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CHERN-SIMONS TERMS IN THE SIGMA MODEL FOR THE HETEROTIC STRING

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ABSTRACT

The gauge anomalous contributions to the two-loop beta functions in the non-linear sigma model for the heterotic string moving in a curved background with torsion, are calculated explicitly, using a well-defined regularization scheme. It is shown that these contributions to both the beta functions for the metric and the antisymmetric tensor field are absorbed exactly by the addition of the full gauge Chern-Simons form to the torsion. The circumstances under which there is an analogous cancellation in the Lorentz sector is discussed and it is shown to depend on the choice of fermionic quantum variables.

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1. - INTRODUCTION

It is well known that the sigma model for the heterotic string¹⁾ in which the right-moving bosonic co-ordinates have fermionic superpartners and there are 16 extra left-moving bosonic co-ordinates which couple to the gauge group, possesses both gauge and Lorentz anomalies²⁾. This is most easily seen in the formalism in which the 16 extra left-moving bosonic co-ordinates are fermionized^{3),4)} to give 32 left-moving fermionic co-ordinates which transform under the gauge group [as a 32 of $\text{Spin}(32)/Z_2$ or a $(1,16)+(16,1)$ representation of the $\text{SO}(16)\times\text{SO}(6)$ subgroup of $E_8\times E_8$]. In two dimensions left- or right-moving fermions correspond to left- or right-helicity states and so there are axial couplings to both the gauge field and the gravitational field.

It has been shown⁵⁾⁻⁹⁾ that these one-loop (in the sigma model) anomalies can be cancelled by adding gauge and Lorentz Chern-Simons three-forms to the torsion H_{ijk} ($\equiv 3/2 \partial_{[i} B_{jk}$] where B_{jk} is the massless antisymmetric tensor field associated with the graviton in closed string theories), which are precisely the same as the Chern-Simons terms which occur in the low-energy supergravity field theory limit (in which the string tension, α' , goes to zero) and have been shown by Green and Schwarz¹⁰⁾ to be necessary for the cancellation of gauge and gravitational anomalies in such field theories. It has moreover been suggested¹¹⁾ that for a heterotic string propagating in a background gravitational and gauge field, the calculation of the sigma model beta functions at two-loop order acquire contributions due to these anomalies which are exactly cancelled by the addition of these Chern-Simons forms to the torsion and substituting in the expressions for the beta functions at the one-loop level.

There are, however, a number of details which have not been considered in this analysis and it is the purpose of this paper to address them. References. 9) and 11) concentrate on the gauge sector and obtain part of the gauge Chern-Simons corrections to the torsion which is quadratic in the fields [Ref. 9) explicitly limits its analysis to Abelian gauge fields for which the cubic term in the Chern-Simons form is absent]. In Section 2 it is shown that a systematic expansion of the gauge sector of the heterotic string sigma model and an explicit calculation of the contributions arising from the axial couplings within a well-defined regularization scheme for handling the ultra-violet divergences, leads to an anomalous contribution to the two-loop beta functions for both the metric (β^G) and the antisymmetric tensor field (β^B), which are absorbed by adding the full Chern-Simons form (including the term cubic in the gauge fields) to the torsion in the one-loop beta functions. The Lorentz sector has hitherto been considered

