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SYMMETRY REDUCTION AND SIMPLE SUPERSYMMETRIC MODELS*

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ABSTRACT

The reduction by the subgroups of the conformal group space time of the Wess-Zumino model and the four-dimensional simple ($N = 1$) supersymmetric Yang-Mills theories is considered. It is shown for both models that the translations are the only subgroups of the Poincaré group which leave a supersymmetric residual system on the reduced space.

I. INTRODUCTION

In previous papers, a study of the geometric formulation of invariant gauge fields under smooth group actions^{1,2} has been used to reduce gauge^{3,4} and matter-coupled gauge^{5,6} systems under subgroups of the conformal group of space-time. Hamiltonian interpretations of the residual systems have been given and a number of invariant solutions found. As a next step in these investigations, we address ourselves to the symmetry reduction of supersymmetric models.

The (manifest) supersymmetries of the S -matrix in more than $1 + 1$ dimensions have been classified by W. Nahm.⁷ He has also obtained the representations of the flat space-time supersymmetries and presented examples with supermultiplets containing a vector as their highest spin component. In this article, we start instead from a definite representation of a supersymmetry to generate supersymmetric models in lower dimension by symmetry reduction. This approach has been successfully applied in flat space-time by L. Brink, J. H. Schwarz and J. Scherk⁸ who derived the extended supersymmetric Yang-Mills models in $3 + 1$ dimensions from higher dimensional $N = 1$ supersymmetric Yang-Mills models.

In the following, we examine the Wess-Zumino model⁹ and the four-dimensional $N = 1$ supersymmetric Yang-Mills system^{10,11}. Our purpose is to identify group actions leading to supersymmetric reduced systems. First, we present in section II the invariance conditions for the fields of the (on-shell) supersymmetric multiplet involved in each model. In section III, we show that the translations are the only subgroups of the invariance group of the Wess-Zumino model which leave a supersymmetric residual system on the reduced space. Next, we devote the section IV to the investigation of the $N = 1$ supersymmetric Yang-Mills theories. We look for the subgroups of the conformal group ($C(3,1)$) leading to reduced supersymmetry. It is found that only the translations fulfill this requirement among the subgroups of the Poincaré group ($P(3,1)$). We also give a condition for the subgroups of $C(3,1)$. In the Appendix, a reduction of a supersymmetric Yang-Mills system without residual supersymmetry is discussed.

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II. INVARIANCE CONDITIONS

The action of a symmetry group G on Minkowski space (M) generates a number of strata on which the invariance conditions are solved, each stratum being the set of orbits with conjugate isotropy subgroups (G_0). For tensor and spinor fields, the invariance conditions under the action of the subgroups of the conformal group are respectively given in Refs. [12] and [13]. However, for a multiplet of Dirac spinor fields transforming under a representation (D) of the gauge group (H), we have the following expression:⁵

$$\Psi(gx) = D({}^g J(x)) \otimes \tilde{D}(\rho^{-1}(g, x)) \Psi(x), \quad (2.1)$$

for every element $g \in G \subset C(3, 1)$, where $x \in M$, D is the representation of the Jacobian (J) of the group transformation g (which is for an orthonormal frame the product of a conformal factor and the direct sum of the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ basic representations of $SL(2, C)$), and $\rho : G \times M \rightarrow H$, is the "transformation function"¹ which characterizes the group action on the gauge bundle. In order to determine the invariant fields, we have first to solve the isotropy condition, i.e. the set of linear algebraic equations derived from the invariance condition at the isotropy point x_0 of each orbit for each element (g_0) of the isotropy subgroup at this point ($G_0(x_0)$). The isotropy condition for the spinor fields multiplet derived from (2.1) is:

$$\Psi(x_0) = D({}^{g_0} J(x_0)) \otimes \tilde{D}(\lambda(g_0)) \Psi(x_0), \quad (2.2)$$

where $\lambda(g_0) = \rho^{-1}(g_0, x_0)$, corresponds to the homomorphism of G_0 into the gauge group classifying the gauge bundles admitting a lift of the group action (see Ref. [1]). We obtain the invariant field on each stratum by acting with G on the solution to the isotropy condition (2.2) and by smoothly (C^∞) joining (if possible) on the stratum the resulting fields on each orbit (for details and examples, see Refs. [12] and [13]).

