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CONFORMAL MAPPINGS AND THE THREE STRING BOSONIC VERTEX

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ABSTRACT

A new approach to the three string bosonic vertex is proposed and a more powerful method of derivation of its gauge properties is given.

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Recently there has been considerable progress in the development of a gauge covariant theory of strings. In our opinion, the best (up to now) formulation of the free open bosonic string can be found in references [1] [2] and [3]. It has a particularly simple form, namely XQX . The interacting open bosonic string was found in two different formulations, one by the authors of this paper in references [4] and [5] hereafter referred to as I and II and by the Kyoto group in reference [6], and the other by E. Witten in reference [2]. The relation between these formulations is unknown although they both recover many of the known results of the dual model. The closed bosonic string was found in II and ref [6].

This paper is one of the series of papers which will find the gauge covariant interactions of superstring theories. The scattering of three Neveu-Schwarz superstrings will be given in the next paper [7]. In this paper we develop some new techniques for deriving and finding the properties of the well known 3 string vertex for open bosonic strings.

In the past literature this vertex was found [8] [9] within the context of the dual model by factorizing the n point scattering amplitude for tachyons. The relation of this vertex to the 3 string vertex which occurs in the gauge covariant action and transformation rules was discussed in II. It was found that they were related by the conformal mapping which takes the upper half plane to the Mandelstam strip. In this paper we work within the context of the dual model, that is the upper half plane, and show how to derive the 3 string vertex. We also establish the gauge identities for this vertex, that is how objects such as L_n and α_n^μ are transmitted and reflected on the three legs.

One advantage of a further understanding of the properties of vertices is that it will help in putting a name to the local gauge group, given in II, which underlies string theories. It would seem likely that for the closed bosonic string this is associated in some way with the unique Lorentzian self dual lattice in 26 dimensions. Clarification of this point will also help understand the remarkable results which relate vertex operators to group generators [10] [11]. In fact, the group generating vertex operators of [11] and [12] can be recovered as special cases of the space-time dependent closed string theory gauge transformation laws given in II.

It will be useful to first summarize some of the literature on the 3 string scattering vertex which was found by factorizing the npt scattering amplitude for tachyon scattering to be.

$$V^{CSV} = \langle o_{1,2,3} | \exp - \sum_{r=1}^3 \sum_{m=0}^{\infty} \sum_{n=1}^m \alpha_m^{\mu r} \alpha_n^{\mu r+1} \frac{(-1)^m \Gamma(n)}{\Gamma(m+1)\Gamma(1+n-m)} \quad (1)$$

We can rewrite this vertex in the form

$$V^{CSV} = \langle o_{1,2,3} | \exp - \frac{1}{2\pi i} \sum_{r=1}^3 \oint_{z_r=0} \frac{dz_r}{z_r} p^r(z_r) Q_+^{r+1}(z_{r+1})$$

where

$$Q_+^{\mu}(z) \equiv \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n^{\mu} z^{-n}, \quad p^{\mu}(z) \equiv \sum_{n=-\infty}^{\infty} \alpha_n^{\mu} z^{-n}, \quad z_{r+1} \equiv \frac{1}{1-z_r}$$

$$\langle o_{1,2,3} | = \langle \downarrow o | \langle \downarrow o | \langle \downarrow o | \quad (2)$$

if An operator $R(z)$ is said to have conformal dimension

$$g R(z) g^{-1} = R(z^1) \left[\frac{z}{z^1} \frac{dz^1}{dz} \right]^d \quad (3)$$

where g is a conformal transformation and so is of the form

$g = \exp \sum_{n=-\infty}^{\infty} a_n L_n$. In fact the transformation $z \rightarrow z^1 = z^1(z)$ is the same as the one found by naively applying g when L_n is represented by

$$L_n = z^{n+1} \frac{d}{dz} \quad (4)$$

For an infinitesimal transformation this reduces to the usual definition [12] [13]

$$[L_n, R(z)] = z^n \left(z \frac{d}{dz} + nd \right) R(z) \quad (5)$$

we recall that $p^{\mu}(z), L(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n}$, and

$$Q^{\mu} = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} \alpha_n^{\mu} - p^{\mu} \ln z + q^{\mu} \text{ where } [q^{\mu}, \alpha_0^{\nu}] = \eta^{\mu\nu}$$

have conformal dimensions +1, +2 and 0 respectively.

