Appendix A
Dirac Delta Function

In 1880 the self-taught electrical scientist Oliver Heaviside introduced the following function

\[ \Theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases} \]  
(A.1)

which is now called Heaviside step function. This is a discontinuous function, with a discontinuity of first kind (jump) at \( x = 0 \), which is often used in the context of the analysis of electric signals. Moreover, it is important to stress that the Heaviside step function appears also in the context of quantum statistical physics. In fact, the Fermi-Dirac function (or Fermi-Dirac distribution)

\[ F_\beta(x) = \frac{1}{e^{\beta x} + 1}, \]  
(A.2)

proposed in 1926 by Enrico Fermi and Paul Dirac to describe the quantum statistical distribution of electrons in metals, where \( \beta = 1/(k_B T) \) is the inverse of the absolute temperature \( T \) (with \( k_B \) the Boltzmann constant) and \( x = \epsilon - \mu \) is the energy \( \epsilon \) of the electron with respect to the chemical potential \( \mu \), becomes the function \( \Theta(-x) \) in the limit of very small temperature \( T \), namely

\[ \lim_{\beta \to +\infty} F_\beta(x) = \Theta(-x) = \begin{cases} 0 & \text{for } x > 0 \\ 1 & \text{for } x < 0 \end{cases}. \]  
(A.3)

Inspired by the work of Heaviside, with the purpose of describing an extremely localized charge density, in 1930 Paul Dirac investigated the following “function”

\[ \delta(x) = \begin{cases} +\infty & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases} \]  
(A.4)
imposing that
\[ \int_{-\infty}^{+\infty} \delta(x) \, dx = 1. \quad (A.5) \]

Unfortunately, this property of \( \delta(x) \) is not compatible with the definition \((A.4)\). In fact, from Eq. \((A.4)\) it follows that the integral must be equal to zero. In other words, it does not exist a function \( \delta(x) \) which satisfies both Eqs. \((A.4)\) and \((A.5)\). Dirac suggested that a way to circumvent this problem is to interpret the integral of Eq. \((A.5)\) as
\[ \int_{-\infty}^{+\infty} \delta(x) \, dx = \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \delta_\epsilon(x) \, dx, \quad (A.6) \]

where \( \delta_\epsilon(x) \) is a generic function of both \( x \) and \( \epsilon \) such that
\[ \lim_{\epsilon \to 0^+} \delta_\epsilon(x) = \begin{cases} +\infty & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases}, \quad (A.7) \]
\[ \int_{-\infty}^{+\infty} \delta_\epsilon(x) \, dx = 1. \quad (A.8) \]

Thus, the Dirac delta function \( \delta(x) \) is a “generalized function” (but, strictly-speaking, not a function) which satisfy Eqs. \((A.4)\) and \((A.5)\) with the caveat that the integral in Eq. \((A.5)\) must be interpreted according to Eq. \((A.6)\) where the functions \( \delta_\epsilon(x) \) satisfy Eqs. \((A.7)\) and \((A.8)\).

There are infinite functions \( \delta_\epsilon(x) \) which satisfy Eqs. \((A.7)\) and \((A.8)\). Among them there is, for instance, the following Gaussian
\[ \delta_\epsilon(x) = \frac{1}{\epsilon \sqrt{\pi}} e^{-x^2/\epsilon^2}, \quad (A.9) \]
which clearly satisfies Eq. \((A.7)\) and whose integral is equal to 1 for any value of \( \epsilon \). Another example is the function
\[ \delta_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} & \text{for } |x| \leq \epsilon/2 \\ 0 & \text{for } |x| > \epsilon/2 \end{cases}, \quad (A.10) \]
which again satisfies Eq. \((A.7)\) and whose integral is equal to 1 for any value of \( \epsilon \). In the following we shall use Eq. \((A.10)\) to study the properties of the Dirac delta function.

