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THE SIX - FERMION AMPLITUDE IN THE SUPERSTRING

V. Alan Kostelecky¹, Olaf Lechtenfeld,
Stuart Samuel², Dirk Versteegen³, Satoshi Watamura
CERN - Geneva

and

Deshdeep Sahdev
ICTP, Miramare, Trieste

ABSTRACT

Using covariant spin fields, the tree amplitude for six massless fermions in the open ten - dimensional superstring is calculated by superconformal field - theoretic methods. The amplitude is presented in a manifestly Lorentz - covariant form.

¹ On leave of absence from: Physics Department, Indiana University, Bloomington, IN 47405 USA

² A Sloan Foundation Fellow. On leave of absence from: Physics Department, City College of New York, 138th and Convent Ave., NY 10031 USA

³ Member of the FNRS (Belgium).

I. INTRODUCTION

Supersymmetric string theories are rich and esthetic mathematical structures that may offer a new way of unifying all four forces, while possibly being finite.¹ Finding a renormalizable or finite theory of gravity has been an outstanding problem for decades.

In its first – quantized form, the string is governed by massless two – dimensional fields: $X^\mu(\sigma, \tau)$ and its fermionic counterpart. Such theories are conformally invariant and, because the conformal algebra, i.e., the Virasoro algebra, in two dimensions is infinite – dimensional, conformal symmetry imposes severe constraints. The theory is, to a large extent, determined by conformal considerations. A number of interesting results in two dimensions have been obtained using conformal methods [2 – 12].

The extension to superstrings and hence to the superconformal group was recently carried out in refs.[3,4,8,10,11]. This is particularly important, given the current interest in superstrings. The work of D. Friedan, E. Martinec and S. Shenker [3] will be the basis of the present work and any reader interested in understanding the details of the present calculation should consult it.

In refs.[8,12], superconformal methods were used to calculate covariantly the scattering of four fermions in the open superstring. This calculation has an interesting history [13 – 20,8,12], being among one of the most difficult calculations in the days of the dual – resonance models. The first simple derivation [19] came with the introduction of the Green – Schwarz light – cone spinor field, S_α [19 – 21]. Starting with a two – dimensional light – cone action, they quantized S_α and used it to compute the four – fermion scattering amplitude. S_α is a free field, which makes calculations with it relatively easy. There has been a search for the covariant analog of S_α . Most attempts [22,23] consist of trying to find a covariant version of the two – dimensional Green – Schwarz light – cone action. A key idea is to incorporate ghosts into S_α to obtain its covariant version [3,8,12,24]. It is an interacting field in the sense that vacuum expectation values of several spin fields cannot be computed using Wick's theorem. This suggests that any attempt to construct a covariant spin field via a two – dimensional action will involve the problems of quantizing an interacting field.

The methods of ref.[3] are, indeed, powerful. They make the covariant computation of the four – fermion amplitude a relatively easy task. The real test of a new method, however, is whether it can be used to obtain something that has not previously been computed. This is the purpose of the present letter: to calculate covariantly the scattering of six fermions in the open superstring. There have been some attempts to obtain this amplitude [8,25]. In the next section, we will explicitly present it. Considering the monumental effort [13 – 18] which went into the original

¹ See, for example, ref. [1] and references therein.

computation of the four – fermion process, the computation of the six – fermion process demonstrates the power of conformal methods and the progress that theoretical physics has achieved in the last decade. Essentially, conformal techniques reduce the problem to computing vacuum expectations of free fields. In another letter [26], we will present the application of these methods to the $O(16) \times O(16)$ heterotic string [27,28].

We have adopted the notational conventions of ref.[29] except for the metric, $\eta^{\mu\nu}$, which we choose to be $(- + + + + + + +)$. Our gamma matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The massless spectrum of the ten – dimensional open superstring consists of a vector field and a Majorana – Weyl fermion. Dotted and undotted indices are associated with left – handed and right – handed degrees of freedom, respectively, so that $\gamma_{11} S^{\dot{\alpha}} = -S^{\dot{\alpha}}$. We choose our massless fermion to be left – handed. Since gamma matrices flip chirality, they have one dotted and one undotted index: $\gamma^{\mu\alpha}_{\dot{\beta}}$ or $\gamma^{\mu\dot{\alpha}}_{\beta}$. We raise and lower indices with the charge conjugation matrix,

