SYMmetry REDUCTION AND SIMPLE SUPERsYMMETRIC MODELS

Martin Légaré

Center for Theoretical Physics
Laboratory for Nuclear Science
and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139 U.S.A.

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1. INTRODUCTION

In previous papers, a study of the geometric formulation of invariant gauge fields under smooth group actions\textsuperscript{1,2} has been used to reduce gauge\textsuperscript{3,4} and matter-coupled gauge\textsuperscript{5,6} systems under subgroups of the conformal group of space-time. Hamiltonian interpretations of the residual systems have been given and a number of invariant solutions found. As a next step in these investigations, we address ourselves to the symmetry reduction of supersymmetric models.

The (manifest) supersymmetries of the $S$-matrix in more than 1+1 dimensions have been classified by W. Nahm.\textsuperscript{7} He has also obtained the representations of the flat space-time supersymmetries and presented examples with supermultiplets containing a vector as their highest spin component. In this article, we start instead from a definite representation of a supersymmetry to generate supersymmetric models in lower dimension by symmetry reduction. This approach has been successfully applied in flat space-time by L. Brink, J. H. Schwarz and J. Scherk\textsuperscript{8} who derived the extended supersymmetric Yang-Mills models in 3+1 dimensions from higher dimensional $N = 1$ supersymmetric Yang-Mills models.

In the following, we examine the Wess-Zumino model\textsuperscript{9} and the four-dimensional $N = 1$ supersymmetric Yang-Mills system\textsuperscript{10,11}. Our purpose is to identify group actions leading to supersymmetric reduced systems. First, we present in section II the invariance conditions for the fields of the (on-shell) supersymmetric multiplet involved in each model. In section III, we show that the translations are the only subgroups of the invariance group of the Wess-Zumino model which leave a supersymmetric residual system on the reduced space. Next, we devote the section IV to the investigation of the $N = 1$ supersymmetric Yang-Mills theories. We look for the subgroups of the conformal group {\((C(3,1))\) leading to reduced supersymmetry. It is found that only the translations fulfill this requirement among the subgroups of the Poincaré group {\((P(3,1))\). We also give a condition for the subgroups of $C(3,1)$. In the Appendix, a reduction of a supersymmetric Yang-Mills system without residual supersymmetry is discussed.

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we find that the condition 

for all \( \phi \in C \) and \( \gamma \in R \)

In addition, the invariance condition:

\[ (x) \phi = (x) \phi \gamma \]

and

\[ (x) \gamma = (x) \phi \gamma \]

are the compatible invariance conditions and the compatible transformations.

The invariance condition and the compatible transformation (\( \phi, \gamma \)) stands for the basic representation (\( \phi, \gamma \)) where \( \phi \) and \( \gamma \) are the reduced representations.

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The invariance condition and the compatible transformation (\( \phi, \gamma \)) stands for the basic representation (\( \phi, \gamma \)) where \( \phi \) and \( \gamma \) are the reduced representations.
Moreover, the invariance condition for $\phi$, the transformation (3.3) and the condition (3.7) imply that:
\[
\phi^g(z) \left( I_2 - D(1) \langle \tau J(x) \rangle \right) = 0 ,
\]
for all $g \in G$. Assuming a non-trivial invariant scalar field $\phi$, we deduce from (3.10) that the spinor representation of the Jacobian must be the identity at any point of the stratum. Although the invariance group of the model is respectively the Poincaré group or the conformal group for massive or massless fields, only the translations (neglecting the discrete subgroups) satisfy the above constraint. These subgroups also ensure the compatibility between the conditions (3.8) and (3.9). Consequently, the translations are the only (continuous) space-time transformations allowing reductions of the Wess-Zumino model with residual supersymmetry. For example, the invariance under two-dimensional space translations yields a supersymmetric system composed of a non-interacting Dirac spinor field with a scalar field in 1 + 1 dimensions. The residual supersymmetry transformations are easily obtained by substituting the invariant fields in the formulas (3.2) and (3.3).