According to the approach of Dirac, the integral involving \( \delta(x) \) must be interpreted as the limit of the corresponding integral involving \( \delta_\epsilon(x) \), namely
\[ \int_{-\infty}^{+\infty} \delta(x) \, f(x) \, dx = \lim_{\epsilon \to 0^+} \int_{-\infty}^{+\infty} \delta_\epsilon(x) \, f(x) \, dx, \quad (A.11) \]
for any function \( f(x) \). It is then easy to prove that
\[
\int_{-\infty}^{+\infty} \delta(x) f(x) \, dx = f(0).
\] (A.12)

by using Eq. (A.10) and the mean value theorem. Similarly one finds
\[
\int_{-\infty}^{+\infty} \delta(x-c) f(x) \, dx = f(c).
\] (A.13)

Several other properties of the Dirac delta function \( \delta(x) \) follow from its definition. In particular
\[
\delta(-x) = \delta(x),
\] (A.14)
\[
\delta(ax) = \frac{1}{|a|} \delta(x) \quad \text{with} \ a \neq 0,
\] (A.15)
\[
\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|} \delta(x-x_i) \quad \text{with} \ f(x_i) = 0.
\] (A.16)

Up to now we have considered the Dirac delta function \( \delta(x) \) with only one variable \( x \). It is not difficult to define a Dirac delta function \( \delta^{(D)}(\mathbf{r}) \) in the case of a \( D \)-dimensional domain \( \mathbb{R}^D \), where \( \mathbf{r} = (x_1, x_2, \ldots, x_D) \in \mathbb{R}^D \) is a \( D \)-dimensional vector:
\[
\delta^{(D)}(\mathbf{r}) = \begin{cases} +\infty & \text{for} \ \mathbf{r} = \mathbf{0} \\ 0 & \text{for} \ \mathbf{r} \neq \mathbf{0} \end{cases}
\] (A.17)

and
\[
\int_{\mathbb{R}^D} \delta^{(D)}(\mathbf{r}) \, d^D \mathbf{r} = 1.
\] (A.18)

Notice that sometimes \( \delta^{(D)}(\mathbf{r}) \) is written using the simpler notation \( \delta(\mathbf{r}) \). Clearly, also in this case one must interpret the integral of Eq. (A.18) as
\[
\int_{\mathbb{R}^D} \delta^{(D)}(\mathbf{r}) \, d^D \mathbf{r} = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^D} \delta^{(D)}_{\epsilon}(\mathbf{r}) \, d^D \mathbf{r},
\] (A.19)

where \( \delta^{(D)}_{\epsilon}(\mathbf{r}) \) is a generic function of both \( \mathbf{r} \) and \( \epsilon \) such that
\[
\lim_{\epsilon \to 0^+} \delta^{(D)}_{\epsilon}(\mathbf{r}) = \begin{cases} +\infty & \text{for} \ \mathbf{r} = \mathbf{0} \\ 0 & \text{for} \ \mathbf{r} \neq \mathbf{0} \end{cases},
\] (A.20)
\[
\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^D} \delta^{(D)}_{\epsilon}(\mathbf{r}) \, d^D \mathbf{r} = 1.
\] (A.21)
Several properties of $\delta(x)$ remain valid also for $\delta^{(D)}(\mathbf{r})$. Nevertheless, some properties of $\delta^{(D)}(\mathbf{r})$ depend on the space dimension $D$. For instance, one can prove the remarkable formula

$$
\delta^{(D)}(\mathbf{r}) = \begin{cases} 
\frac{1}{2\pi} \nabla^2 (\ln |\mathbf{r}|) & \text{for } D = 2 \\
-\frac{1}{D(D-2)V_D} \nabla^2 \left( \frac{1}{|\mathbf{r}|^{D-2}} \right) & \text{for } D \geq 3 
\end{cases}
$$

(A.22)

where $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_D^2}$ and $V_D = \pi^{D/2} / \Gamma(1 + D/2)$ is the volume of a $D$-dimensional hypersphere of unitary radius, with $\Gamma(x)$ the Euler Gamma function. In the case $D = 3$ the previous formula becomes

$$
\delta^{(3)}(\mathbf{r}) = -\frac{1}{4\pi} \nabla^2 \left( \frac{1}{|\mathbf{r}|} \right),
$$

(A.23)

which can be used to transform the Gauss law of electromagnetism from its integral form to its differential form.
Appendix B
Fourier Transform

It was known from the times of Archimedes that, in some cases, the infinite sum of decreasing numbers can produce a finite result. But it was only in 1593 that the mathematician Francois Viete gave the first example of a function, \( f(x) = 1/(1-x) \), written as the infinite sum of power functions. This function is nothing else than the geometric series, given by

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1. \tag{B.1}
\]

In 1714 Brook Taylor suggested that any real function \( f(x) \) which is infinitely differentiable in \( x_0 \) and sufficiently regular can be written as a series of powers, i.e.