by observing that when referred to the tangent space the sigma model action for the right-moving fermions is identical to that of the left-moving fermions with the gauge field, A_i , replaced by $\omega_i^{(-)}$ (the metric spin connection minus the torsion). It is pointed out in Section 3 that great care must be taken when doing this. In the first place, the usual technique of normal co-ordinate expansion¹²⁾ is not applicable in this case, since the choice of such a co-ordinate system is only permitted in a theory in which general co-ordinate invariance is not broken at the quantum level by Lorentz anomalies and in any case, since such an expansion only produces gauge invariant terms in the action, it will never expose the relevant coupling between the spin connection and the (anomalous) covariant divergence of the fermion current. One is therefore obliged to perform an ordinary linear expansion of the bosonic co-ordinates around a background string, and it is shown that if one chooses to quantize the theory by demanding that the right-moving quantum bosonic co-ordinates and the fermionic co-ordinates transform into each other under world sheet supersymmetry, then there can be extra anomalous contributions to the beta functions which do not look like Chern-Simons forms added to the torsion. If, on the other hand, one chooses to define the quantum field of the right-moving fermion in the tangent frame, thereby ignoring interactions arising from the expansion of the vielbein around the background string, the remaining interactions are indeed identical to those obtained in the gauge sector. We thus see that in the case of the heterotic string propagating in a curved background, the theory is not fully defined in terms of the classical action, but one must specify one's choice of quantum co-ordinates. Finally, it is pointed out that if one replaces the gauge field by $\omega_i^{(+)}$ (the metric spin connection plus torsion) so that the heterotic string reduces to that of the superstring¹³⁾, the anomalous contributions from the left- and right- moving fermions do not exactly cancel, since they generate Chern-Simons forms calculated from $\omega_i^{(+)}$ and $\omega_i^{(-)}$ respectively. The results are summarized in Section 4.

In order to establish the notation used in this paper we write down the action for the non-linear sigma model representing a heterotic string propagating in a background gravitational field (with torsion) and a background gauge field. As in Refs. 4) and 11), we write this action using the NSR spinning string formalism¹⁴⁾ for which world sheet supersymmetry of the right-moving co-ordinates is manifest

$$\begin{aligned}
 S = \frac{1}{2\pi\alpha'} \int d\tau \sqrt{g} \left\{ \frac{1}{2} (G_{ij} g^{\mu\nu} + B_{ij} \varepsilon^{\mu\nu}) \partial_\mu X^i \partial_\nu X^j \right. \\
 + i G_{ij} \bar{\lambda}^i \gamma^\mu (\partial_\mu \lambda^j + \Gamma_{\mu\kappa}^j \partial_\mu X^\kappa \lambda^j + H_{\mu\kappa}^j \varepsilon_\mu^\nu \partial_\nu X^\kappa \lambda^j) \quad (1.1) \\
 + i \bar{\psi}^\alpha \gamma^\mu (\partial_\mu \psi^\alpha - i \tau_{\alpha\beta}^a A_\mu^a \partial_\mu X^i \psi^\beta) + \bar{\psi}^\alpha \gamma^\mu \tau_{\alpha\beta}^a \psi^\beta F_{ij}^a \bar{\lambda}^i \gamma^\mu \lambda^j \\
 \left. + 2\pi\alpha' R^{(2)} \Phi \right\}
 \end{aligned}$$

where the middle Latin indices run over the ten dimensions of the space-time manifold, M , middle Greek indices run over the two dimensions of the world sheet (parametrized by co-ordinates $\eta \equiv \sigma, \tau$) and early Latin and Greek indices are gauge group indices for the adjoint representation and the representation under which the fermions transform respectively. G_{ij} is the metric on the space-time manifold (Γ_{ij}^k is the corresponding affine connection), B_{ij} is the antisymmetric tensor field (the torsion, H_{ij}^k is the corresponding field strength) and Φ is the dilaton field, A_i^a is the background gauge field (F_{ij}^a is the corresponding field strength). $g^{\mu\nu}$ is the metric on the world sheet [$R^{(2)}$ is the corresponding curvature scalar] and γ^μ are two-dimensional gamma matrices. X^i are the bosonic co-ordinates, λ^i are right-moving fermionic co-ordinates [$(\partial_\sigma + \partial_\tau) \lambda^i = 0$] which transform as space-time vectors and ψ^α are left-moving fermionic co-ordinates [$(\partial_\sigma - \partial_\tau) \psi^\alpha = 0$] transforming as a 32-dimensional representation of the gauge group, but which are space-time scalars. We shall assume that the background dilaton field is a constant so that the last term in Eq. (1.1)¹⁵⁾ is just a string loop counting constant, and furthermore since the β functions (in a critical number of dimensions) come from short-distance behaviour, we can restrict the world-sheet topology to be S^2 .