It will be useful to recover the usual result for the scattering of 2 strings and the tachyonic state. This involves multiplying one leg by the "twist" factor

$$\Omega = e^{-L_1(-1)R} \quad (6)$$

where $R = L_0 - \frac{1}{2}(\alpha_0^\mu)^2$. One may easily verify that $\Omega^2 = 1$ and it clearly leaves on-shell states invariant up to a phase factor. Consequently, we define

$$V(1,2) = V^{CSV} |0\rangle_3 e^{-L_1^{(2)}(-1)R^{(2)}} \quad (7)$$

Under a conformal transformation one may verify that

$$e^{L_1} Q_+(z) e^{-L_1} = (Q_+(\frac{z}{1-z}) - Q_+(-1)). \quad (8)$$

and [12] [13]

$$\langle 0| e^{+zL_1} = \langle 0| \exp + \{ \alpha_0^\mu Q_+(\frac{1}{z}) \}. \quad (9)$$

Hence we find that

$$\begin{aligned} V(1,2) &= \langle 0| \langle 0| e^{-L_1^{(2)}(-1)R^{(2)}} \exp - \frac{1}{2\pi i} \left\{ \oint_{z=0} \frac{dz}{z} P'(z) (Q_+^2(\frac{1}{z}) \right. \\ &\quad \left. - Q_+^2(1)) + \oint_{z=0} \frac{dz}{z} \alpha_0^\mu Q_+(\frac{1}{1-z}) \right\} \\ &= \langle 0| \langle 0| \exp \left\{ - \frac{1}{2\pi i} \oint \frac{dz}{z} P'(z) Q_+^2(\frac{1}{z}) - \alpha_0^\mu Q_+^1(1) \right. \\ &\quad \left. + (\alpha_0^{\mu 1} + \alpha_0^{\mu 2}) Q_+^2(1) \right\} \\ &= \langle 0| \langle 0| \exp \left\{ - \sum_{n=1}^{\infty} \frac{1}{n} \alpha_n^{\mu 1} \alpha_n^{\mu 2} - \alpha_0^\mu (Q_+^1(1) + Q_+^2(1)) \right\} \end{aligned} \quad (10)$$

After turning 2 around and identifying the 1 and 2 oscillators we recover the well known result. The vertex for an excited level on line 3 can be evaluated in a similar way.

Although the factorization of the general n point vertices was discussed in the literature [16], there is little discussion of the gauge identities they satisfy. In a recent paper, II it was shown by explicit computation that V^{CSV} obeyed the identity

$$V^{CSV} \{ A(z) p^{1\mu}(\frac{1}{1-z}) + B(z) p^{2\mu}(\frac{z-1}{z}) + C(z) p^{3\mu}(z) \} = 0 \quad (11)$$

where A,B and C satisfy the relation

$$\frac{A}{[-z]} + B + \frac{C}{[z-1]} = 0 \quad (12)$$

In order that the series involved in checking equation (11) converge, one of the coefficients A,B,C must vanish. A more detailed discussion of this point is given below. We now wish to show that an operator $R(z)$ of conformal dimension d satisfies the following identity.