\[
f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n, \tag{B.2}
\]

where the coefficients \( c_n \) are given by

\[
c_n = \frac{1}{n!} f^{(n)}(x_0), \tag{B.3}
\]

with \( f^{(n)}(x) \) the \( n \)-th derivative of the function \( f(x) \). The series (B.2) is now called Taylor series and becomes the so-called Maclaurin series if \( x_0 = 0 \). Clearly, the geometric series (B.1) is nothing else than the Maclaurin series, where \( c_n = 1 \). We observe that it is quite easy to prove the Taylor series: it is sufficient to suppose that Eq. (B.2) is valid and then to derive the coefficients \( c_n \) by calculating the derivatives of \( f(x) \) at \( x = x_0 \); in this way one gets Eq. (B.3).

In 1807 Jean Baptiste Joseph Fourier, who was interested on wave propagation and periodic phenomena, found that any sufficiently regular real function function \( f(x) \) which is periodic, i.e. such that
where $L$ is the periodicity, can be written as the infinite sum of sinusoidal functions, namely

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{2\pi}{L} x \right) + b_n \sin \left( \frac{2\pi}{L} x \right) \right], \quad (B.5)$$

where

$$a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(y) \cos \left( \frac{2\pi}{L} y \right) \, dy, \quad (B.6)$$

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(y) \sin \left( \frac{2\pi}{L} y \right) \, dy. \quad (B.7)$$

It is quite easy to prove also the series (B.5), which is now called Fourier series. In fact, it is sufficient to suppose that Eq. (B.5) is valid and then to derive the coefficients $a_n$ and $b_n$ by multiplying both sides of Eq. (B.5) by $\cos \left( \frac{2\pi}{L} x \right)$ and $\cos \left( \frac{2\pi}{L} x \right)$ respectively and integrating over one period $L$; in this way one gets Eqs. (B.6) and (B.7).

It is important to stress that, in general, the real variable $x$ of the function $f(x)$ can represent a space coordinate but also a time coordinate. In the former case $L$ gives the spatial periodicity and $2\pi/L$ is the wavenumber, while in the latter case $L$ is the time periodicity and $2\pi/L$ the angular frequency.

Taking into account the Euler formula

$$e^{i n \frac{2\pi}{T} x} = \cos \left( n \frac{2\pi}{L} x \right) + i \sin \left( n \frac{2\pi}{L} x \right) \quad (B.8)$$

with $i = \sqrt{-1}$ the imaginary unit, Fourier observed that his series (B.5) can be re-written in the very elegant form

$$f(x) = \sum_{n=-\infty}^{+\infty} f_n e^{i n \frac{2\pi}{T} x}, \quad (B.9)$$

where

$$f_n = \frac{1}{L} \int_{-L/2}^{L/2} f(y) e^{-i n \frac{2\pi}{T} y} \, dy \quad (B.10)$$

are complex coefficients, with $f_0 = a_0/2$, $f_n = (a_n - ib_n)/2$ if $n > 0$ and $f_n = (a_{-n} + ib_{-n})/2$ if $n < 0$, thus $f_n^* = f_{-n}$.

The complex representation (B.9) suggests that the function $f(x)$ can be periodic but complex, i.e. such that $f: \mathbb{R} \to \mathbb{C}$. Moreover, one can consider the limit $L \to +\infty$ of infinite periodicity, i.e. a function which is not periodic. In this limit
Eq. (B.9) becomes the so-called Fourier integral (or Fourier anti-transform)
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(k) e^{ikx} \, dk \]  
(B.11)

with
\[ \tilde{f}(k) = \int_{-\infty}^{+\infty} f(y) e^{-iky} \, dy \]  
(B.12)

the Fourier transform of \( f(x) \). To prove Eqs. (B.11) and (B.12) we write Eq. (B.9) taking into account Eq. (B.10) and we find
\[ f(x) = \sum_{n=-\infty}^{+\infty} \left( \frac{1}{L} \int_{-L/2}^{L/2} f(y) e^{-i\frac{2\pi}{L} y} \, dy \right) e^{in\frac{2\pi}{L} x}. \]  
(B.13)

Setting
\[ k_n = n \frac{2\pi}{L} \quad \text{and} \quad \Delta k = k_{n+1} - k_n = \frac{2\pi}{L} \]  
(B.14)

the previous expression of \( f(x) \) becomes
\[ f(x) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left( \int_{-L/2}^{L/2} f(y) e^{-i\frac{2\pi}{L} y} \, dy \right) e^{i\frac{2\pi}{L} x} \Delta k. \]  
(B.15)

