameters.

Fig.5
Values of $C_{8}\langle O_{8} \rangle$ obtained from the FESR with the same
input as in curves (a) and (b) of Fig.4.

Fig.6
Values of $C_{10}\langle O_{10} \rangle$ obtained from the FESR with the same
input as in curves (a) and (b) of Fig.4.

Fig.7
The Gauss-Weierstrass transform $U^+(s,\tau)$ at $\tau = 0.5 \text{ GeV}^4$.
Dashed line: right-hand side of Eq. (52) using $\rho(s)$ of
Eq.(51) as in Fig.4(a). Solid line: QCD prediction
using the parameters found in the duality window.

Fig.8
Same as Fig.7 except that now $\tau = 1.5 \text{ GeV}^4$. 

-19-
(a)

(b)

\[ C_8 < O_8 > \]

Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 6
Fig. 8