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TWO-LOOP MODULAR INVARIANCE AND SPIN-STATISTICS

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Abstract

Within the context of a class of fermionic closed string theories I explicitly show some relationships between two-loop modular invariance, factorization, correct spin-statistics and the existence of a massless gravity multiplet.

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The purpose of this letter is to elucidate the connection between two-loop modular invariance and a "physically sensible projection" that is used in [1] and [2] to classify ten and four dimensional closed fermionic string theories. I use the notation of [1] as far as possible. The relevant concepts have been discussed in [1-7] so I will not repeat them here.

Let us first consider models without world sheet supersymmetry. The conformal anomaly cancels if we have 26-d non-compact bosonic co-ordinates and 2d Majorana fermions on the world sheet. Let $i=1, \dots, 2d$ label the left moving fermions and $i=2d+1, \dots, 4d$ the right movers. The spin structure is labelled by a vector $\underline{a}=\{a_i\}$ for each of the a and b cycles on the world sheet. In this paper we only allow $a_i=0$ (this is the anti-periodic or Neveu-Schwarz sector if we are referring to a torus) or $a_i=1$ ("periodic Ramond sector"), and can work with $a_i \bmod 2$.

Following [1] the fermionic contribution to the one-loop partition function is

$$Z^{(1)} = \sum_{\underline{a}, \underline{b}} C_{\underline{b}}^{\underline{a}} Z_{\underline{b}}^{\underline{a}} = \sum_{a_i, b_i = 0 \text{ or } 1} C_{b_1, \dots}^{a_1, \dots} Z_{b_1}^{a_1} Z_{b_2}^{a_2} \dots \quad (1)$$

where $Z_{\underline{b}}^{\underline{a}}$ is the partition function for boundary conditions $\underline{a}$ in the σ direction and $\underline{b}$ in the τ direction. As in [1] we make the restriction that the net (left minus right) number of fermions with a given spin structure is a multiple of 8. The $C_{\underline{b}}^{\underline{a}}$ are some set of coefficients that are restricted by one loop modular invariance to satisfy

$$C_{\underline{b}}^{\underline{a}} = C_{\underline{a}}^{\underline{b}} = C_{\underline{b} + \underline{a} + \underline{1}}^{\underline{a}} \quad (2)$$

where $\underline{1}$ is the vector with all components set to one.

In fact in terms of theta functions [4,6,8], for the left movers we have

$$\left(Z_{\underline{b}_i}^{\underline{a}_i} \right)^8 = (-1)^{\underline{a}_i + \underline{b}_i} \bar{\theta}^4 \left[\begin{smallmatrix} \underline{a}_i \\ \underline{b}_i \end{smallmatrix} \right] (0|\tau) \quad (3)$$

with the complex conjugate for the right movers; spin structure independent terms have been ignored.

For a genus two world sheet the period matrix τ becomes a 2 by 2 complex symmetric matrix

$$\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} \quad (4)$$

However the spin structure dependence is still given by theta functions [6,9] and so we then have

$$Z_{\underline{b}^1 \underline{b}^2}^{\underline{a}^1 \underline{a}^2} = (-1)^{\frac{1}{8}(\underline{a}^1 + \underline{b}^1 + \underline{a}^2 + \underline{b}^2)} \prod_{i=1}^{2d} \bar{\theta}^{\frac{1}{2}} \left[\begin{smallmatrix} \underline{a}_i^1 & \underline{a}_i^2 \\ \underline{b}_i^1 & \underline{b}_i^2 \end{smallmatrix} \right] (0|\tau) \prod_{i=2d+1}^{4d} \theta^{\frac{1}{2}} \left[\begin{smallmatrix} \underline{a}_i^1 & \underline{a}_i^2 \\ \underline{b}_i^1 & \underline{b}_i^2 \end{smallmatrix} \right] (0|\tau) \quad (5)$$

where upper indices refer to cycles about the first and second loops. The factor of $1/8$ arises because we have Z^8 in (3). The dot product is Lorentzian in that it includes a minus sign for the left movers. The phase has been chosen so that as $\tau_{12} \rightarrow 0$ then

$$Z_{\underline{b}^1 \underline{b}^2}^{\underline{a}^1 \underline{a}^2}(\tau) \rightarrow Z_{\underline{b}^1}^{\underline{a}^1}(\tau_{11}) Z_{\underline{b}^2}^{\underline{a}^2}(\tau_{22}) \quad (6)$$

We write the two loop vacuum amplitude in the form

$$Z^{(2)} = \sum_{\underline{a}^1, \underline{b}^1, \underline{a}^2, \underline{b}^2} C_{\underline{b}^1 \underline{b}^2}^{\underline{a}^1 \underline{a}^2} Z_{\underline{b}^1 \underline{b}^2}^{\underline{a}^1 \underline{a}^2} \quad (7)$$

In the $\tau_{12} \rightarrow 0$ limit the leading term of $Z^{(2)}$ should split into a product of one loop amplitudes and a (divergent) propagator, (for an example of this see [10]), so

$$Z^{(2)} = \left(\sum_{\underline{a}^1, \underline{b}^1} C_{\underline{b}^1}^{\underline{a}^1} Z_{\underline{b}^1}^{\underline{a}^1} \right) \cdot \text{prop} \cdot \left(\sum_{\underline{a}^2, \underline{b}^2} C_{\underline{b}^2}^{\underline{a}^2} Z_{\underline{b}^2}^{\underline{a}^2} \right) \quad (8)$$

whence

$$C_{\underline{b}^1 \underline{b}^2}^{\underline{a}^1 \underline{a}^2} = C_{\underline{b}^1}^{\underline{a}^1} C_{\underline{b}^2}^{\underline{a}^2} \quad (9)$$

In general factorization may not be so straightforward, however all that is needed is that the spin structure dependence factorizes. So in this paper factorization will just mean that (9) holds.