In the limit \( L \to +\infty \) one has \( \Delta k \to dk \), \( k_n \to k \) and consequently
\[ f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{+\infty} f(y) e^{-iky} \, dy \right) e^{ikx} \, dk, \]  
(B.16)

which gives exactly Eqs. (B.11) and (B.12). Note, however, that one gets the same result (B.16) if the Fourier integral and its Fourier transform are defined multiplying them respectively with a generic constant and its inverse. Thus, we have found that any sufficiently regular complex function \( f(x) \) of real variable \( x \) which is globally integrable, i.e. such that
\[ \int_{-\infty}^{+\infty} |f(x)| \, dx < +\infty, \]  
(B.17)

can be considered as the (infinite) superposition of complex monocromatic waves \( e^{ikx} \). The amplitude \( \tilde{f}(k) \) of the monocromatic wave \( e^{ikx} \) is the Fourier transform of \( f(x) \).
The Fourier transform $\tilde{f}(k)$ of a function $f(x)$ is sometimes denoted as $\mathcal{F}[f(x)](k)$, namely
\[
\tilde{f}(k) = \mathcal{F}[f(x)](k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx.
\] (B.18)

The Fourier transform $\mathcal{F}[f(x)](k)$ has many interesting properties. For instance, due to the linearity of the integral the Fourier transform is clearly a linear map:
\[
\mathcal{F}[a \, f(x) + b \, g(x)](k) = a \, \mathcal{F}[f(x)](k) + b \, \mathcal{F}[g(x)](k).
\] (B.19)

Moreover, one finds immediately that
\[
\mathcal{F}[f(x - a)](k) = e^{-ika} \mathcal{F}[f(x)](k),
\] (B.20)
\[
\mathcal{F}[e^{ik_0 x} f(x)](k) = \mathcal{F}[f(x)](k - k_0).
\] (B.21)
\[
\mathcal{F}[x \, f(x)](k) = i \, \tilde{f}'(k),
\] (B.22)
\[
\mathcal{F}[f^{(n)}(x)](k) = (ik)^n \tilde{f}(k),
\] (B.23)

where $f^{(n)}(x)$ is the $n$-th derivative of $f(x)$ with respect to $x$.

In the Table we report the Fourier transforms $\mathcal{F}[f(x)](k)$ of some elementary functions $f(x)$, including the Dirac delta function $\delta(x)$ and the Heaviside step function $\Theta(x)$. We insert also the sign function $sgn(x)$ defined as: $sgn(x) = 1$ for $x > 0$ and $sgn(x) = -1$ for $x < 0$. The table of Fourier transforms clearly shows that the Fourier transform localizes functions which is delocalized, while it delocalizes functions which are localized. In fact, the Fourier transform of a constant is a Dirac delta function while the Fourier transform of a Dirac delta function is a constant. In general, it holds the following uncertainty theorem

<table>
<thead>
<tr>
<th>$f(x)$</th>
<th>$\mathcal{F}<a href="k">f(x)</a>$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>$2\pi \delta(k)$</td>
</tr>
<tr>
<td>$\delta(x)$</td>
<td>1</td>
</tr>
<tr>
<td>$\Theta(x)$</td>
<td>$\frac{1}{i k} + \pi \delta(k)$</td>
</tr>
<tr>
<td>$e^{ik_0 x}$</td>
<td>$2\pi \delta(k - k_0)$</td>
</tr>
<tr>
<td>$e^{-x^2/(2a^2)}$</td>
<td>$a\sqrt{2\pi}e^{-a^2k^2/2}$</td>
</tr>
<tr>
<td>$e^{-a</td>
<td>x</td>
</tr>
<tr>
<td>$sgn(x)$</td>
<td>$\frac{i}{k}$</td>
</tr>
<tr>
<td>$\sin(k_0 x)$</td>
<td>$\pi/2 [\delta(k - k_0) - \delta(k + k_0)]$</td>
</tr>
<tr>
<td>$\cos(k_0 x)$</td>
<td>$\pi [\delta(k - k_0) + \delta(k + k_0)]$</td>
</tr>
</tbody>
</table>