As for the invariant gauge fields, they satisfy for every $g \in G$ the condition:¹

$$(f_g^* \omega)(x) = \text{Ad } \rho^{-1}(g, x) \omega(x) + \rho^{-1}(g, x) d\rho(g, x), \quad (2.3)$$

where ω is a gauge field (1-form connection) and Ad denotes the adjoint representation of the gauge group. Their isotropy condition is simply:

$$(f_{g_0}^* \omega)(x_0) = \text{Ad } \lambda(g_0) \omega(x_0), \quad (2.4)$$

Let us note that the compatibility between the invariance conditions and the gauge transformations requires that a gauge field ω' , obtained from a gauge transformation $h(x)$ on ω , obeys the invariance condition (2.3) with a transformation function $\rho'(g, x)$ given by:

$$\rho'^{-1}(g, x) = h^{-1}(x) \rho^{-1}(g, x) h(x). \quad (2.5)$$

III. WEISS-ZUMINO MODEL

The (on-shell) Weiss-Zumino model is formed of one complex scalar field (ϕ) and one anticommuting Majorana spinor field (Weyl component: ψ) with the following Lagrangian density, written in the Weyl spinor notation (see Ref. [14]):

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi) + \frac{1}{2} (i \bar{\psi} \sigma^\mu \partial_\mu \bar{\psi} + i \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi - m \bar{\psi} \psi - m \bar{\psi} \bar{\psi}). \quad (3.1)$$

The (on-shell) supersymmetry transformations leaving the corresponding action invariant are:

$$\delta_\epsilon \phi = 2\epsilon \phi, \quad (3.2)$$

and

$$\delta_\epsilon \psi = -(i \sigma^\mu \bar{\epsilon} \partial_\mu \phi + m \epsilon \phi^*), \quad (3.3)$$

where ϵ and $\bar{\epsilon}$ are defined as the $SL(2, C)$ components of the infinitesimal anticommuting Majorana spinor parameter.

Since the reduced systems are required to be supersymmetric, the fields (ϕ' and ψ') resulting from any supersymmetry transformation must also satisfy the invariance conditions imposed on the original fields ϕ and ψ :

$$\phi'(gx) = \phi(x), \quad (3.4)$$

and

$$\psi'(gx) = D(\frac{1}{2}, 0)({}^g J(x)) \psi'(x), \quad (3.5)$$

where: $\phi' = \phi + \delta_\epsilon \phi$, $\psi' = \psi + \delta_\epsilon \psi$, and $D(\frac{1}{2}, 0)$ stands for the $(\frac{1}{2}, 0)$ basic representation of $SL(2, C)$. The invariance conditions and the supersymmetry transformations are then compatible if:

$$(\partial_\epsilon \phi)(gx) = (\partial_\epsilon \phi)(x), \quad (3.6)$$

and

$$(\delta_\epsilon \psi)(x) = D({}^g J(x)) (\delta_\epsilon \psi)(x), \quad (3.7)$$

for all $g \in G$ and all ϵ .

In addition to the invariance condition:

$$\psi(gx) = D(\frac{1}{2}, 0)({}^g J(x)) \psi(x), \quad (3.8)$$

we find from the transformation (3.2) and the condition (3.6) that the spinor ψ is subjected to the constraint:

$$\psi(gx) = \psi(x). \quad (3.9)$$

Moreover, the invariance condition for ϕ , the transformation (3.3) and the condition (3.7) imply that:

$$\phi^*(x) \left(1_2 - D(\frac{1}{2}, 0) ({}^g J(x)) \right) = 0, \quad (3.10)$$

for all $g \in G$. Assuming a non-trivial invariant scalar field ϕ , we deduce from (3.10) that the spinor representation of the Jacobian must be the identity at any point of the stratum. Although the invariance group of the model is respectively the Poincaré group or the conformal group for massive or massless fields, only the translations (neglecting the discrete subgroups) satisfy the above constraint. These subgroups also ensure the compatibility between the conditions (3.8) and (3.9). Consequently, the translations are the only (continuous) space-time transformations allowing reductions of the Wess-Zumino model with residual supersymmetry. For example, the invariance under two-dimensional space translations yields a supersymmetric system composed of a non-interacting Dirac spinor field with a scalar field in 1+1 dimensions. The residual supersymmetry transformations are easily obtained by substituting the invariant fields in the formulas (3.2) and (3.3).