2. - THE GAUGE SECTOR

The gauge sector part of the action consists of the terms in Eq. (1.1) which depend on the fermions ψ^α . The term which is quartic in fermion fields does not contribute to the beta functions at two-loop order and may therefore be ignored. When the gauge field, $A_i(X)$, is expanded around some background, $A_i(X_0)$, the

interactions will generate corrections to the effective action which are ultra-violet divergent. It is therefore necessary to accompany the classical action with a regularization prescription for handling these divergences. The most convenient

such regularization scheme is dimensional regularization, in which the world sheet is considered to be in $2-\epsilon$ dimensions. However, the concept of "left-moving" or "right-moving" fields only exists in two dimensions and one has to give a prescription for extending this away from two dimensions. [It is for this reason that we avoided using light-cone notation ∂_+X , ∂_-X , etc., in Eq. (1.1)]. In two dimensions left- and right-moving fermions have left and right helicities respectively. We choose to maintain the helicity properties away from two

dimensions. This implies that outside two dimensions the left-handed fermions, ϕ^α , have a component moving in the opposite direction which decouples in the two-dimensional limit. Indeed one might have thought that since the ϕ^α are left-moving, they could only give rise to corrections to the effective action proportional to $\partial_+X^i \partial_-X^j$ rather than the required $\partial_+X^i \partial_+X^j$. However, for ultra-violet contributions the opposite moving component of the fermion field can effect transitions between ∂_+X^i and ∂_-X^i (in an alternative regularization scheme such as Pauli-Villars, the regulator fermion fields have a mass so that one is introducing regulator fields with the opposite helicity and consequently moving in the opposite direction).

We begin by discussing the chiral anomaly¹⁶⁾ using this regularization scheme. In two dimensions there is an ambiguity in the chiral anomaly. This is because the axial current is dual to the vector current ($J_A^\mu = \epsilon^{\mu\nu} j_\nu$) so one can choose to keep the vector current conserved and have the anomaly in the axial current or vice versa. We fix this ambiguity [as suggested by Sen⁹⁾] by requiring that vector theories (in which both left and right helicity fermions are present) are anomaly-free. Although in two dimensions only the two-point function is superficially ultra-violet divergent, non-Abelian anomalies, which involve the covariant derivative of the axial current, involve both the two- and three-point functions. We wish to calculate

$$D_\mu \langle J_A^\mu \rangle_A^a = \partial_\mu \langle J_A^\mu \rangle_A^a + f^{abc} A_\mu^b \langle J_A^\mu \rangle_A^c \quad (2.1)$$

where the subscript A means in the presence of an external gauge field, A_μ^a ($\equiv A_1(X) \partial_\mu X^1$), which couples to the vector current. Expanding up to second order in the gauge field gives

$$\begin{aligned}
 i \partial_\mu \langle J_A^{\mu a}(\underline{1}) J_V^{\nu b}(\underline{1}') \rangle_0 A_\nu^b(\underline{1}') &= \frac{1}{2} \partial_\mu \langle J_A^{\mu a}(\underline{1}) J_V^{\nu b}(\underline{1}') J_V^{\rho c}(\underline{1}'') \rangle_0 A_\nu^b(\underline{1}') A_\rho^c(\underline{1}'') \\
 &- \frac{i}{2} f^{abd} \delta^2(\underline{1}-\underline{1}') \langle J_A^{\mu d}(\underline{1}) J_V^{\nu c}(\underline{1}'') \rangle_0 A_\mu^b(\underline{1}') A_\nu^c(\underline{1}'') \\
 &+ \frac{i}{2} f^{abc} \delta^2(\underline{1}-\underline{1}'') \langle J_A^{\mu d}(\underline{1}) J_V^{\nu c}(\underline{1}') \rangle_0 A_\mu^b(\underline{1}') A_\nu^c(\underline{1}'') + \dots
 \end{aligned} \quad (2.2)$$