Table: Fourier transforms $\mathcal{F}[f(x)](k)$ of simple functions $f(x)$, where $\delta(x)$ is the Dirac delta function, $sgn(x)$ is the sign function, and $\Theta(x)$ is the Heaviside step function.
\[ \Delta x \Delta k \geq \frac{1}{2}, \]  
(B.24)

where
\[ \Delta x^2 = \int_{-\infty}^{\infty} x^2 |f(x)|^2 \, dx - \left( \int_{-\infty}^{\infty} x |f(x)|^2 \, dx \right)^2 \]  
(B.25)

and
\[ \Delta k^2 = \int_{-\infty}^{\infty} k^2 |\tilde{f}(k)|^2 \, dk - \left( \int_{-\infty}^{\infty} k |\tilde{f}(k)|^2 \, dk \right)^2 \]  
(B.26)

are the spreads of the wavepackets respectively in the space \( x \) and in the dual space \( k \). This theorem is nothing else than the uncertainty principle of quantum mechanics formulated by Werner Heisenberg in 1927, where \( x \) is the position and \( k \) is the wavenumber. Another interesting and intuitive relationship is the Parseval identity, given by
\[ \int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 \, dk. \]  
(B.27)

It is important to stress that the power series, the Fourier series, and the Fourier integral are special cases of the quite general expansion
\[ f(x) = \sum_{\alpha} f_{\alpha} \phi_{\alpha}(x) \, d\alpha \]  
(B.28)

of a generic function \( f(x) \) in terms of a set \( \phi_{\alpha}(x) \) of basis functions spanned by the parameter \( \alpha \), which can be a discrete or a continuous variable. A large part of modern mathematical analysis is devoted to the study of Eq. (B.28) and its generalization.

The Fourier transform is often used in electronics. In that field of research the signal of amplitude \( f \) depends on time \( t \), i.e. \( f = f(t) \). In this case the dual variable of time \( t \) is the frequency \( \omega \) and the fourier integral is usually written as
\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(\omega) e^{-i\omega t} \, dk \]  
(B.29)

with
\[ \tilde{f}(\omega) = \mathcal{F}[f(t)](\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} \, dt \]  
(B.30)

the Fourier transform of \( f(t) \). Clearly, the function \( f(t) \) can be seen as the Fourier anti-transform of \( \tilde{f}(\omega) \), in symbols
\[ f(t) = \mathcal{F}^{-1}[\tilde{f}(\omega)](t) = \mathcal{F}^{-1}[\mathcal{F}[f(t)](\omega)](t), \]  
(B.31)
which obviously means that the composition $F^{-1} \circ F$ gives the identity.

More generally, if the signal $f$ depends on the 3 spatial coordinates $\mathbf{r} = (x, y, z)$ and time $t$, i.e. $f = f(\mathbf{r}, t)$, one can introduce Fourier transforms from $\mathbf{r}$ to $\mathbf{k}$, from $t$ to $\omega$, or both. In this latter case one obviously obtains

$$f(\mathbf{r}, t) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} \tilde{f}(\mathbf{k}, \omega) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \, d^3k \, d\omega$$  \hspace{1cm} (B.32)

with

$$\tilde{f}(\mathbf{k}, \omega) = \mathcal{F}[f(\mathbf{r}, t)](\mathbf{k}, \omega) = \int_{\mathbb{R}^4} f(\mathbf{r}, t) e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \, d^3r \, dt.$$  \hspace{1cm} (B.33)

Also in this general case the function $f(\mathbf{r}, t)$ can be seen as the Fourier anti-transform of $\tilde{f}(\mathbf{k}, \omega)$, in symbols

$$f(\mathbf{r}, t) = \mathcal{F}^{-1}[\tilde{f}(\mathbf{k}, \omega)](\mathbf{r}, t) = \mathcal{F}^{-1}[\mathcal{F}[f(\mathbf{r}, t)](\mathbf{k}, \omega)](\mathbf{r}, t).$$  \hspace{1cm} (B.34)
Appendix C

Laplace Transform

The Laplace transform is an integral transformation, similar but not equal to the Fourier transform, introduced in 1737 by Leonard Euler and independently in 1782 by Pierre-Simon de Laplace. Nowadays the Laplace transform is mainly used to solve non-homogeneous ordinary differential equations with constant coefficients.