IV. SIMPLE SUPERSYMMETRIC YANG-MILLS SYSTEMS

A simple ($N = 1$) supersymmetric Yang-Mills theory (on-shell) in 3+1 dimensions consists of a Yang-Mills field minimally coupled to a multiplet of anticommuting Majorana spinor fields transforming under the adjoint representation of the gauge group. Its Lagrangian density, expressed in the Weyl spinor notation (see Ref. [14]), is :

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \frac{i}{2} [\bar{\psi}^a \sigma^{\mu\nu} (D_\mu \psi)^a + \psi^a \sigma^{\mu\nu} (D_\mu \bar{\psi})^a], \quad (4.1)$$

where $F_{\mu\nu}$ represents the field strength corresponding to the gauge field ω_μ . D_μ the covariant derivative with respect to the gauge field, and ψ the $(\frac{1}{2}, 0)$ Weyl component of the Majorana spinor-field. The corresponding action is left invariant under the following (on-shell) supersymmetry transformations, which form a representation of the super Poincaré algebra:

$$\delta_\epsilon \omega_\mu^a = i \bar{\epsilon} \gamma_\mu \bar{\Psi}^a = i (\bar{\epsilon} \sigma_\mu \psi^a + \epsilon \sigma_\mu \bar{\psi}^a), \quad (4.2)$$

and

$$\delta_\epsilon \psi^a = -\frac{1}{2} F_{\mu\nu}^a \sigma^{\mu\nu} \epsilon, \quad (4.3)$$

where ϵ and $\bar{\epsilon}$ are the $SL(2, C)$ components of the anticommuting Majorana four-spinor parameter $\bar{\epsilon}$, and $\bar{\Psi}$ is the four-component Majorana spinor field with values in the Lie algebra of the gauge group. (Use of the four-spinor notation will sometimes be made for brevity¹⁴.)

As in section III, a reduced supersymmetry is ensured if the invariance conditions are also imposed on the gauge and spinor fields coming from any supersymmetry transformations (4.2) and (4.3). This means that:

$$(j_\theta^* \omega^\lambda)(x) = \text{Ad } \rho^{-1}(g, x) \omega^\lambda(x) + \rho^{-1}(g, x) d\rho(g, x), \quad (4.4)$$

and

$$\psi'(gx) = D(\frac{1}{2}, 0) ({}^g J(x)) \otimes \text{Ad } \rho^{-1}(g, x) \psi'(x), \quad (4.5)$$

where: $\omega' = \omega + \delta_\epsilon \omega$ and $\psi' = \psi + \delta_\epsilon \psi$. Let us remark that a supersymmetric residual system is also produced if the fields ω' and ψ' are gauge equivalent to invariant fields. We discuss this case at the end of this section.

From the equations (4.4) and (4.5) and the invariance conditions for the gauge field (ω) and the spinor field (ψ), we get:

$$(f_\theta^* \delta_\epsilon \omega)(x) = \text{Ad } \rho^{-1}(g, x) \delta_\epsilon(x), \quad (4.6)$$

and

$$(\delta_\epsilon \psi)(gx) = D(\frac{1}{2}, 0) ({}^g J(x)) \otimes \text{Ad } \rho^{-1}(g, x) \delta_\epsilon \psi(x). \quad (4.7)$$

Since these conditions hold for every Majorana spinor ϵ , it follows that:

$$\bar{\sigma}_\mu \psi(gx) {}^g J(x)^\mu{}_\nu = \bar{\sigma}_\nu \text{Ad } \rho^{-1}(g, x) \psi(x), \quad (4.8)$$

and

$$\sigma^{\mu\nu} F_{\mu\nu}(gx) = D(\frac{1}{2}, 0) ({}^g J(x)) \sigma^{\mu\nu} \text{Ad } \rho^{-1}(g, x) F_{\mu\nu}(x), \quad (4.9)$$

At the isotropy point x_0 , the above expressions simplify to:

$$\bar{\sigma}_\mu \psi(x_0) {}^{g_0} J(x_0)^\mu{}_\nu = \bar{\sigma}_\nu \text{Ad } \lambda(g_0) \psi(x_0), \quad (4.10)$$

and

$$\sigma^{\mu\nu} F_{\mu\nu}(x_0) = D(\frac{1}{2}, 0) ({}^{g_0} J(x_0)) \sigma^{\mu\nu} \text{Ad } \lambda(g_0) F_{\mu\nu}(x_0), \quad (4.11)$$