$$V^{CSV} \{A(z) R^1\left(\frac{1}{1-z}\right) + B(z) R^2\left(\frac{z-1}{z}\right) + C(z) R^3(z)\} = 0 \quad (13)$$

provided

$$\frac{A(z)}{[-z]^d} + B(z) + \frac{C(z)}{[z-1]^d} = 0 \quad (14)$$

Let us first consider the transformation $z \rightarrow z^1 = \frac{1}{1-z}$

which is generated by $g_r = e^{L_1^r} e^{-L_{-1}^r}$. Essentially this transformation maps string 3 into string 2, string 2 into string 1 and string 1 into string 3 i.e. it cycles the amplitude. We notice that

$$\begin{aligned} & g \{A(z) R^1\left(\frac{1}{1-z}\right) + B(z) R^2\left(\frac{z-1}{z}\right) + C(z) R^3(z)\} g^{-1} \\ &= \{A(z) \left(\frac{1}{-z}\right)^d R^1\left(\frac{z-1}{z}\right) + B(z) (z-1)^d R^2(z) \\ &+ C(z) \left(\frac{-z}{z-1}\right)^d R^3\left(\frac{1}{1-z}\right)\} \end{aligned} \quad (15)$$

where $g = \prod_{r=1}^3 g_r$

If we take $R = P^\mu$ use equations (11) and (12) and the fact that V^{CSV} is cyclic then

$$V^{CSV} g^{-1} \{\bar{A}(z) R^1\left(\frac{1}{1-z}\right) + \bar{B}(z) R^2\left(\frac{z-1}{z}\right) + \bar{C}(z) R^3(z)\} = 0 \quad (16)$$

where $\bar{A}$, $\bar{B}$ and $\bar{C}$ can read off from equation (15) and still satisfy equation (12). Consequently V^{CSV} and $V^{CSV} g^{-1}$ satisfy the same P^μ identity. However, as we will see later this identity straight-forwardly determines the vertex uniquely and so

$$V^{CSV} g^{-1} = V^{CSV} \quad (17)$$

This equation may also be verified using identities similar to equations (8) and (9) and the discussion that follows it. We now sketch, using the machinery set up in reference [12], an alternative proof of equation (17). We may write V^{CSV} in the generic form.

$$V^{CSV} = \langle 0_{1,2,3} | \exp -(\alpha^1 \Omega \alpha^2 + \alpha^2 \Omega \alpha^3 + \alpha^3 \Omega \alpha^1) \quad (18)$$

where Ω is an infinite dimensional matrix which corresponds to the transformation $z \rightarrow 1-z$ and hence to the matrix

$$\Omega \rightarrow \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \quad (19)$$

We now apply g^{-1} which corresponds to $z \rightarrow 1 - \frac{1}{z}$ and so to the matrix

$$g^{-1} \rightarrow \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} = S \quad (20)$$

Applying the rules of reference [12] we find that

$$V^{CSV} g^{-1} \langle 0_{1,2,3} | \exp -(\alpha^1 \tilde{S} \Omega S \alpha^2 + \alpha^2 \tilde{S} \Omega S \alpha^3 + \alpha^3 \tilde{S} \Omega S \alpha^1) \quad (21)$$

Now one may verify that the matrix corresponding to $\tilde{S}$ is given by

$$\tilde{S} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} S^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = S \quad (22)$$

and hence we find

$$\tilde{S} \Omega S = \Omega \quad (23)$$

which establishes the result. In fact, one may show that

$$S = \tilde{\Omega} \Omega.$$

Since R is constructed from α oscillators a little thought shows that it will satisfy an identity such as equation (13) which explains how R 's are transmitted and reflected on each leg provided a certain identity holds. This identity can be written in the form

$$\frac{A}{f(z)} + B + \frac{C}{g(z)} = 0 \quad (24)$$

as we may scale it by an arbitrary function of z .

However from equation (15)

$$\bar{A} = C(z) \left[\frac{-z}{z-1} \right]^d, B(z-1)^d = \bar{C}, \frac{A(z)}{(-z)^d} = \bar{B} \quad (25)$$

must satisfy the same identity as A, B and C since v^{CSV} cannot satisfy two identities of this type. This determines $f(z) = (-z)^d$ and $g(z) = (z-1)^d$ establishing the desired result.

One could use the above chain of arguments to derive the vertex v^{CSV} . Any vertex is bound to have a rule of the form of equation (18) for the transmission and reflection of α_n 's on the three legs. If we then assume it satisfies equation (17): we may then derive equations (11) and (12) which uniquely determines the form of v^{CSV} .

