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Ten dimensional Heterotic Strings from Niemeier Lattices

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Abstract

It is shown how all known ten dimensional heterotic string theories with rank 16 gauge groups can be derived in a direct way from Niemeiers classification of Euclidean even self-dual lattices in 24 dimensions. This result is based on a recent construction of heterotic strings from odd self-dual Lorentzian lattices. A natural embedding of all these theories in the 26 dimensional closed bosonic string is suggested by this construction.

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1. Introduction

Although the consistency requirements which string theories have to satisfy are quite restrictive, it has become clear that there are more solutions than one originally expected [1] [2] [3] [4]. At the same time, it has become clear that many string theories are closely related. In particular, it has been shown in [5] that a certain class of heterotic strings with rank 16 gauge group is related to odd Lorentzian self-dual lattices. The uniqueness of those lattices guarantees that any such lattice can be Lorentz rotated into any other.

Although the possibility of making Lorentz rotations suggests a continuous infinity of new ten dimensional theories (as in [6]), there is actually only a discrete set of theories that makes physical sense, as we will explain below. The purpose of this paper is to construct all lattices corresponding to sensible ten dimensional heterotic string theories. There is no need to consider GSO-projections or lattice shifts at all. We begin with a brief summary of the results obtained in [5].

2. Odd self-dual lattices.

The heterotic strings considered in [5] are constructed out of the usual ten dimensional left and right moving bosonic coordinates X^{μ} plus 16 left and 6 right moving bosons, compactified on a lattice $\Gamma_{16;5,1}$, with signature $(16(-),5(+),-)$. The first

16 bosons produce a gauge group G by the Frenkel Kac mechanism. The last six produce covariant Neveu-Schwarz or Ramond theories, and can be thought of as a bosonization of NS or R world sheet fermions and ghosts [7]. We can write the vectors on the lattice as $\vec{w}=(\vec{\lambda}_L,\vec{\lambda}_R,q)$, where $\vec{\lambda}_L$ is a 16-dimensional weight vector of G, $\vec{\lambda}_R$ a 5-dimensional weight of the Wick rotated Lorentz group SO(10), and q the ghost charge. The lattice contains far too many many states, because classes of states whose ghost charges differ by an integer are in fact equivalent. The light cone partition function for the physical states (not including oscillators) can be obtained by dividing the partition function of $\Gamma_{16;5,1}$ by the partition function of the even Lorentzian self-dual lattice $\Gamma_{1,1}$. These partition functions are given by [5]

$$\vartheta_{16;5,1}(\tau,\bar{\tau}) = \sum_{w\in\Gamma_{16;5,1}} e^{-\pi i\bar{\tau}\lambda_L^2}\, e^{\pi i\tau(\lambda_R^2-q^2-2q-2)}\, e^{2\pi i q}$$

$$\vartheta_{1,1}(\tau) = \sum_{w\in\Gamma_{1,1}} e^{\pi i\tau w^2} \tag{2.1}$$

Notice that $\theta_{1,1}$ depends only on τ (not $\bar{\tau}$), because it only describes right movers. Although these two functions are separately ill-defined because of the Lorentzian metric for the right movers, their ratio is just the light cone partition function. After dividing by $\theta_{1,1}$, the states remaining in the light cone partition function are those for which the last two entries of $\vec{w}$ have fixed values [8] :

$$(\ldots 0,-1) \quad \text{or} \quad (\ldots,1/2,-1/2) \tag{2.2}$$

Thus we can select the physical states out of the lattice by keeping only states with these last two entries. The masses of these states are given by $m^2 = m_L{}^2/2 + m_R{}^2/2$,

$$
\begin{aligned}
\frac{1}{8} m_L^2 &= \frac{1}{2} \lambda_L^2 + N_L - 1 \\
\frac{1}{8} m_R^2 &= \frac{1}{2} \lambda_R^2 + N_R - \frac{1}{2}
\end{aligned}
\tag{2.3}
$$

where $\vec{u}$ contains the first four entries of $\vec{\lambda}_R$, and we have included the occupation numbers N_L and N_R of the bosonic oscillator modes. Furthermore, physical states have to satisfy $m_L{}^2 = m_R{}^2$.