where $g_0 \in G_0$. Substituting, respectively, the isotropy conditions for the spinor field and the field strength in the equations (4.10) and (4.11), we find the constraints:

$$[\bar{\sigma}^\mu {}^{g_0} J(x_0)^\mu{}_\nu - \bar{\sigma}_\nu D^{-1} ({}^{g_0} J(x_0))] \psi(x_0) = 0, \quad (4.12)$$

and

$$[\sigma^{\mu\nu} - D ({}^{g_0} J(x_0)) \sigma^{\lambda\gamma} {}^{g_0} J(x_0)^\mu{}_\lambda {}^{g_0} J(x_0)^\nu{}_\rho] F_{\mu\nu}(x_0) = 0, \quad (4.13)$$

where $\psi(x_0)$ and $F_{\mu\nu}(x_0)$ are assumed to be non-trivial.

We next look for the symmetry subgroups of $P(3, 1)$ obeying to (4.8) and (4.9) with the invariant fields ψ and $F_{\mu\nu}$. Choosing an orthonormal frame on Minkowski space, the Jacobian matrix ${}^{g_0} J(x_0)$ induces a homomorphism of the isotropy subgroup G_0 into the Lorentz group $SO(3, 1)$. Therefore, the equation (4.12) can be written as:

$$[D(\frac{1}{2}, 0) ({}^{g_0} \Lambda^{-1})^\dagger] \bar{\sigma}_\nu D(\frac{1}{2}, 0) ({}^{g_0} \Lambda^{-1}) \psi(x_0) = 0, \quad (4.14)$$

with ${}^{g_0} \Lambda \in SO(3, 1)$. However, no one-parameter subgroup of $SO(3, 1)$ satisfies (4.14). Since the condition (4.14) is preserved by conjugations under $SO(3, 1)$, this can be shown by considering only the three one-parameter subgroups of $SO(3, 1)$ up to conjugations by $SO(3, 1)$: L_3 , K_3 , and $L_2 + K_1$,¹⁵ where L_i stands for the generator of the rotations around the x^i -axis, and K_i , for the generator of the boosts along

the x^i -axis. But no non-vanishing $\psi(x_0)$ fulfills this condition for each of these one-parameter subgroups of $SO(3, 1)$. Hence, the image of the Jacobian homomorphism at x_0 is the identity element (since we ignore the discrete subgroups), which means that $G_0(x_0)$ is included in the kernel of $J(x_0)$ ($G_0(x_0) \subseteq \text{Ker } J(x_0)$). For the action of any subgroup of $P(3, 1)$ on Minkowski space, the kernel of $J(x_0) \subseteq T(4)$ (the pure translations in four dimensions), and thus: $G_0 \subseteq T(4)$. This implies that $G_0(x_0)$ is the identity element. Consequently, the homomorphism λ is trivial and from the corollary 3 or Ref. [1], the transformation function equals the identity ($\rho^{-1}(g, x) = e$).

Returning to the equation (4.8), the value of ρ enables us to write

$$\partial_\mu \psi(gx) {}^g \Lambda^\mu{}_\nu = \partial_\nu \psi(x) . \quad (4.15)$$

With use of the invariance condition (2.1) for the spinor field, this relation becomes:

$$\left[D(\tfrac{1}{2}, 0) ({}^g \Lambda^{-1})^t - \mathbf{1}_2 \right] \partial_\nu \psi(x) = 0 , \quad (4.16)$$

for every value of the index ν and at every point x of the stratum. But there exists no non-trivial spinor field $\psi(x)$ which obeys to this equation for any of the one-parameter subgroups of $SO(3, 1)$. As in the isotropy case, one can arrive at this result by looking at the three above-mentioned one-parameter subgroups of $SO(3, 1)$. We are thus left with the translations. The compatibility condition (4.9) for the invariant field strength $F_{\mu\nu}$ is also satisfied for these subgroups. Thence, only the reductions by translations permit a supersymmetry preserving the set of invariant fields. An example of such reductions is detailed in L. Brink *et al.*⁸ for two-dimensional space translations where a Yang-Mills system interacting with a complex scalar field and a Dirac spinor field in $1 + 1$ dimensions is derived.