It is often more useful to write identities of the type discussed above in an integral format. In carrying out the following steps it is important to take careful account of the conditions for convergence. What must be avoided is an infinite sum of creation operators with coefficients which belong to a divergent series. Only operators well defined when acting on any state consisting of a finite number of creation oscillators on the vacuum should be considered. As a result equations (13) and (14) are only valid for

$|z| > 1$ if $C = 0$, $|z-1| < |z|$ if $B = 0$ and $|1-z| > |z|$ if $A = 0$.
Let us consider then

$$\oint_{z=0} \frac{dz}{z} C(z) R^3(z).$$

where $C(z)$ is arbitrary. We can deform this contour which is understood to be in the region $|z| \ll 1$ to be around $z=1$ and $z=\infty$ provided we swoop C for B and A around 1 and ∞ respectively and use equation (14). Carrying out this manoeuvre we find the result

$$\begin{aligned} v^{CSV} \left\{ \oint_{z=0} \frac{dz}{z} \phi(z) \frac{(z-1)^{d-1}}{(-z)^{d-1}} R^3(z) + \oint_{z=1} \frac{dz}{z(z-1)} \frac{\phi(z)}{(-z)^{d-1}} R^2\left(\frac{z-1}{z}\right) \right. \\ \left. - \oint_{z=\infty} \frac{dz}{(z-1)} \phi(z) R^1\left(\frac{1}{1-z}\right) \right\} = 0 \end{aligned} \quad (26)$$

Introducing $z_1 = \frac{1}{1-z}$, $z_3 = \frac{1}{1-z_2}$, $z_1 = z$, we may write the above equation as

$$v^{CSV} \left\{ \oint_{z=0} \frac{dz}{z} R^3(z) \phi(z) \frac{(z-1)^{d-1}}{(-z)^{d-1}} \right.$$

$$+ \oint_{\mathfrak{Z}_2=0} \frac{d\mathfrak{Z}_2}{\mathfrak{Z}_2} \frac{\phi(z)}{[-z]^{d-1}} R^z(\mathfrak{Z}_2) + \oint_{\mathfrak{Z}_1=0} \frac{d\mathfrak{Z}_1}{\mathfrak{Z}_1} \phi(z) R^1(\mathfrak{Z}_1) \} = 0 \quad (27)$$

where

$$\mathfrak{Z}_1 = \frac{1}{1-z}, \mathfrak{Z}_2 = 1 - \frac{1}{z}. \quad (28)$$

Two examples of the above identities are

$$V^{CSV} \{ \alpha_{-n}^{\mu} + \sum_{p=0}^n \binom{n}{p} (-1)^p \alpha_p^{\mu} - \sum_{p=0}^{\infty} (-1)^p \binom{-n}{p} \alpha_{p+n}^{\mu} \} = 0 \quad (29)$$

and

$$V^{CSV} \{ L_{-n}^1 - L_0^1 + \sum_{p=1}^{n+1} \frac{(n+1)!}{(n+1-p)!p!} (-1)^p L_{p-1}^3 + L_0^3 - (-1)^n \sum_{p=0}^{\infty} \binom{-n+1}{p} L_{n+p}^2 + L_0^2 - L_1^2 \} = 0 \quad n \geq 1 \quad (30)$$

The first of these equations easily determines V^{CSV} .

It is straightforward to transform equation (2.27) from upper half plane to any other region of the complex plane by a conformal transformation. Of particular interest is the strip using the conformal transformation of reference [14] take $z_1 = z = 3$, then one finds that $z_2 = 3_2$ and $z_3 = 3_3$, in accord with the fact that $z \rightarrow \frac{1}{1-z}$ takes the three strings into each other. On the strip equation (27) becomes

$$V \sum_{r=1}^3 \oint_{\mathfrak{Z}_r=0} \frac{d\mathfrak{Z}_r}{\mathfrak{Z}_r} \phi(z) (z - z_{int})^{d-1} \frac{R^r(\mathfrak{Z}_r)}{[\alpha_r]^{d-1}} = 0 \quad (31)$$

$$\text{where } V = V^{CSV} \exp - \left(\sum_{n=1}^{\infty} a_n L_n \right)$$

For $R = p^\mu$ we have $d = 1$ and as explained in II we can easily find the α part of the vertex of gauge covariant string theory since the latter vertex lives on the strip. The above derivation also explains in more detail the subtraction procedure advocated in II.