We can now use two loop modular transformations to generate restrictions on the one loop coefficients $C_{\underline{b}}^{\underline{a}}$. Dehn twists about the a and b cycles give (2) again. For full modular invariance we also need to consider c twists [5,6] under which

$$\theta^4 \begin{bmatrix} \underline{a}^1 & \underline{a}^2 \\ \underline{b}^1 & \underline{b}^2 \end{bmatrix} \rightarrow (-1)^{\underline{a}^1 + \underline{a}^2} \theta^4 \begin{bmatrix} \underline{a}^1 & \underline{a}^2 \\ \underline{b}^1 + \underline{a}^1 + \underline{a}^2 + 1 & \underline{b}^2 + \underline{a}^1 + \underline{a}^2 + 1 \end{bmatrix} \quad (10)$$

This gives a restriction on the two loop and hence one loop coefficients which can be simplified using (2) to yield

$$C_{\underline{b}^1}^{\underline{a}^1} C_{\underline{b}^2}^{\underline{a}^2} = (-1)^{\frac{1}{8}(\underline{a}^1 + \underline{a}^2) \cdot 1} C_{\underline{b}^1 + \underline{a}^2}^{\underline{a}^1} C_{\underline{b}^2 + \underline{a}^1}^{\underline{a}^2} \quad (11)$$

If we demand factorization (9) and one loop modular invariance (2) then (11) is the necessary and sufficient condition for two loop modular invariance.

Let us now return to projection operators. For a state defined by boundary conditions $\underline{a}$ in the σ direction and fermion numbers $\underline{n}$, the coefficient $\eta_{\underline{n}}^{\underline{a}}$ of its contribution to $Z^{(1)}$ is given by summing over all possible τ boundary conditions

$$\eta_{\underline{a}}^{\underline{a}} = \sum_{\underline{b}} (-1)^{\frac{1}{8}(\underline{a}+\underline{b}) \cdot \underline{1} + \underline{b} \cdot \underline{a}} C_{\underline{b}}^{\underline{a}} \quad (12)$$

We have no space-time spinors so in this case a "physically sensible projection" means a state should either be projected out or should give a positive contribution i.e. after appropriate normalization we must have

$$\eta_{\underline{a}}^{\underline{a}} = 0 \text{ or } 1 \quad (13)$$

for all $\underline{a}$ and $\underline{a}$.

To show that (11) leads to (13) we first note that (2) and (11) give

$$C_{\underline{b}^1}^{\underline{a}} C_{\underline{b}^2}^{\underline{a}} = (-1)^{\frac{1}{8}(\underline{a}+\underline{b}^2) \cdot \underline{1}} C_{\underline{b}^1+\underline{b}^2}^{\underline{a}} C_{\underline{b}^2}^{\underline{0}} \quad (14)$$

using this one can easily show that

$$(\eta_{\underline{a}}^{\underline{a}})^2 = \eta_{\underline{a}}^{\underline{a}} \eta_{\underline{a}}^{\underline{a}} \quad (15)$$

so $\eta_{\underline{a}}^{\underline{a}} = 0$ or $\eta_{\underline{0}}^{\underline{0}}$.

We cannot take $\eta_{\underline{0}}^{\underline{0}}=0$ because this would leave us with no physical states. Thus we can always rescale the $C_{\underline{b}}^{\underline{a}}$ to obtain $\eta_{\underline{0}}^{\underline{0}}=1$, and hence (13).

It is interesting that $\eta_{\underline{0}}^{\underline{0}}$ refers to the pure Neveu-Schwarz sector with an even number of fermionic raising operators and so in particular includes $\alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |0\rangle_{\text{NS}}$ which is the gravity multiplet. In other words states are either projected out or have the same statistics as the graviton/dilaton.

To show that (13) implies (11) we consider the analog of equation (3.17) in [1] by which (13) implies

$$C_{\underline{b}}^{\underline{a}} = C_{\beta \cdot \underline{V}}^{\alpha \cdot \underline{V}} = 2^{-r} (-1)^{-\frac{1}{8}(\alpha+\beta) \cdot \underline{V} \cdot \underline{1}} (-1)^{\beta \cdot \gamma(\alpha)} \quad (16)$$

One loop invariance demands that $\beta.\gamma(\alpha)=\alpha.\gamma(\beta)$ and then (11) follows trivially.