$C^{\alpha\dot{\beta}}$ or $(C^{\dot{\alpha}\beta})$ and $C_{\alpha\dot{\beta}}$ (or $C_{\dot{\alpha}\beta}$). Thus, $\gamma^{\mu\alpha\beta} \equiv \gamma^{\mu\alpha}_{\dot{\gamma}} C^{\dot{\gamma}\beta}$. C satisfies

$$C\gamma^\mu C^{-1} = -\gamma^{\mu t} \text{ (t for transpose). } \gamma^{\mu\nu} \text{ is defined to be } (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)/2.$$

II. THE SIX – FERMION SCATTERING AMPLITUDE

This section describes the computation of the scattering of six fermions in the open ten – dimensional superstring. The result is given in Eqs.(3 – 6,9).

One of the important ingredients in obtaining this scattering amplitude is the construction of the spin field, S^α . It is the covariant analogue of the Green – Schwarz field, which was also denoted by S_α but which was subject to a Majorana – Weyl constraint as well as $\gamma^+ S_\alpha = 0$ [19 – 21]. As has been pointed out in refs.[3,4,8,12,24], ghosts play an important role in the construction of a covariant S^α . The spin field, alone, has fractional conformal weight $5/8$ [30 – 33]. The authors of ref.[3] bosonize the (bosonic) ghost γ and antighost β , which are the two – dimensional supersymmetric partners of the reparametrization ghost c and antighost b , by writing them as $\gamma = \exp(\phi) \eta$ and $\beta = \partial\xi \exp(-\phi)$, where ϕ is a free boson field. When the spin field is multiplied by $\exp(-\phi/2)$ or $\exp(\phi/2)$, conformal fields of dimension 1 and 0 ensue. In physical applications spin fields always appear in such combinations, and it is natural to define covariant spin fields incorporating the fractional ghost operators.

The spin fields can be defined via bosonization [3,8,12]. This is accomplished by parametrizing the Lorentz currents non – linearly on the maximal torus of $SO(10)$ after Wick rotation. Hence, we need five bosons, denoted $\phi^1 \dots \phi^5$. The Neveu – Schwarz – Ramond fermion field, ψ^μ , and the spin field may be defined as conformal fields of highest weight in the vector and spinor representations of the $SO(10)$ current algebra. For the covariant spin field including ghosts, the result is

$$S^\alpha = \exp\left(\sum_{k=1}^5 \alpha^k \phi^k\right) \exp\left(+\frac{1}{2} \phi\right) C_\alpha, \quad (1)$$

$$S^{\dot{\alpha}} = \exp\left(\sum_{k=1}^5 \dot{\alpha}^k \phi^k\right) \exp\left(-\frac{1}{2} \phi\right) C_{\dot{\alpha}},$$

where α represents $(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$ with an even number of $-$ signs (positive eigenvalue of γ_{11}) and $\dot{\alpha}$ represents $(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$ with an odd number of $-$ signs (negative eigenvalue of γ_{11}). The spinor index takes on $2^5 = 32$ values, the correct number for a ten-dimensional fermion. The post-factor, c_α , is an operator that generates cocycle factors [34,35] in operator products. We will call the c_α cocycle operators. Ref.[12] has given the cocycle operators by appealing to the mathematical literature on the existence of such operators for even Lorentzian lattices.² A simple way of representing them will be presented in a separate publication [36]. The cocycle operator naturally breaks up into two operators, one associated with the spin field and one with the ghost factor. The latter must be carefully chosen so that the bosonization procedure destroys neither associativity of operator products nor the Lorentz covariance of the theory.

Ref.[36] will evaluate the scattering amplitude in the open superstring for an arbitrary number of bosons and fermions in a non-manifestly Lorentz-covariant form. In this letter, we use them to compute the six-fermion scattering process.

Ref.[3] contains a description of fermion-scattering processes using ghost-emitting vertex operators, $V_{(-1/2)}$ and $V_{(+1/2)}$, the violation in γ -ghost number being respectively $-1/2$ and $+1/2$. Expressions for $V_{(-1/2)}$ and $V_{(+1/2)}$ can be found in refs.[3,37]. For six external particles, the relevant correlation function is

$$\langle V_{(+1/2)}(z_6) V_{(-1/2)}(z_5) V_{(-1/2)}(z_4) V_{(-1/2)}(z_3) V_{(-1/2)}(z_2) V_{(-1/2)}(z_1) \rangle \quad (2)$$

It violates γ -ghost number by -2 units, which is necessary to have a non-zero matrix element [3]. To compute it, the Green function for five covariant $S^{\dot{\alpha}}$'s and one S^α (coming from $V_{(+1/2)}(z_6)$) is needed. The calculation will be briefly described below.