One could also derive V^{CSV} by starting on the strip and assuming the vertex is an overlap δ function. As a result of which equation (31) with $d=1$ holds for $R = p^\mu$. Mapping onto the upper half plane we can derive equation (12) for $R = P^\mu$ and hence the form of V^{CSV} .

The Fubini Veneziano vertex takes the form of equation (10) when multiplied by $z^{-L_0^1} z^{L_0^2}$. The more usual formulation is found by turning one line into a ket and identifying the oscillators of this line with those of the other line. This vertex is an object of well defined conformal weight, namely k^2 , as well as giving by contour integration over z , a physical vertex. The latter property means that if one leg is on-shell the second leg is automatically on-shell. In order to generalise these properties to the three point vertex let us consider

$$V = \oint_{z=0} \frac{dz}{z} V^{CSV}(z) \quad (32)$$

where

$$V^{CSV}(z) = V^{CSV} \left(z^{L_0^1} \left(\frac{1}{1-z} \right)^{L_0^2} \left(\frac{z-1}{z} \right)^{L_0^3} \right) \quad (33)$$

We observe that if any two legs are on-shell then the third leg is automatically projected to be on-shell. From equation (30) it follows that

$$V^{CSV}(z) \{ L_{-n}^1 z^{-n} - (L_0^1 + n - 1) \} |1\rangle_z |1\rangle_s = 0 \quad (34)$$

if $|1\rangle_z$ and $|1\rangle_s$ are on-shell consequently

$$V |1\rangle_z |1\rangle_s L_{-n} = 0, n \geq 1; V |1\rangle_z |1\rangle_s (L_0 - 1) = 0 \quad (35)$$

We may also generalise the property of conformal weight to $V(z)$. The identity of equation (30) can be exponentiated to yield the equation

$$V(z) g'_1 g'_2 g'_3 = V(\bar{z}) \quad (36)$$

where $g'_1 = \exp \sum_{n=-\infty}^{\infty} a_n L_n^1$ and maps z to $\bar{z}$

$$\begin{aligned} g'_2 &= ((1-z)^{L_0^2} e^{-L_1^2 (-1)^{L_0^2} z^{-L_0^2}} \bar{z}^{L_0^2} (-1)^{L_0^2} e^{L_1^2 (1-\bar{z})} e^{-L_0^2} \\ g'_3 &= \left(\left[\frac{z}{z-1} \right]^{L_0^3} e^{-L_{-1}^3} z^{-L_0^3} e^{\sum a_n L_n^3} \bar{z}^{L_0^3} e^{L_{-1}^3} \left[\frac{\bar{z}-1}{\bar{z}} \right]^{L_0^3} \right) \end{aligned} \quad (37)$$

The reader may verify that g'_2 and g'_3 are just such as to induce the same conformal mapping on z as g .

In II a new vertex involving the ghost oscillators of the form $V^{CSV} V(\beta, \bar{\beta})$ was found. Clearly the above discussion can also be extended to consider $V(\beta, \bar{\beta})$. We will report on this and the supersymmetric vertices elsewhere.

In II it is shown explicitly that

$$V \sum_{r=1}^3 \frac{Q^r}{\alpha^r} = 0 \quad (38)$$

where Q is the BRST change [18]. However, since Q is unaffected by the conformal transformation mapping the covariant vertex V to the upper half plane, we must find that

$$[V^{CSV} V(\beta, \bar{\beta})] \sum_{r=1}^3 Q^r = 0. \quad (39)$$

It will be interesting to discover if the techniques used to derive V^{CSV} can be extended to give a simple derivation of the higher point vertices which occurred in the literature.

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