Let us make a brief comment on the possible reductions for the subgroups of the conformal group $C(3, 1)$. With a choice of orthonormal frames, the Jacobian at x_0 is a homomorphism of G_0 into the direct product of the Lorentz group with the group of dilatations:

$${}^{g_0} J(x_0) : g_0 \in G_0(x_0) \mapsto e^{\lambda_{g_0}}(x_0) {}^{g_0} \Lambda(x_0) \in \mathbf{R}^+ \times SO(3, 1) , \quad (4.17)$$

where $\lambda_{g_0}(x_0) \in \mathbf{R}$ and ${}^{g_0} \Lambda \in SO((3, 1))$. The isotropy condition (4.10) then reads:

$$\left[e^{\lambda_{g_0}(x_0)} {}^{g_0} \Lambda(x_0)^\mu{}_\nu - e^{-\frac{2}{3}\lambda_{g_0}(x_0)} \partial_\nu D(\tfrac{1}{2}, 0) ({}^{g_0} \Lambda^{-1}(x_0)) \right] \psi(x_0) = 0 . \quad (4.18)$$

It turns out that all the one-parameter subgroups of $\mathbf{R}^+ \times SO(3, 1)$ force the spinor field at the isotropy point to vanish. Thus we conclude that: $G_0(x_0) \subseteq \text{Ker } J(x_0)$.

Finally, in the hope of finding new subgroups leading to supersymmetric systems, we relax the compatibility conditions (4.4) and (4.5) by replacing the transformation function ρ by a gauge equivalent one (see Eq. (2.5)):

$$(f_g^* \omega')(x) = \text{Ad } \rho'^{-1}(g, x) \omega'(x) + \rho'^{-1}(g, x) d\rho'(g, x) , \quad (4.19)$$

and

$$\Psi'(gx) = D({}^g J(x)) \otimes \text{Ad } \rho'^{-1}(g, x) \Psi'(x) . \quad (4.20)$$

It corresponds to allow each invariant field transformed by a supersymmetry to be equivalent up to a gauge transformation (which can be a function of the Majorana spinor parameter $\mathcal{E}$) to a field of the set of invariant fields under the symmetry group G . On the other hand, these conditions involve explicitly the transformation function ρ .

For $G \subset P(3, 1)$, the following expressions in four-spinor notation are deduced from (4.19) and (4.20) at the isotropy point (for every value of the Majorana spinor $\mathcal{E}$):

$$X_\mu(x_0, \mathcal{E}) {}^{g_0} \Lambda^\mu{}_\nu = \text{Ad } \lambda(g_0) X_\nu(x_0, \mathcal{E}) , \quad (4.21)$$

and

$$D^{-1}({}^{g_0} \Lambda) \Phi(x_0, \mathcal{E}) = \text{Ad } \lambda(g_0) \Phi(x_0, \mathcal{E}) , \quad (4.22)$$

with the definitions:

$$X_\mu(x_0, \mathcal{E}) \equiv \text{Ad } h(x_0, \mathcal{E}) [\tilde{\mathcal{E}} \gamma_\mu \Psi(x_0) - \omega_\mu(x_0)] , \quad (4.23)$$

and

$$\Phi(x_0, \mathcal{E}) \equiv \text{Ad } h(x_0, \mathcal{E}) [F^{\mu\nu}(x_0) \sigma_{\mu\nu} \mathcal{E} - \Psi(x_0)] . \quad (4.24)$$

If $G_0 = \{e\}$, then $\rho' = \rho = \{e\}$, and therefore, no symmetry groups other than the translations are admissible. For $G_0 \neq \{e\}$, the equations (4.21) and (4.22) for $X_\mu(x_0, \mathcal{E})$ and $\Phi(x_0, \mathcal{E})$ are respectively equivalent to the isotropy conditions for the gauge field ω_μ and the spinor field Ψ , and thus the solution X_μ has the same form as $\omega_\mu(x_0)$, and the solution Φ , the same as Ψ . Since $\Psi(x_0)$ and $\omega(x_0)$ span different subspaces of the gauge algebra (when $G_0 \neq \{e\}$), there exists no $\text{Ad } h(x_0, \mathcal{E})$ automorphism of the gauge algebra such that (4.23) and (4.24) are satisfied, and therefore no new supersymmetric residual system is obtained. In the Appendix, we look at the $SO(3)$ invariant gauge and spinor fields of the $SU(3)$ supersymmetric Yang-Mills system as an example of the above discussion.