(with implied integrations over $\underline{1}'$, $\underline{1}''$). The terms higher order in the gauge field are anomaly-free since they are all ultra-violet finite. Using the above-mentioned regularization scheme, the two-point function is given by

$$\langle J_A^{\mu a}(\underline{1}) J_V^{\nu b}(\underline{1}') \rangle_0 = -i \frac{\delta^{ab}}{4\pi} \left[\varepsilon^{\mu\nu} - (\varepsilon^{\nu e} \partial^\mu \partial_e + \varepsilon^{\mu e} \partial^\nu \partial_e) \frac{1}{\square} \right] \delta^2(\underline{1}-\underline{1}') \quad (2.3)$$

where $1/\square$ is the Neumann function on the world sheet. The divergence of the three-point function is given by

$$\begin{aligned}
 \partial_\mu \langle J_A^{\mu a}(\underline{1}) J_V^{\nu b}(\underline{1}') J_V^{\rho c}(\underline{1}'') \rangle_0 &= \frac{f^{abc}}{4\pi} \left[\delta^2(\underline{1}-\underline{1}') (\varepsilon^{\rho\sigma} \partial^\nu \partial_\sigma + \varepsilon^{\nu\sigma} \partial^\rho \partial_\sigma) \frac{1}{\square} \delta^2(\underline{1}-\underline{1}'') \right. \\
 &\quad \left. - \delta^2(\underline{1}-\underline{1}'') (\varepsilon^{\rho\sigma} \partial^\nu \partial_\sigma + \varepsilon^{\nu\sigma} \partial^\rho \partial_\sigma) \frac{1}{\square} \delta^2(\underline{1}-\underline{1}') \right] \quad (2.4)
 \end{aligned}$$

Inserting this into Eq. (2.2), we arrive at the standard anomaly result

$$(\mathcal{D}_\mu J_A^\mu)^a = \frac{1}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu}^a \quad (2.5)$$

where

$$F_{\mu\nu}^a = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c) \partial_\mu X^i \partial_\nu X^j$$

In the treatment of Ref. 11), the gauge part of the action was expanded up to first order in the quantum correction (to X^i), ξ^i , obtaining a term proportional to the left-hand side of Eq. (2.5). This was then substituted by the right-hand side of Eq. (2.5) which was further expanded to give rise to an anomalous contribution to the two-loop beta functions (β^G), which can be absorbed

by adding part of the Chern-Simons three-form to the torsion and substituting in the one-loop beta function. A systematic treatment of the quantum corrections in the gauge sector requires an expansion of the action up to a given order in ξ without exploiting the known result for the axial anomaly. One must then calculate all possible diagrams up to the required order which can contribute to the beta functions. The fermion loops will have an anomalous contribution due to the axial couplings. The calculation of all such diagrams gives the full correction to the beta functions including terms which are absorbed by adding the complete Chern-Simons forms to the torsion.

In order to construct all two-loop diagrams contributing to the beta functions, it is necessary to expand the action up to second order in the quantum corrections, ξ^i . Applying gauge completion (i.e., adding and subtracting terms in order to convert ordinary derivatives into covariant derivatives or derivatives of the gauge field into field strengths) we obtain, for the gauge sector