In models with world sheet supersymmetry the immediate problem is that the Rarita-Schwinger determinant is not yet known in an explicit form except at one loop. However it is known [3] that the heterotic string in which all fermions have the same spin structure α (giving an $O(32)$ tachyonic theory) is modular invariant for arbitrary genus. Thus we write (c.f. [10])

$$Z_\alpha \approx \left(\det_\alpha \nabla_{1/2} \right)_L^{32} \frac{\left(\det_\alpha \nabla_{1/2} \right)_R^{10}}{\left(\det_\alpha \nabla_{3/2} \right)_R} \approx \bar{\theta}_{[\alpha]}^{16} \theta_{[\alpha]}^4 Y_\alpha \quad (17)$$

where $Y_\alpha = (\det_\alpha \nabla_{1/2})^2 (\det_\alpha \nabla_{3/2})^{-1}$ and ∇ is the appropriate kinetic operator for the "spin" 1/2 and 3/2 fermions. As usual I ignore α independent terms, supermoduli, etc. We are only interested in phases and these are always one for θ^{16} so we may as well ignore the left movers. At one loop we can take $Y_\alpha=1$ and can then form the usual modular covariant sum

$$\sum_{a,b} (-1)^{a+b} \theta^4 \begin{bmatrix} a \\ b \end{bmatrix} \quad (18)$$

If Y_α were absent then this would come from factorizing the two loop expression

$$\sum_{a^1, a^2, b^1, b^2} (-1)^{a^1+b^1+a^2+b^2} \theta^4 \begin{bmatrix} a^1 & a^2 \\ b^1 & b^2 \end{bmatrix} \quad (19)$$

but this is not covariant under the c twists of (10) and so it is clear that Y_α now plays a non-trivial role. In particular this explains why one cannot do a GSO projection on a set of eight internal (as opposed to space-time) fermions. Two dimensional supersymmetry suggests that the spin structures in Y_α transform identically to those of a theta function and so

this just leaves freedom in the phases induced by Dehn twists. Hence if Y_α should pick up a phase under c twists which cancels that of θ^4 , then (19) with θ^4 replaced by $\theta^4 Y$ is covariant. We are assuming that Y_α factorizes in the same fashion as a theta function. I conclude that world sheet supersymmetry just leads to an extra factor of $\exp \pi i(a^1 + a^2)$ in (11), where a^1 and a^2 refer to the world sheet fermions with Lorentz indices corresponding to the non-compact space-time directions.

For the case considered in [1] it follows that (11) becomes

$$C_{\underline{b}^1}^{\underline{a}^1} C_{\underline{b}^2}^{\underline{a}^2} = (-1)^{\frac{1}{8} \sum_{i=1}^{32} (a_i^1 + a_i^2)} C_{\underline{b}^1 + \underline{a}^2}^{\underline{a}^1} C_{\underline{b}^2 + \underline{a}^1}^{\underline{a}^2} \quad (20)$$

The space-time or NSR fermions in this case are labelled by $i=0$ and the distinction between $i=0$ and $i=1, \dots, 32$ is precisely because of the phase arising from the presence of Y .

As for the bosonic case it is easy to show that (20) implies and is implied by a "physically sensible projection" (in the sense that space-time spinors have $\eta=0$ or -1 and all other states have $\eta=0$ or $+1$). The phase induced by Y_α turns out to have a double role. Firstly it turns the states from the Ramond sector of the NSR fermions into space-time fermions by supplying the appropriate minus sign in the partition function. Secondly it generates a shift of fermion number for the NSR fermions. One still finds (15) but now $\eta_{\underline{0}}^0$ refers to the class of states which includes $b_{-1/2}^\mu \tilde{\alpha}_{-1}^\nu |0\rangle_{NS}$ i.e. the graviton/dilaton multiplet again.

Finally we note that from equation (3.20) of [1] for $\beta=0$ we have that $\chi(0)=0$ so that in particular $\eta_{\underline{0}}^0=1$. This is consistent with the above and shows that a "physically sensible projection" implies that the graviton/dilaton cannot be projected out of the physical spectrum.

In summary, we have briefly studied a class of fermionic closed string theories with periodic or antiperiodic boundary conditions on the fermions. After assumptions about factorization and the imposition of one loop modular invariance we find that the theory is two loop modular invariant if and only if the path integral weights at one loop are such as to give a "physically sensible projection" (correct spin-statistics for the physical states). Either of these conditions then imply that the graviton and dilaton must be present in the spectrum.

However the calculation presented here does not offer any physical insight into why these relationships are present. A natural candidate for the link is locality. The spin-statistics relation is a result of demanding that the field (anti-)commutators be zero for any space-like interval, however small, so it is just a consequence of dealing with point particles and local interactions. In string theories two loop modular invariance is necessary for consistency, however factorization is just the assumption that the two loop diagram can be split into two one loop diagrams joined by propagators for point particles, and joined to the world sheets by local vertex operators. (See also [11] in which it is the GSO projection that renders the vertex operators local.) Related issues have been discussed in [4].

Theories with more general boundary conditions (as in [2]) will be dealt with elsewhere [12], hopefully this will also give a clearer description of the assumptions involved.

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