V. SUMMARY

We have investigated in this paper the reduction by symmetry of two simple supersymmetric models. We have found that the Wess-Zumino model can only be reduced to supersymmetric systems by the translation subgroups of the conformal group by looking at the compatibility between the supersymmetry transformations and the invariance conditions. The residual supersymmetry is derived by substitution of the invariant fields in the supersymmetry transformations of the model. We have also studied the $N = 1$ four-dimensional supersymmetric Yang-Mills system. It has been shown that no residual supersymmetry is left for any subgroup of the Poincaré group except the pure translations, and that a reduced supersymmetry requires at least a trivial Jacobian homomorphism of the isotropy subgroup for any subgroup of the conformal group. No other subgroup of $P(3, 1)$ produces such reductions even if we relax the compatibility conditions by allowing each invariant field transformed by

supersymmetry to be equivalent up to a gauge transformation to a field of the set of invariant fields under the symmetry group chosen. Finally, the $SO(3)$ invariant gauge and spinor-fields of the $N = 1$ supersymmetric Yang-Mills system with gauge group $SU(3)$ are discussed in the Appendix as an example of a reduction which leaves no supersymmetry.

In future work, it would be interesting to study the symmetry reduction of higher dimensional supersymmetric systems to four-dimensional supersymmetric models. For instance, this method can be applied to derive the $SO(8)$ supergravity model from simple supergravity in 11 dimensions.¹⁶ Further results have been published recently by N. S. Manton¹⁷ who considered non-trivial coset spaces as extra-dimensional compact manifolds.

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APPENDIX

SO(3) INVARIANCE OF THE SU(3) ON-SHELL SIMPLE SUPER-YANG-MILLS MULTIPLY

In the following, we give an example of a reduction leading to a residual system which is not supersymmetric. For simplicity, we look at the $N = 1$ supersymmetric Yang-Mills system with gauge group $SU(3)$ reduced by symmetry $SO(3)$. The $SO(3)$ reduction of the same model with gauge group $SU(2)$ leaves only a Yang-Mills field and a multiplet of scalar fields. We use for that purpose the Gell-Mann representation of the generators λ_a of $SU(3)$, with $a = 1, \dots, 8$, and their adjoint representation: $(\lambda_a^{Ad})_b^c = -(f_a)_{bc}$, where the quantities f_{ab}^c are the structure constants of the Lie algebra $su(3)$. The orbits generated under the action of $SO(3)$ are two-dimensional spheres which form a stratum parametrized by the time $t \in \mathbb{R}$ and the radii $r \in \mathbb{R}^+$ of the spheres, leaving the origin as a singular orbit.

The invariant gauge field is obtained either by the invariance condition (2.3) or equivalently by the theorem 2 of Ref. [5]. In the former case, the transformation function can be written as an extension of the homomorphism λ of the isotropy group $SO(2)$ into $SU(3)$. For any isotropy point x_0 along the z -axis, except the origin, the isotropy subgroup consists of the rotations $e^{i\chi L_3}$. Accordingly, we choose the following homomorphism (up to a Z_2 factor) of $SO(3)$ for the transformation function (ρ^{-1}):

$$\rho^{-1} : g = e^{\phi L_3} e^{\theta L_2} e^{\chi L_3} \in SO(3) \mapsto \begin{bmatrix} e^{\phi + i\theta} e^{i\chi} & 0 \\ 0 & 1 \end{bmatrix} \in SU(2) \subset SU(3), \quad (\text{A.1})$$

where ϕ, θ and χ are the rotation angles, $\{L_i\}$ the generators of $SO(3)$, and $\{\tau_i \equiv \frac{\sigma_i}{2}\}$ the generators of $SU(2)$. Let us mention that the isomorphism of $SO(3)$ onto an $SO(3)$ subgroup of $SU(3)$ implies a zero invariant spinor field and that the trivial homomorphism $\lambda(G_0) = \{e\}$ permits only a pure (zero curvature) invariant Yang-Mills field. (As for the other homomorphisms λ , they can not be extended to global homomorphisms (ρ^{-1}) including the origin, and we ignore them.)