$$\begin{aligned}
 S^{\text{gauge}} = & \frac{i}{2\pi\alpha'} \int d^2x \int_0^1 d\tau \left[\bar{\Psi}^\alpha \gamma^\mu (\partial_\mu \Psi^\beta - i \tau_{\alpha\beta}^a A_\mu^a(X_0) \partial_\mu X_0^i \Psi^\beta) \right. \\
 & + (\bar{\Psi}^\alpha \gamma^\mu \Psi^\beta \tau_{\alpha\beta}^a F_{\mu\lambda}^a(X_0) \partial_\mu X_0^\lambda - A_\mu^a(X_0) D_\mu^{ab} \bar{\Psi}^\alpha \gamma^\mu \tau_{\alpha\beta}^b \Psi^\beta) \xi^k \\
 & + \frac{1}{2} (\bar{\Psi}^\alpha \tau_{\alpha\beta}^a \gamma^\mu \Psi^\beta) (\partial_\mu X_0^k \nabla_\lambda F_{\mu\lambda}^a(X_0) \xi^l \xi^m + F_{\lambda k}^a(X_0) \xi^l \nabla_\lambda \xi^k) \\
 & - \frac{1}{2} \nabla_\lambda A_\mu^a(X_0) \xi^m \xi^l D_\mu^{ab} (\bar{\Psi}^\alpha \gamma^\mu \tau_{\alpha\beta}^b \Psi^\beta) \\
 & \left. - \frac{1}{2} \bar{\Psi}^\alpha \gamma^\mu \tau_{\alpha\beta}^a \Psi^\beta (\xi^{abc} A_\mu^b(X_0) A_\mu^c(X_0) \xi^l \nabla_\mu \xi^k \right. \\
 & \left. + \xi^{abc} A_\mu^b(X_0) \nabla_\mu A_\mu^c(X_0) \partial_\mu X_0^k \xi^l \xi^m) \right] \quad (2.6)
 \end{aligned}$$

where $X_0^i(\eta)$ is the background string, ∇_μ is the covariant derivative on the space-time manifold, and $\nabla_\mu \xi^k = \partial_\mu \xi^k + \Gamma_{ij}^k \partial_\mu X_0^i \xi^j$. The diagrams contributing to β^G at the two-loop level are shown in Fig. 1 and those contributing to β^B are shown in Fig. 2. Figures 1a and 1b were considered in Ref. 11). They give contributions to β_{ij}^G

$$-\frac{\alpha'}{8\pi} F_{ki}^a F^{a,k}_j \quad (2.7)$$

and

$$\frac{\alpha'}{4\pi} H_i{}^{pm} A_m^a F_{pj}^a \quad (2.8)$$

(we absorb the eigenvalue of the quadratic Casimir operator into the definition of the gauge field and assume $\text{Tr } \tau^a \tau^b = \delta^{ab}$). Note that the term multiplying the torsion in (2.8) is not the full gauge Chern-Simons term, firstly because the part which is cubic in the gauge field is missing and secondly because it is antisymmetrized in only two of its three space-time indices. The remaining contribution comes from Fig. 1c involving a four-point coupling between two bosonic and two fermionic fields. Since axial couplings induce anomalies, one can no longer use the gauge invariance and dimensionality argument to neglect diagrams with gauge field insertions on the propagators [i.e., by gauge invariance the lowest dimension operator which can have an ultra-violet divergent coefficient is $F_{\mu\nu} F^{\mu\nu}$, but since this is dimension four, it is ultra-violet convergent in two dimensions¹⁷⁾] and in Fig. 1c we have the gauge field inserted once on the fermion line. This gives rise to a contribution to β_{ij}^G of

$$-\frac{\alpha'}{8\pi} H_i{}^{pm} (A_j^a F_{pm}^a - \frac{1}{3} f^{abc} A_p^a A_m^b A_j^c) \quad (2.9)$$

Adding this to Eqs. (2.7) and (2.8) gives a total gauge field contribution to β_{ij}^G at order α'

$$\beta_{ij}^G(\text{gauge}) = \frac{-\alpha'}{8\pi} F_{ik}^a F_{j}^a{}^k - \frac{3\alpha'}{2} H_i{}^{pm} (A_j^a F_{pm}^a - \frac{1}{3} f^{abc} A_p^a A_m^b A_j^c) \quad (2.10)$$

The second term which arises out of the axial couplings is cancelled exactly by adding

$$\frac{3\alpha'}{4} \omega_{(3)}^{ym}{}_{jpm} \equiv \frac{3\alpha'}{4} (A_j^a F_{pm}^a - \frac{1}{3} f^{abc} A_p^a A_m^b A_j^c)$$

to the torsion H_{jpm} and substituting in the one-loop expression for β_{ij}^G in the presence of torsion.