To check modular invariance of these theories it suffices to consider the transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$. The partition function is invariant under $\tau \to \tau + 1$ if [5]

$$\vec{\lambda}_L{}^2 - \vec{\lambda}_R{}^2 + q^2 + 2q = 0 \bmod 2 \tag{2.4}$$

For the second transformation one finds that the partition function transforms properly if $\Gamma_{16;5,1}$ is self-dual, apart from a phase factor $e^{4\pi i q}$, which is irrelevant only if q is integer or half-integer.

It is thus clear that the theory is not modular invariant for arbitrary SO(17,5) Lorentz transformations of $\vec{w}$. Now, to impose sensible space time properties we furthermore require that q is integer if $\vec{\lambda}_R$ is a vector weight of SO(10), and half-integer if $\vec{\lambda}_R$ is a spinor weight. The factor $e^{2\pi i q}$ in (2.1) ensures then that space time bosons and fermions contribute with a relative minus sign to the partition function, so that the spin statistics theorem is satisfied. This, together with (2.4), is all we need to know to classify the sensible solutions.

3. Classification

We can write the SO(10) weight $\vec{\lambda}_R$ as $(\vec{u},p)$, where $\vec{u}$ is a SO(8) weight and p is an integer or half-integer. Because of the spin-statistics relation mentioned above, p and q are either both integer or half-integer. Since a self-dual lattice contains all vectors having integer dot product with all lattice vectors, it must certainly contain $(\vec{\lambda}_L,\vec{u},p,q) = (0,0,\pm 1,\pm 1)$. By adding linear combinations of these vectors we can always restrict p and q to have the values (0,0), $(0,-1)$, $(1/2,-1/2)$ and (1/2,1/2). These vectors can be regarded as representatives of "conjugacy classes" of the lattice $\Gamma_{1,1}$. We denote these classes as (0), (v), (s) and (c), respectively. Notice that because of the physical state selection rule (2.2) only classes (v) and (s) contain physical states. In exactly the same way one can see that the lattice must contain the vectors $(0,\vec{r},0,0)$ of the SO(8) root lattice, so that we can restrict $\vec{u}$ to be a representative of the SO(8) conjugacy classes, which we again denote as (0), (v), (s) and (c). The lattice thus consists of the following sets of vectors

$$
\begin{array}{ll}
(\Delta_{oo},0,0) & (\Delta_{ov},0,v) \\
(\Delta_{vo},v,0) & (\Delta_{vv},v,v) \\
(\Delta_{sc},s,c) & (\Delta_{ss},s,s) \\
(\Delta_{cc},c,c) & (\Delta_{cs},c,s)
\end{array}
\tag{3.1}
$$

Here $\Delta_{\alpha\beta}$ denotes a set of vectors $\vec{\lambda}_L$ associated with the SO(8) conjugacy class α and the $\Gamma_{1,1}$ conjugacy class β.

It is now easy to see that (2.4) is satisfied if and only if Δ_{vo} and Δ_{ov} contain only vectors of odd length and all other $\Delta_{\alpha\beta}$ contain only vectors of even length. In

this case, (3.1) defines indeed an odd, integral lattice : for example, (Δ_{ss},s,s) contains only vectors of odd length.

To construct a self-dual lattice $\Gamma_{16;5,1}$ with the required properties we have to find a combination of sets of vectors $\Delta_{\alpha\beta}$ with

$$\begin{aligned}
&1.\ (\Delta_{\alpha\beta},\Delta_{\gamma\delta}) - (\alpha,\gamma) - <\beta,\delta> \in Z \\
&2.\ \Delta_{ov}, \Delta_{vo} \text{ odd, all others even} \qquad (3.2)\\
&3.\ \text{if } \vec{w} = (\vec{\lambda}_L,\vec{u},\vec{s}) \text{ and } (\vec{\lambda}_L,\Delta_{\alpha\beta}) - (\vec{u},\alpha) - <\vec{s},\beta> \ \in \mathbb{Z} \\
&\text{for all } \alpha \text{ and } \beta, \text{ then } \vec{w}\in\Gamma_{16;5,1}
\end{aligned}$$

Here (,) denotes an Euclidean and < , > a Lorentzian dot product with metric $(1,-1)$. The crucial observation is now that in $\Gamma_{1,1}$ the conjugacy classes have the following mutual dot products

$$\begin{aligned}
<0,\alpha> &\in Z \qquad (\alpha=0,v,c,s)\\
<v,s> &\in Z+1/2\\
<v,c> &\in Z+1/2 \qquad (3.3)\\
<s,c> &\in Z+1/2
\end{aligned}$$