IV. SIMPLE SUPERSYMMETRIC YANG-MILLS SYSTEMS

A simple ($N = 1$) supersymmetric Yang-Mills theory (on-shell) in 3 + 1 dimensions consists of a Yang-Mills field minimally coupled to a multiplet of anticommuting Majorana spinor fields transforming under the adjoint representation of the gauge group. Its Lagrangian density, expressed in the Weyl spinor notation (see Ref. [14]), is:
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \frac{i}{2} \bar{\psi} \gamma^\mu (D_\mu \psi) \psi + \bar{\psi} \gamma^\mu (D_\mu \psi) \psi ,
\]
where $F_{\mu\nu}$ represents the field strength corresponding to the gauge field $\omega_\mu$, $D_\mu$ the covariant derivative with respect to the gauge field, and $\psi$ the $(\frac{1}{2}, 0)$ Weyl component of the Majorana spinor field. The corresponding action is left invariant under the following (on-shell) supersymmetry transformations, which form a representation of the super Poincaré algebra:
\[
\delta \omega_\mu^a = \bar{\xi} \gamma_\mu \psi^a = i \left( \bar{\psi} \gamma_\mu \psi^a + \bar{\psi} \gamma_\mu \psi^a \right) ,
\]
and
\[
\delta \psi^a = -\frac{1}{2} F^{a\mu\nu} \sigma_{\mu\nu} \xi ,
\]
where $\epsilon$ and $\xi$ are the $SL(2, C)$ components of the anticommuting Majorana four-spinor parameter $\xi$, and $\psi$ is the four-component Majorana spinor field with values in the Lie algebra of the gauge group. (Use of the four-spinor notation will sometimes be made for brevity.)

As in section III, a reduced supersymmetry is ensured if the invariance conditions are also imposed on the gauge and spinor fields coming from any supersymmetry transformations (4.2) and (4.3). This means that:
\[
(f^g \omega^g)(x) = A d \rho^{-1}(g, x) \omega^g(x) + \rho^{-1}(g, x) d\omega(x) ,
\]
and
\[
\psi^g(x) = D(1) \langle \tau J(x) \rangle \otimes A d \rho^{-1}(g, x) \psi^g(x) ,
\]
where $\omega^g = \omega + \delta \omega$ and $\psi^g = \psi + \delta \psi$. Let us remark that a supersymmetric residual system is also produced if the fields $\omega^g$ and $\psi^g$ are gauge equivalent to invariant fields. We discuss this case at the end of this section.

From the equations (4.4) and (4.5) and the invariance conditions for the gauge field ($\omega$) and the spinor field ($\psi$), we get:
\[
(f^g \delta \omega)(x) = A d \rho^{-1}(g, x) \delta \omega(x) ,
\]
and
\[
(f^g \delta \psi)(x) = D(1) \langle \tau J(x) \rangle \otimes A d \rho^{-1}(g, x) \delta \psi(x) .
\]
Since these conditions hold for every Majorana spinor $\xi$, it follows that:
\[
\delta \omega^g(\phi) = A d \rho^{-1}(g, x) \delta \omega(x) ,
\]
and
\[
\delta \psi^g(\phi) = D(1) \langle \tau J(x) \rangle A d \rho^{-1}(g, x) \delta \psi(x) .
\]
At the isotropy point $x_0$, the above expressions simplify to:
\[
\delta \omega^g(\phi) = A d \rho^{-1}(g, x_0) \delta \omega(x_0) ,
\]
and
\[
\delta \psi^g(\phi) = D(1) \langle \tau J(x_0) \rangle A d \rho^{-1}(g, x_0) \delta \psi(x_0) .
\]
where $\rho_0 \in G_0$. Substituting, respectively, the isotropy conditions for the spinor field and the field strength in the equations (4.10) and (4.11), we find the constraints:
\[
[\sigma^{\mu\nu} \partial_\mu J(x_0) \rho_0^\nu - \partial_\nu D^{-1}(\rho_0 J(x_0))] \psi(x_0) = 0 ,
\]
and
\[
[\sigma^{\mu\nu} D(\rho_0 J(x_0)) \sigma^{\rho\sigma} \partial_\rho J(x_0) \rho_0^\sigma \rho_0^\nu] F_{\mu\nu}(x_0) = 0 ,
\]
where $\psi(x_0)$ and $\psi(x_0)$ are assumed to be non-trivial.

Next we look at the symmetry group representations of $P(3, 1)$ obeying to (4.8) and (4.9) with the invariant fields $\phi$ and $F_{\mu\nu}$. Choosing an orthonormal frame on Minkowski space, the Jacobian matrix $\rho_0 J(x_0)$ induces a homomorphism of the isotropy subgroup $G_0$ into the Lorentz group $SO(3, 1)$. Therefore, the equation (4.12) can be written as:
\[
[D(1) (\rho_A^{-1})] (\rho_0 J(x_0)) \rho_A^{-1}(x_0) \psi(x_0) = 0 ,
\]
with $\rho_A \in SO(3, 1)$. However, no one-parameter subgroup of $SO(3, 1)$ satisfies (4.14). Since the condition (4.14) is preserved by conjugations under $SO(3, 1)$, this can be shown by considering only the three one-parameter subgroups of $SO(3, 1)$ up to conjugations by $SO(3, 1)$: $I_3$, $K_3$, and $I_3 + K_3$, where $I_3$ stands for the generator of the rotations around the $x^3$-axis, and $K_3$ for the generator of the boosts along
SUMMARY

The model of the stochastic process of the output of the system is described in the paper. The model is a continuous-time, continuous-state Markov process. The state of the system at any time is described by a function of the system's history. The model is used to study the behavior of the system over time.