The diagrams of Fig. 2 show the two-loop contributions to β^B . Although the fermion loop always introduces a contribution which is antisymmetric in space-time indices, there is no contribution to β^B from Fig. 1a by virtue of the antisymmetry property of F_{ij}^a . Figure 2a is the diagram corresponding to Fig. 1b

(but with the torsion insertion missing on the boson line and a derivative acting on the right-hand vertex). There are no diagrams corresponding to Figs 1b and 1c with the torsion replaced by the affine connection, since these diagrams do not involve Lorentz anomalies, and so we can exploit gauge invariance of general coordinate transformations to eliminate counterterms involving the insertions of $\Gamma_{jk}^i \partial_\mu X_0^k$ on the boson propagators. The contribution from Fig. 2a can either be computed directly, remembering that the fermion is propagating in a background gauge field so that (by power counting) we must include the diagram with one gauge field insertion on the fermion propagators, or by using the fact that the fermion loop may be written

$$\xi(\eta) \xi'(\eta') A_m^a(x(\eta)) F_{jc}^b(x(\eta')) \partial_\mu X_0^c(\eta') \frac{\delta}{\delta A_j^b(x(\eta'))} D_\nu^{ac} \langle j_A^{\nu,c} \rangle_A \quad (2.11)$$

and using the expression for the gauge anomaly. Either way it gives rise to a contribution to β_{ij}^B of

$$-\frac{\alpha'}{4\pi} A^{a,k} (\nabla_i F_{jk}^a - \nabla_j F_{ik}^a + f^{abc} A_{[i}^b F_{j]k}^c) \quad (2.12)$$

Figure 2b involves $D_\nu j_A^\nu$ directly and gives a contribution

$$\frac{\alpha'}{8\pi} \nabla^m A_m^a F_{ij}^a \quad (2.13)$$

whereas Fig. 2c, which is the other diagram involving the four-point coupling, gives

$$-\frac{\alpha'}{4\pi} A_i^a (\nabla^m F_{mj}^a - f^{abc} A^{b,m} \nabla_m A_j^c) \quad (2.14)$$

Adding these contributions and performing some algebraic manipulations including the Bianchi identity for the field strength ($D_{[i}^{ab} F_{jk]}^b = 0$), we end up with

$$\beta_{ij}^B = \frac{3\alpha'}{4\pi} \nabla^m (A_{[i}^a F_{j]m}^a - \frac{1}{3} f^{abc} A_i^a A_j^b A_m^c) \quad (2.15)$$

which is again cancelled exactly by adding $(3\alpha'/4)\omega_{(3)j\text{mp}}^{\text{YM}}$ to the torsion and substituting in the one-loop expression for β^B .

Finally, we observe that there are no contributions from diagrams in which the vertex $\xi_{i\mu}^{A,Dab}(\bar{\psi}\gamma^\mu\tau_{ab}\psi)$ acts twice. For such diagrams either one or both of the currents are vector currents and hence covariantly conserved, or they are both axial currents, but by the requirement that theories with both left- and right-moving fermions are anomaly-free, a fermion loop with two axial current insertions can be replaced by a loop with two vector insertions.

3. - THE LORENTZ SECTOR

The usual treatment of the bosonic or supersymmetric non-linear sigma model consists of expanding the (super)field about some background ($X_0^i(\eta)$) along a geodesic and expressing the tangent vector to the geodesic in terms of Riemann normal co-ordinates¹²⁾ for which the connection vanishes locally at the background point. In the case of a theory with Lorentz anomalies this procedure is not suitable. Indeed, it would not be possible to obtain a term involving the spin connection multiplied by the covariant divergence of the fermion current, and thereby obtaining contributions to β^G and β^B analogous to those obtained in Section 2 for the gauge sector (this is simply because the spin connection is locally set to zero). The Lorentz anomalies in fact mean that general co-ordinate invariance is effectively lost in the quantum theory, and one can no more choose normal co-ordinates to treat the Lorentz sector than one could in the gauge sector have chosen a gauge in which the gauge field vanishes locally at the background point (such a gauge exists, but the gauge anomalies imply a breakdown of gauge invariance so that we cannot confine ourselves to such a gauge in order to calculate beta functions). One is therefore led to perform a linear expansion of the bosonic co-ordinates

$$X^i(\eta) = X_0^i(\eta) + \xi^i(\eta) \quad (3.1)$$

and sacrifice the use of the geometric properties which leads to interactions expressible in terms of the Riemann tensor and covariant derivatives [i.e., ξ^i in Eq. (3.1) does not transform as a space-time vector].