These are precisely the same as the corresponding Euclidean dot products of the SO(8) conjugacy classes. This is not so for the lengths of the vectors. Here we find for $\Gamma_{1,1}$ and SO(8), respectively

$$\begin{aligned}
<0,0> &\in 2Z & (0,0) &\in 2Z\\
<v,v> &\in 2Z+1 & (v,v) &\in 2Z+1\\
<s,s> &\in 2Z & (s,s) &\in 2Z+1 \qquad (3.4)\\
<c,c> &\in 2Z & (c,c) &\in 2Z+1
\end{aligned}$$

We can use these facts to relate the problem of satisfying (3.2) to a problem already solved in the mathematical literature. Suppose one has a lattice (3.1) obeying all three conditions. Now regard the third component of $(\Delta_{\alpha\beta},\alpha,\beta)$ not as a $\Gamma_{1,1}$ weight, but as a SO(8) weight in the "same" conjugacy class. Furthermore, replace the metric by an Euclidean one on the whole lattice. Then the resulting lattice Γ_{24} is an even self-dual lattice of dimension 24. It is even, because in going from $\Gamma_{1,1}$ to S0(8) all weights change in precisely the right way. It is integral, because all three terms in (3.2) remain unchanged, and since they are integer or half-integer, the relative signs do not matter. It is self- dual, because if $\vec{v}'\notin\Gamma_{24}$ has integral dot products with all $\vec{v}\in\Gamma_{24}$, one can easily construct a vector $\vec{w}'\notin\Gamma_{16;5,1}$ with integral dot product with all $\vec{w}\in\Gamma_{16;5,1}$, contradicting the fact that $\Gamma_{16;5,1}$ was self-dual.

All 24-dimensional even Euclidean self-dual lattices have been classified by Niemeier [9]. To use these lattices for our purpose we must decompose each vector $\vec{w}$ on such a lattice as $(\vec{\lambda},\vec{\alpha},\vec{\beta})$, where $\vec{\alpha}$ and $\vec{\beta}$ are SO(8) weights and $\vec{\lambda}$ is a weight of the remaining symmetry (gauge) group. This is only possible if the group G associated with the Niemeier lattice contains SO(8)$\times$SO(8) in a regular embedding. In particular, this rules out all embeddings in SU(N). Furthermore, only 8 of the 16 conjugacy classes of SO(8)$\times$SO(8) should appear because of the spin statistics relation. This rules out any embedding of the two SO(8)'s in separate simple factors of G, for the following reason. Every embedding of SO(8) produces at least the conjugacy class (0). If in addition it contains (s) or (c), then combination with the second SO(8) is certain to produce the inacceptable combinations (0,s) or (0,c). Hence the only acceptable SO(8) decompositions of a simple factor of G are (0) or (0+v).

However, embedding two SO(8) factors in this way, we would get a lattice consisting only of vectors $(\Delta_{oo},0,0)$,$(\Delta_{ov},0,v)$, $(\Delta_{vo},v,0)$ and (Δ_{vv},v,v). But such a lattice cannot be self-dual : since Δ_{oo} and Δ_{vv} have even vectors and Δ_{ov} and Δ_{vo} only odd ones, the vectors $(\Delta_{vv},0,v)$, $(\Delta_{vo},0,0)$, etc. are examples of vectors which are not on the lattice, but have integer dot products with it.

Thus the only remaining possibility is the embedding $SO(2N) \supset SO(2N-16)\times SO(16)$ or $E_8 \supset SO(16)$, with $SO(16) \supset SO(8)\times SO(8)$. Consequently, the conjugacy classes of the two SO(8) are correlated, and the lattice can be written in the form $(\Delta_o,0) + (\Delta_v,v) + (\Delta_s,s) + (\Delta_c,c)$, where the second entry denotes a SO(16) conjugacy class. Comparing with (3.1) we have, therefore

$$\begin{aligned} \Delta_{oo} &= \Delta_{vv} = \Delta_o \\ \Delta_{ov} &= \Delta_{vo} = \Delta_v \\ \Delta_{ss} &= \Delta_{cc} = \Delta_s \\ \Delta_{sc} &= \Delta_{cs} = \Delta_c \end{aligned} \tag{3.5}$$