The model is defined by the following equations:

\[ P(X_t = x | X_s = y) = \begin{cases} \pi(x, y) & \text{if } s < t \leq y, \\ 0 & \text{otherwise} \end{cases} \]

where \(\pi(x, y)\) is the transition probability from state \(x\) to state \(y\) at time \(t\).

The model is used to study the behavior of the system over time. The model is used to study the behavior of the system over time.
supersymmetry to be equivalent up to a gauge transformation to a field of the set of invariant fields under the symmetry group chosen. Finally, the $SO(3)$ invariant gauge and spinor-fields of the $N = 1$ supersymmetric Yang-Mills system with gauge group $SU(3)$ are discussed in the Appendix as an example of a reduction which leaves no supersymmetry. In future work, it would be interesting to study the symmetry reduction of higher dimensional supersymmetric systems to four-dimensional supersymmetric models. For instance, this method can be applied to derive the $SO(8)$ supergravity model from simple supergravity in 11 dimensions. Further results have been published recently by N. S. Manton who considered non-trivial coset spaces as extra-dimensional compact manifolds.

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REFERENCES


APPENDIX

SO(3) INVARIANCE OF THE SU(3) ON-SHELL SIMPLE SUPER-YANG-MILLS MULTIPLE

In the following, we give an example of a reduction leading to a residual system which is not supersymmetric. For simplicity, we look at the $N = 1$ supersymmetric Yang-Mills system with gauge group $SU(3)$ reduced by symmetry $SO(3)$. The $SO(3)$ reduction of the same model with gauge group $SU(2)$ leaves only a Yang-Mills field and a multiplet of scalar fields. We use for that purpose the Gell-Mann representation of the generators $\lambda^a$ of $SU(3)$, with $a = 1, \ldots, 8$, and their adjoint representation:

\[(\lambda^a)^b_c = -\langle f_{ab} \rangle c^e, (a, b, c = 1, \ldots, 8),\]

where the quantities $f_{ab}^c$ are the structure constants of the Lie algebra $su(3)$. The orbits generated under the action of $SO(3)$ are two-dimensional spheres which form a stratum parametrized by the time $t \in \mathbb{R}$ and the radius $r \in \mathbb{R}^+$ of the spheres, leaving the origin as a singular orbit.

The invariant gauge field is obtained either by the invariance condition (2.3) or equivalently by the theorem 2 of Ref. [5]. In the former case, the transformation function can be written as an extension of the homomorphism $\lambda$ of the isotropy group $SO(2)$ into $SU(3)$. For any isotropy point $x_0$ along the $z$-axis, except the origin, the isotropy subgroup consists of the rotations $\varepsilon^z r_0 z$. Accordingly, we choose the following homomorphism (up to a $Z_3$ factor) of $SO(3)$ for the transformation function ($\rho^{-1}$):

\[\rho^{-1} : g = e^{t\phi \varepsilon^z} e^{\lambda \chi z} \in SO(3) \rightarrow \begin{pmatrix} e^{-\varepsilon^z t} & 0 \\ 0 & 1 \end{pmatrix} \in SU(2) \subset SU(3),\]

(A.1)

where $\phi, \theta$ and $\chi$ are the rotation angles, $\{r_0\}$, the generators of $SO(3)$, and $\{t_0 \equiv \frac{\chi}{\pi}\}$, the generators of $SU(2)$. Let us mention that the isomorphism of $SO(3)$ onto an $SO(3)$ subgroup of $SU(3)$ implies a zero invariant spinor field and that the trivial homomorphism ($\lambda(G_0) = \{e\}$) permits only a pure (zero curvature) invariant Yang-Mills field. (As for the other homomorphisms $\lambda$, they can not be extended to global homomorphisms ($\rho^{-1}$) including the origin, and we ignore them.)

The isotropy condition for the gauge field (2.4) is explicitly:

\[\omega(x_0) = Ad(\lambda(g_0))\omega(x_0),\]

(A.2)

for every $g_0 \in G_0$. Its solution at the isotropy point $x_0 = (0, 0, 0, z \neq 0)$ is given in matrix form by

\[\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & E \\ A & B & 0 & 0 & 0 & 0 \\ -B & A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & F \end{pmatrix},\]

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