Given a sufficiently regular function $f(t)$ of time $t$, the Laplace transform of $f(t)$ is the function $F(s)$ such that

$$F(s) = \int_{0}^{\infty} f(t) e^{-st} dt,$$  \hspace{1cm} (C.1)

where $s$ is a complex number. Usually the integral converges if the real part $\text{Re}(s)$ of the complex number $s$ is greater than critical real number $x_c$, which is called abscissa of convergence and unfortunately depends on $f(t)$. The Laplace transform $F(s)$ of a function $f(t)$ is sometimes denoted as $\mathcal{L}[f(t)](s)$, namely

$$F(s) = \mathcal{L}[f(t)](s) = \int_{0}^{\infty} f(t) e^{-st} dt.$$  \hspace{1cm} (C.2)

For the sake of completeness and clarity, we write also the Fourier transform $\tilde{f}(\omega)$, denoted as $\mathcal{F}[f(t)](\omega)$, of the same function

$$\tilde{f}(\omega) = \mathcal{F}[f(t)](\omega) = \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt.$$  \hspace{1cm} (C.3)

First of all we notice that the Laplace transform depends on the behavior of $f(t)$ for non-negative values of $t$, while the Fourier transform depends also on the behavior of $f(t)$ for negative values of $t$. This is however not a big problem, because we can set $f(t) = 0$ for $t < 0$ (or equivalently we can multiply $f(t)$ by the Heaviside step function $\Theta(t)$), and then the Fourier transform becomes
\[ \tilde{f}(\omega) = \mathcal{F}[f(t)](\omega) = \int_{0}^{\infty} f(t) e^{i\omega t} \, dt. \]  

Moreover, it is important to observe that, comparing Eq. (C.2) with Eq. (C.4), if both \( \mathcal{F} \) and \( \mathcal{L} \) of \( f(t) \) exist, we obtain

\[ F(s) = \mathcal{L}[f(t)](s) = \mathcal{F}[f(t)](is) = \tilde{f}(is), \]  

or equivalently

\[ \tilde{f}(\omega) = \mathcal{F}[f(t)](\omega) = \mathcal{L}[f(t)](-i\omega) = F(-i\omega). \]  

Remember that \( \omega \) is a real variable while \( s \) is a complex variable. Thus we have found that for a generic function \( f(t) \), such that \( f(t) = 0 \) for \( t < 0 \), the Laplace transform \( F(s) \) and the Fourier transform \( \tilde{f}(\omega) \) are simply related to each other.

In the following Table we report the Laplace transforms \( \mathcal{L}[f(t)](s) \) of some elementary functions \( f(t) \), including the Dirac delta function \( \delta(t) \) and the Heaviside step function \( \Theta(t) \), forgetting about the possible problems of regularity.

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( \mathcal{L}<a href="s">f(t)</a> )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>( \delta(t - \tau) )</td>
<td>( e^{-\tau s} )</td>
</tr>
<tr>
<td>( \Theta(t - \tau) )</td>
<td>( \frac{e^{-\tau s}}{s} )</td>
</tr>
<tr>
<td>( t^n )</td>
<td>( \frac{n!}{s^{n+1}} )</td>
</tr>
<tr>
<td>( e^{-at} )</td>
<td>( \frac{1}{s+a} )</td>
</tr>
<tr>
<td>( e^{-a</td>
<td>t</td>
</tr>
<tr>
<td>( \sin(at) )</td>
<td>( \frac{a}{s^2+a^2} )</td>
</tr>
<tr>
<td>( \cos(at) )</td>
<td>( \frac{s}{s^2+a^2} )</td>
</tr>
</tbody>
</table>

**Table.** Laplace transforms \( \mathcal{L}[f(t)](s) \) of simple functions \( f(t) \), where \( \delta(t) \) is the Dirac delta function and \( \Theta(t) \) is the Heaviside step function, and \( \tau > 0 \) and \( n \) positive integer.

We now show that writing \( f(t) \) as the Fourier anti-transform of \( \tilde{f}(\omega) \) one can deduce the formula of the Laplace anti-transform of \( F(s) \). In fact, one has

\[ f(t) = \mathcal{F}^{-1}[\tilde{f}(\omega)](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-i\omega t} \, d\omega. \]  