The isotropy condition for the gauge field (2.4) is explicitly:

$$\omega_\mu(x_0)^{g_0} J(x_0)^\mu_\nu = \text{Ad } \lambda(g_0) \omega_\nu(x_0), \quad (\text{A.2})$$

for every $g_0 \in G_0$. Its solution at the isotropy point $x_0 = (0, 0, 0, z \neq 0)$ is given in matrix form by

$$[\omega(x_0)]_\mu^\alpha = \begin{bmatrix} 0 & 0 & C & 0 & 0 & 0 & E \\ A & B & 0 & 0 & 0 & 0 & 0 \\ -B & A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & D & 0 & 0 & 0 & F \end{bmatrix}, \quad (\text{A.3})$$

where $A_1, \dots, F$ are constants. Inserting this expression in the invariance condition (2.3) and letting the parameters $A, \dots, F$ vary smoothly (C^∞) on the stratum, we obtain the most general $SO(3)$ invariant $SU(3)$ gauge field (for λ):

$$\omega_\mu(x) = \delta_{\mu 0} (C x^i \lambda'_i + E \lambda'_8) + \delta_{\mu i} \left[A \lambda'_i + (D - A) \frac{x^i x^j}{r^2} \lambda'_j - B \epsilon^{ijk} \frac{x^j}{r} \lambda'_k + F x^i \lambda'_8 \right], \quad (\text{A.4})$$

where $A, \dots, F$ are functions of r and t , and $\lambda'_i \equiv \frac{\lambda_i}{2}$ ($i, j, k = 1, 2, 3$).

For the spinor field, the isotropy condition (2.2) reads:

$$\Psi(x_0) = \left[D \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ({}^{g_0} J(x_0)) \right] \otimes \text{Ad } \lambda(g_0) \Psi(x_0), \quad (\text{A.5})$$

where $\Psi \in C^{4 \times 8}$ is a Dirac spinor-field. At the isotropy point $x_0 = (0, 0, 0, z)$, the solution to (A.4) has the form:

$$\Psi(x_0) = \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} \lambda'_4 + \begin{bmatrix} -i\alpha \\ i\beta \\ -i\gamma \\ i\delta \end{bmatrix} \lambda'_5 + \begin{bmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{bmatrix} \lambda'_6 + \begin{bmatrix} i\alpha' \\ -i\beta' \\ i\gamma' \\ i\delta' \end{bmatrix} \lambda'_7, \quad (\text{A.6})$$

where $\alpha, \dots, \delta$ and $\alpha', \dots, \delta'$ are constants. Imposing the Majorana constraint requires that:

$$\alpha = -\delta^*, \quad \beta = \gamma^*, \quad \alpha' = -\delta'^*, \quad \text{and} \quad \beta' = \gamma'^*. \quad (\text{A.7})$$

We determine the $SO(3)$ invariant Majorana spinor field with values in the gauge algebra $\mathfrak{su}(3)$ by substituting (A.6) and (A.7) in the invariance condition (2.1) and by allowing $\alpha, \beta, \alpha', \delta'$ to depend smoothly on the parameters r and t of the orbits. For simplicity, we write:

$$\Psi^{\text{Maj}}(t, r, \theta, \phi) = \begin{bmatrix} e^{\phi r_3} e^{\theta r_2} & 0_2 \\ 0_2 & e^{\phi r_3} e^{\theta r_2} \end{bmatrix} \otimes \text{Ad} \left(e^{\phi \lambda'_5} e^{\theta \lambda'_2} \right) \Psi^{\text{Maj}}(t, r), \quad (\text{A.8})$$

where $\Psi^{\text{Maj}}(t, r)$ stands for the spinor (A.6) with the restrictions (A.7). Let us point out that the $SO(3)$ invariant spinor $\Psi^{\text{Maj}}(t, r, \theta, \phi)$ belongs to the subspace of $\mathfrak{su}(3)$ spanned by $\{\lambda_4, \lambda_5, \lambda_6, \lambda_7\}$, and that the $SO(3)$ invariant gauge field (A.3) takes values in the orthogonal subspace (with respect to the Killing form) generated by $\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\}$. From this fact, it is easily seen by introducing the invariant fields (A.3) and (A.8) in the supersymmetry transformations (4.2) and (4.3) that no supersymmetry is left.

However, as mentioned in section IV, the compatibility conditions can be weakened by demanding instead a residual supersymmetry up to a gauge transformation. But even in that case, the conditions (4.1) and (4.2) are still too stringent. The relation (4.21) for $X_\mu(x_0, \mathcal{E})$ is equivalent to the isotropy condition (A.2) for $\omega_\mu(x_0)$, and therefore the solution X_μ has the form (A.3), except that the parameters can depend on the Majorana spinor $\mathcal{E}$. It follows that the expression (4.23) cannot be satisfied even if we add a gauge transformation since $\text{Ad } h(x_0, \mathcal{E})$ is an automorphism of the gauge algebra.