Table 1 summarizes the content of the various sectors. The combination $\Delta = \Delta_o + \Delta_v + \Delta_s + \Delta_c$ must itself form a lattice, and Δ_o must contain the dual of that lattice. If Δ is (part of) a weight lattice of a group G, Δ_o contains the root lattice, and because of $\Delta_o=\Delta_{vv}$ the presence of massless G gauge bosons is automatically guaranteed. The sector (v) is the only one that may contain tachyons; this is the case if Δ_v has vectors of length 1. The massless states in the other sectors are always due to vectors of lenght 2. A theory has chiral fermions if $\Delta_s\neq\Delta_c$ and Δ_s or Δ_c has vectors of length 2. Absence of anomalies is guaranteed by modular invariance [11].

To find all relevant possibilities for Γ_{24}, we consider the Niemeier lattices [9]. Each one of these is characterized by the root lattice of some semi-simple algebra of rank 24, together with certain weights. All such root lattices containing $D_8 \sim SO(16)$ are given in the first column of table 2. The second column gives the conjugacy classes of the additional weights. These are easily obtained from the "glue vectors" tabulated in [12]. The D_8 sublattices, the remaining gauge groups G_{het} and the conjugacy classes of G_{het} appearing in $\Delta_o,\ldots,\Delta_c$ can be read off directly from the first two columns, and are presented in the other columns. We get precisely all the theories obtained with the bosonic construction of [2] and no more. It is a priori clear (since we can obtain only rank 16 groups) that the fermionically constructed E_8 theory of [4] (suggested in [2] and perhaps identical to the heterotic string constructed in [10]) does not fit in our framework.

It is straightforward to apply this method to type II superstrings. One starts with a Lorentzian odd self-dual lattice $\Gamma_{5,1;5,1}$, which can be mapped onto an Euclidean even self-dual lattice Γ_{16}. This lattice can only be $E_8 \times E_8$ or D_{16}. From $E_8 \times E_8$ one obtains the type IIA and type IIB superstrings[1], and from D_{16} the type A and B theories constructed fermionically in [14].

The above construction has an obvious relation to the one presented in [15] [16] for superstrings out of the bosonic string. To make this more precise, write a Niemeier lattice Γ_{24} as "$D_8 \times G_{het}$", and define a new Lorentzian lattice $\Gamma_{16;16} = (G_{het})_L \times (D_8 \times E_8)_R$, with metric (16(+1), 16(−1)). Since D_8 has only integer length weights, this is an even Lorentzian self-dual lattice, on which the bosonic string can be compactified. The construction presented in [16] is immediately appli-

[1] The open superstring related to $\Gamma_{0;5,1}$ [13] can be mapped uniquely onto $\Gamma_8 = E_8$.

cable to this lattice : one removes states belonging to the extra E_8, decomposes SO(16) to $SO(8)_1 \times SO(8)_2$ and keeps only states with fixed weights $(0,0,0,-1)$ and $(1/2,1/2,1/2,-1/2)$ of $SO(8)_2$, and no $SO(8)_2$ excitations [16]. This is precisely the physical state selection rule (2.2) we described earlier for the $\Gamma_{1,1}$ lattice. It obviously would be interesting to make the map from $SO(8)_2$ to $\Gamma_{1,1}$ more precise, since it would open the way to a covariant approach to the embedding of all superstrings in the bosonic string. In any case, as embeddings in the bosonic string all theories in table 2 are clearly on the same footing. It is attractive to think of all of them as "vacua" of the bosonic string.

It is also possible to apply these ideas to compactified $10-d$ dimensional strings, by considering odd self-dual lattices $\Gamma_{16+d;5+d,1}$, which can be mapped onto even self-dual Lorentzian lattices $\Gamma_{16+d;8+d}$. Unfortunately, the lattice classification theorems are not of much help here. Because the left and right vectors need not have integral lengths, we cannot map, in general, this lattice onto an even Euclidean lattice. In particular, for $d=6$ this is manifestly impossible, since the resulting lattice would have dimension 36, and there are no such Euclidean even self-dual lattices. However, an interesting subset of six dimensional theories can be related to 32 dimensional Euclidean even self-dual lattices. It is known that there are huge numbers of such lattices, but for that reason they have not been classified.