The amplitude, $\mathcal{A}$, for the six-fermion process is obtained from this correlation function by incorporating other factors including the measure, $\int d\mu(z_1 \dots z_6)$. The result is:

$$\mathcal{A}(u^i, k^i) \propto \left[\prod_{i=1}^6 u_{\dot{\alpha}_i}^i(k^i) \right] \int d\mu(z_1 \dots z_6) G(z_i, \alpha_i) \quad (3)$$

² ϕ^i and ϕ may be combined to express the covariant spin field in terms of a six-dimensional Lorentzian weight lattice [12].

In this equation, k^i is the momentum of the i^{th} fermion, subject to the on-shell conditions $k^i \cdot k^i = 0$ and $\sum_i k^i = 0$. Also, $u_{\dot{\alpha}_i}^i(k^i)$, which satisfies $u_{\dot{\alpha}_i}^i(k^i) \not{K}^i = 0$,

is the left-handed Majorana-Weyl spinor wave function associated with the i^{th} fermion. The terms in Eq.(3) are explained in the next few paragraphs.

The term $G(z_i, \dot{\alpha}_i)$ is given by

$$G(z_i, \dot{\alpha}_i) = \left[\prod_{j>i}^6 (z_j - z_i)^{k^j \cdot k^i} \right] \sum_{i=1}^5 (z_6 - z_i)^{-1} (k^{i\nu} - \frac{1}{8} k_\mu^6 M_{(i)}^{\mu\nu}) \times \frac{1}{\sqrt{2}} \langle (\gamma_\nu^{\dot{\alpha}_6} S^{\dot{\beta}})(z_6) S^{\dot{\alpha}_5}(z_5) \dots S^{\dot{\alpha}_1}(z_1) \rangle \quad (4)$$

where $M^{\mu\nu}_{(i)}$ is defined in Eq.(8) below and the six-point spin-field correlation function is given by

$$\begin{aligned} \langle (\gamma_\nu^{\dot{\alpha}_6} S^{\dot{\beta}})(z_6) S^{\dot{\alpha}_5}(z_5) \dots S^{\dot{\alpha}_1}(z_1) \rangle = \\ \sigma(1 \dots 5) \left\{ -\frac{1}{80} (\gamma_\nu \gamma^{\rho\lambda})^{\dot{\alpha}_6 \dot{\alpha}_5} \gamma_\rho^{\dot{\alpha}_4 \dot{\alpha}_3} \gamma_\lambda^{\dot{\alpha}_2 \dot{\alpha}_1} \right. \end{aligned} \quad (5)$$

$$+ \frac{1}{24} \gamma_\nu^{\dot{\alpha}_6 \dot{\alpha}_5} \gamma^\rho \gamma^{\dot{\alpha}_4 \dot{\alpha}_3} \gamma_\rho^{\dot{\alpha}_2 \dot{\alpha}_1} \left. \right\}$$

The pentagon and "box-with-triangle" diagrams represent the two pole structures that arise in this correlation function:

$$\begin{aligned} \text{Pentagon diagram} &= [(z_5 - z_4)(z_4 - z_3)(z_3 - z_2)(z_2 - z_1)(z_1 - z_5)]^{-1} \\ \text{Box-with-triangle diagram} &= (z_6 - z_4) [(z_6 - z_5)(z_5 - z_4)(z_4 - z_3) \\ &\quad (z_3 - z_2)(z_2 - z_1)(z_1 - z_4)]^{-1} \end{aligned} \quad (6)$$

Here, the notation is that a solid line corresponds to a pole and a dotted line corresponds to a monomial. Arrows indicate the ordering of z variables.

The term $\sigma(1 \dots 5)$ in Eq.(5) is an antisymmetrization operator that acts on the first five z_i and $\dot{\alpha}_i$ indices:

$$\sigma(1 \dots 5) f(z_i, \alpha_i) = \sum_{\sigma \in S_5} \text{sign}(\sigma) f(z_{\sigma(i)}, \alpha_{\sigma(i)}) \quad (7)$$

Eq.(7) includes a sum over the 120 elements of the permutation group, S_5 . When $\sigma(1 \dots 5)$ acts in Eq.(5), it guarantees that the right-hand side of Eq.(5) is antisymmetric in the first five labels. The left-hand side of Eq.(5) is antisymmetric because the S^α 's are anticommuting objects.