The fermionic part of the Lorentz sector of the action consists of the terms in Eq. (1.1) (once again the term quartic in the fermion field may be ignored). Since the fermions λ^i are right-handed and the vector and axial currents are dual to each other, this may be rewritten

$$S^\lambda = \frac{i}{2\pi\alpha'} \int d^2\eta \sqrt{g} G_{ij}(X) \bar{\lambda}^i \gamma^{\mu} (\partial_\mu \lambda^j + \Gamma^{(-)j}_{kl} \partial_\mu X^k \lambda^l) \quad (3.2)$$

where $\Gamma^{(-)j}_{kl} \equiv \Gamma^{j}_{kl} - H^{j}_{kl}$. Expanding (3.2) up to first order in ξ^i gives two expressions

$$S^\lambda_{1a} = \frac{i}{2\pi\alpha'} \int d^2\eta \sqrt{g} G_{ij} [\bar{\lambda}^i \gamma^{\mu} \lambda^k \xi^m \partial_\mu X^l_0 R^{(-)i}_{kml} - \Gamma^{(-)j}_{km} \xi^m (\nabla^0_\mu \bar{\lambda} \gamma^{\mu} \lambda)^{ik}] \quad (3.3)$$

where

$$(\nabla^0_\mu \bar{\lambda} \gamma^{\mu} \lambda)^{ik} = \partial_\mu \bar{\lambda}^i \gamma^{\mu} \lambda^k + (\Gamma^{(-)i}_{ln} \bar{\lambda}^n \gamma^{\mu} \lambda^k + \Gamma^{(-)k}_{ln} \bar{\lambda}^i \gamma^{\mu} \lambda^n) \partial_\mu X^l_0$$

with the metric, Riemann tensor, and affine connection all calculated at X^i_0 , and

$$S^\lambda_{1b} = \frac{i}{2\pi\alpha'} \int d^2\eta \sqrt{g} (G_{mj} \Gamma^{m}_{ik} + G_{mi} \Gamma^{m}_{jk}) \bar{\lambda}^i \gamma^{\mu} \nabla^0_\mu \lambda^j \xi^k \quad (3.4)$$

S^λ_{1b} is the part obtained by expanding the metric in Eq. (3.2). By referring to the tangent frame at X^i_0 [$\lambda^\alpha = e^\alpha_i(X_0) \lambda^i$, etc.], and using the anticommuting properties of λ^i , Eq. (3.3) may be rewritten

$$S^\lambda_{1a} = \frac{i}{2\pi\alpha'} \int d^2\eta \sqrt{g} [\bar{\lambda}^\alpha \gamma^{\mu} (\tau^a)_{\alpha\beta} \lambda^\beta \xi^m \partial_\mu X^l_0 R^a_{ml} - \omega^{(-)a}_m \xi^m D_\mu^{\alpha\beta} \bar{\lambda}^\alpha \tau^b_{\alpha\beta} \lambda^\beta] \quad (3.5)$$

where $\omega^{(-)a}_m$ is the spin connection calculated with the torsion subtracted from the affine connection, and $\tau^a_{\alpha\beta}$ are the generators of the holonomy group of the space-time manifold. This expression is identical to the terms linear in ξ^i in Eq. (2.6) for the gauge sector, with A^a_i replaced by $\omega^{(-)a}_i$ and is precisely the expression that would have been obtained had we first referred to the tangent frame at every point on the manifold to obtain