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Table 1: Contents of the four sectors

S0(16)	SO(8)×SO(8)	Physical	$m_R^2/8$	Spin
(0)	(0,0)	No		
	(v,v)	Yes	0,1,2,...	Odd Rank Tensor
(v)	(v,0)	No		
	(0,v)	Yes	−1/2,1/2,...	Even Rank Tensor
(s)	(c,c)	No		
	(s,s)	Yes	0,1,2,...	Left Spinor
(c)	(s,c)	No		
	(c,s)	Yes	0,1,2,...	Right Spinor

Table 2: Ten dimensional Heterotic strings

Niemeier Lattice		Heterotic String				
Roots	Weights	Algebra	SO(16) – Sector			
			(0)	(v)	(s)	(c)
$E_8{}^3$		$E_8 \times E_8$	(0+s,0+s)	–	(0+s,0+s)	–
$E_8 \times D_{16}$	(0,s)	D_{16}	(0+s)	–	(0+s)	–
$E_8 \times D_{16}$	(0,s)	$E_8 \times D_8$	(0+s,0)	(0+s,v)	(0+s,s)	(0+s,c)
D_{24}	(s)	D_{16}	(0)	(v)	(s)	(c)
$D_{12} \times D_{12}$	[v,s]	$D_4 \times D_{12}$	(0,0)	(v,0)	(s,v)	(s,c)
	(c,c)		(v,s)	(0,s)	(c,c)	(c,v)
$D_{10} \times E_7{}^2$	(s,1,0)	$D_2 \times E_7{}^2$	(0,0,0)	(v,0,0)	(s,1,0)	(c,1,0)
	(c,0,1)		(v,1,1)	(0,1,1)	(c,0,1)	(s,0,1)
	(v,1,1)					
$D_9 \times A_{15}$	(s,4k+2)	$D_1 \times A_{15}$	(0,4k)	(v,4k)	(s,4k+2)	(c,4k+2)
	(0,4k)					
$D_8{}^3$	[s,v,v]	$D_8 \times D_8$	(0,0)	(s,v)	(v,v)	(c,0)
	[c,0,c]		(c,c)	(v,s)	(s,s)	(0,c)
	(s,s,s)					

In column 2, square brackets indicate cyclic permutation. For E_7, 1 denotes the conjugacy class containing the (56); for A_{15} , an entry m denotes the class containing tensors of rank m (k=0,...,4).

Table 2: Ten dimensional Heterotic strings

Niemeier Lattice			Heterotic String			
Roots	Weights	Algebra	SO(16) – Sector			
			(0)	(v)	(s)	(c)
E_8^3		$E_8 \times E_8$	$(0+s,0+s)$	–	$(0+s,0+s)$	–
$E_8 \times D_{16}$	$(0,s)$	D_{16}	$(0+s)$	–	$(0+s)$	–
$E_8 \times D_{16}$	$(0,s)$	$E_8 \times D_8$	$(0+s,0)$	$(0+s,v)$	$(0+s,s)$	$(0+s,c)$
D_{24}	(s)	D_{16}	(0)	(v)	(s)	(c)
$D_{12} \times D_{12}$	$[v,s]$	$D_4 \times D_{12}$	$(0,0)$	$(v,0)$	(s,v)	(s,c)
	(c,c)		(v,s)	$(0,s)$	(c,c)	(c,v)
$D_{10} \times E_7^2$	$(s,1,0)$	$D_2 \times E_7^2$	$(0,0,0)$	$(v,0,0)$	$(s,1,0)$	$(c,1,0)$
	$(c,0,1)$		$(v,1,1)$	$(0,1,1)$	$(c,0,1)$	$(s,0,1)$
	$(v,1,1)$					
$D_9 \times A_{15}$	$(s,4k+2)$	$D_1 \times A_{15}$	$(0,4k)$	$(v,4k)$	$(s,4k+2)$	$(c,4k+2)$
	$(0,4k)$					
D_8^3	$[s,v,v]$	$D_8 \times D_8$	$(0,0)$	(s,v)	(v,v)	$(c,0)$
	$[c,0,c]$		(c,c)	(v,s)	(s,s)	$(0,c)$
	(s,s,s)					

In column 2, square brackets indicate cyclic permutation. For E_7, 1 denotes the conjugacy class containing the (56); for A_{15} , an entry m denotes the class containing tensors of rank m $(k=0,\ldots,4)$.