The operator $M^{\mu\nu}_{(i)}$ in Eq.(4) acts on the spinor index of the i^{th} fermion. It implements Lorentz rotations. More precisely, it acts on a function, f , with spinor indices by multiplying by $\gamma^{\mu\nu}$ on the i^{th} α :

$$M^{\mu\nu}_{(i)} f^{\alpha_6 \dots \alpha_1} = (\gamma^{\mu\nu})^{\alpha_i}_{\beta} f^{\alpha_6 \dots \alpha_{i-1} \beta \alpha_{i+1} \dots \alpha_1} \quad (8)$$

Finally, the measure is given by:

$$\int d\mu(z_1 \dots z_6) = \int_{\substack{z_j > z_i \\ j > i}} \prod_{i=1}^6 dz_i \frac{(z_m - z_\ell)(z_\ell - z_k)(z_k - z_m)}{dz_m dz_\ell dz_k} \quad (9)$$

$m > \ell > k$

Here and in the above equations, the z_i are real variables. Originally, the correlation functions were computed for complex z_i , but for the open bosonic string, the particle emission from vertex operators occurs at the boundary of the string, which corresponds to z_i on the real axis.³ The range of integration in Eq.(9) is $\infty > z_6 > \dots > z_1 > -\infty$. The symbolic notation of dividing by $dz_m dz_\ell dz_k$ means that these three integrations are to be omitted. The points z_m, z_ℓ and z_k can be fixed to arbitrary values, due to the $SL(2, R)$ invariance. A natural choice is $k=1, \ell=5$ and $m=6$ with $z_1 = 0, z_5=1$ and $z_6=\infty$, in which case one integrates between 0 and 1 over z_2, z_3 and z_4 with these three z 's ordered. The correct measure is naturally generated in the approach of ref.[3] when BRST-invariant vertex operators are used. The factor in the numerator comes from the correlation function of the anticommuting reparametrization ghosts, c .

When Eqs.(4-9) are assembled into Eq.(3), the scattering amplitude is obtained. An internal-symmetry group can be included by using Paton-Chan [38] factors and multiplying Eq.(3) by the appropriate trace of matrices [20]. Our results can readily be extended to closed superstrings.

We now outline the computation of the amplitude. To calculate the six-vertex-operator expectation, we had to compute three kinds of correlation functions: one involving three c -ghost fields, one involving the bosonic X^μ fields, and one involving Neveu-Schwarz-Ramond fields and covariant spin fields. The X^μ and

³ The variable, z , is related to the usual string variables, σ and τ , via $z = \exp(\tau + i\sigma)$.

c -ghost correlation functions are straightforward to compute since free fields are involved. The former generates the momentum dependence in the amplitude. The spinor correlation functions give rise to the gamma-matrix dependence as well as additional pole structures in the z_i . There are two kinds to compute. The first involves six spin fields. Since these spin fields are not free, the result is not immediate. The method we used was to bosonize the spin field as in Eq.(1). Exponentials of the ϕ^i and ϕ appear, as well as the cocycle operators, c_α . Since the bosons are free fields, their contribution is straightforward to obtain. They generate the pole structures in Eq.(5). The cocycle operators in the matrix element can be moved to the right until they act on the vacuum. In the process, commutators of the c_α with exponentials of the boson fields are produced. These commutators generate certain factors and gamma matrices, in a form which is not obviously Lorentz invariant. It took considerable work to recast the expression in a manifestly Lorentz-covariant form. The second correlation function involves $\psi^\mu\psi^\nu$ and six spin fields. Since $\psi^\mu\psi^\nu$ is the operator that implements Lorentz rotations, its effect can be computed using the methods of Sec. II of ref.[7]. Basically, it generates Lorentz rotations on the other fields. This is why $M^{\mu\nu}_{(i)}$ appears in Eq.(4).

III. CONCLUSION

In this letter, we have computed the amplitude for six-ground state fermions in the open ten-dimensional superstring by using the superconformal methods of ref.[3]. It was quite important to have an explicit construction of the cocycle operators, c_α , with which it was easy to work. Without them, it would have been difficult to obtain the correct gamma-matrix structure in Eq.(5). With the covariant version of the light-cone Green-Schwarz spinor field, S^α , it is possible to compute processes involving an arbitrary number of massless fermions and bosons [36]. Superconformal methods are proving to be powerful computation tools for superstring theories.

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