Because \( \tilde{f}(\omega) = F(-i\omega) \) one finds also

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(-i\omega) e^{-i\omega t} \, d\omega. \]
Using \( s = -i\omega \) as integration variable, this integral representation becomes
\[
f(t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} F(s) e^{st} \, ds, \tag{C.9}
\]
where the integration is now a contour integral along any path \( \gamma \) in the complex plane of the variable \( s \), which starts at \( s = -i\infty \) and ends at \( s = i\infty \). What we have found is exactly the Laplace anti-transform of \( F(s) \), in symbols
\[
f(t) = \mathcal{L}^{-1} [F(s)](t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} F(s) e^{st} \, ds. \tag{C.10}
\]
Remember that this function is such that \( f(t) = 0 \) for \( t < 0 \). The fact that the function \( f(t) \) is the Laplace anti-transform of \( F(s) \) can be symbolized by
\[
f(t) = \mathcal{L}^{-1} [F(s)](t) = \mathcal{L}^{-1} [\mathcal{L}[f(t)](s)](t), \tag{C.11}
\]
which means that the composition \( \mathcal{L}^{-1} \circ \mathcal{L} \) gives the identity.

The Laplace transform \( \mathcal{L}[f(t)](s) \) has many interesting properties. For instance, due to the linearity of the integral the Laplace transform is clearly a linear map:
\[
\mathcal{L}[af(t) + bg(t)](s) = a\mathcal{L}[f(t)](s) + b\mathcal{L}[g(t)](s). \tag{C.12}
\]
Moreover, one finds immediately that
\[
\mathcal{L}[f(t-a)\Theta(t-a)](s) = e^{-as} \mathcal{L}[f(t)](s), \tag{C.13}
\]
\[
\mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f(t)](s-a). \tag{C.14}
\]

For the solution of non-homogeneous ordinary differential equations with constant coefficients, the most important property of the Laplace transform is the following
\[
\mathcal{L}[f^{(n)}(t)](s) = s^n \mathcal{L}[f(t)](s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0) \tag{C.15}
\]
where \( f^{(n)}(t) \) is the \( n \)-th derivative of \( f(t) \) with respect to \( t \). For instance, in the simple cases \( n = 1 \) and \( n = 2 \) one has
\[
\mathcal{L}[f'(t)](s) = s F(s) - f(0), \tag{C.16}
\]
\[
\mathcal{L}[f''(t)](s) = s^2 F(s) - sf'(0) - f(0), \tag{C.17}
\]
by using \( F(s) = \mathcal{L}[f(t)](s) \). The proof of Eq. (C.15) is straightforward performing integration by parts. Let us prove, for instance, Eq. (C.16):
We now give a simple example of the Laplace method to solve ordinary differential equations by considering the differential problem

\[ f'(t) + f(t) = 1 \quad \text{with} \quad f(0) = 2. \]  \hfill (C.19)

We apply the Laplace transform to both sides of the differential problem

\[ \mathcal{L}[f'(t) + f(t)](s) = \mathcal{L}[1](s) \]  \hfill (C.20)

obtaining

\[ sF(s) - 2 + F(s) = \frac{1}{s}. \]  \hfill (C.21)

This is now an algebraic problem with solution

\[ F(s) = \frac{1}{s(s + 1)} + \frac{2}{s + 1} = \frac{1}{s} - \frac{1}{s + 1} + \frac{2}{s + 1} = \frac{1}{s} + \frac{1}{s + 1}. \]  \hfill (C.22)

Finally, we apply the Laplace anti-transform

\[ f(t) = \mathcal{L}^{-1} \left[ \frac{1}{s} + \frac{1}{s + 1} \right](t) = \mathcal{L}^{-1} \left[ \frac{1}{s} \right](t) + \mathcal{L}^{-1} \left[ \frac{1}{s + 1} \right](t). \]  \hfill (C.23)

By using our Table of Laplace transforms we find immediately the solution

\[ f(t) = 1 + e^{-t}. \]  \hfill (C.24)

The Laplace transform can be used to solve also integral equations. In fact, one finds

\[ \mathcal{L} \left[ \int_{0}^{t} f(y) \, dy \right](s) = \frac{1}{s} F(s), \]  \hfill (C.25)

and more generally

\[ \mathcal{L} \left[ \int_{-\infty}^{t} f(y) g(t - y) \, dy \right](s) = \frac{I_0}{s} + F(s) G(s), \]  \hfill (C.26)
where
\[ I_0 = \int_{-\infty}^{0} f(y) g(t - y) \, dy. \] (C.27)

Notice that the integral which appears in Eq. (C.25) is, by definition, the convolution \((f \ast g)(t)\) of two functions, i.e.
\[ (f \ast g)(t) = \int_{0}^{t} f(y) g(t - y) \, dy. \] (C.28)
Bibliography

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