$$S^\lambda = \frac{i}{2\pi\alpha'} \int d^2\sigma \sqrt{g} \bar{\lambda}^\alpha \gamma^\mu (\partial_\mu \delta_{\alpha\beta} + \omega_{\alpha}^{(-)a}(\chi) \partial_\mu \chi^i \tau_a^i{}_\beta) \hat{\lambda}^\beta \quad (3.6)$$

where $\hat{\lambda}^\alpha = e_i^\alpha(X) \lambda^i$, and then expanded the spin connection $\omega_{\alpha}^{(-)a}(\chi) \partial_\mu \chi^i$ about X_0^i as we did in the gauge sector. However, the expression for S_{1b}^λ , which comes from expanding the metric, leads in general to extra interactions and corresponds to the expansion of the vielbein $e_i^\alpha(X)$ about X_0^i in the definition of $\hat{\lambda}^\alpha$ above.

S_{1a}^λ then clearly leads to contributions to the beta functions analogous to Eqs. (2.10) and (2.15) with the gauge field replaced by the spin connection ($\omega_m^{(-)a}$) and the opposite sign for the anomalous contributions. However, the extra interactions from expanding the metric (or vielbein) can lead to extra contributions to the β -functions. For example, at order v^2 there is a term obtained from expanding the metric up to order ξ^i and the rest up to order ξ^i , such as

$$i(G_{mj} \Gamma_{ik}^m + G_{mi} \Gamma_{jk}^m) \bar{\lambda}^i \gamma^\mu \lambda^\rho \partial_\mu \chi_\rho^L R^{(-)j}{}_{pml} \xi^k \xi^m \quad (3.7)$$

The axial part of the fermion loop gives rise to a contribution to β_{ij}^B of

$$\frac{\alpha'}{2\pi} G_{mn} \Gamma_{lk}^m R^{(-)npk} [\bar{\lambda}^i \gamma^\mu \lambda^l]_p \quad (3.8)$$

This expression has the wrong symmetry properties to be referred to the tangent frame at X_0^i and expressed in terms of the spin connection and identified as part of a Chern-Simons form. It is quite possible that in higher orders in α' there are discrepancies between the beta functions in the gauge and Lorentz sectors which involve the vector as well as axial vector couplings.

The solution to this "puzzle" lies in the choice of fermion variables in which to quantize. Although the classical action actions (3.2) and (3.6) are identical, when one quantizes a theory (i.e., defines a path integral), one can choose whether to define the fermionic part of the measure by $d[\lambda^i]$ or $d[\hat{\lambda}^\alpha]$. In the former case, one is maintaining the right-handed supersymmetry between the quantum corrections ξ^i and λ^i

$$\delta \xi^i = i\sqrt{2} \bar{\epsilon} \lambda^i \quad ; \quad \delta \lambda^i = -\frac{1}{\sqrt{2}} \partial_- X^i \epsilon \quad (3.9)$$

whereas in the latter case the supersymmetry relation

$$\delta X^i = i\sqrt{2} \bar{\epsilon} \lambda^i, \quad \delta \hat{\lambda}^\alpha = \frac{1}{\sqrt{2}} (e_i^\alpha(X) \partial_- X^i + e_\rho^j(X) \hat{\lambda}^\rho \omega_j^\alpha(X) (\tau^a)^{\gamma\delta} \hat{\lambda}^\delta) \quad (3.10)$$

mixes orders of α' through the expansion of the vielbein and spin connection about X_0^i . The former is the (implied) measure in any calculation which uses manifest world-sheet supersymmetry [e.g., Ref. 18)], whereas it is the latter measure which must be used if one is to conclude that the Lorentz sector is analogous to the gauge sector with the gauge connection replaced by the spin connection. In that case one treats $\hat{\lambda}^\alpha$ as a quantum variable and ignores the fact that in terms of the original fermionic superpartners of the bosonic coordinates, λ^i , the $\hat{\lambda}^\alpha$ have an expansion in order α' due to the expansion of the vielbein. At order ξ^i this is equivalent to ignoring S_{1b}^λ above. The measures are related to each other by $\det(e_i^\alpha(X))$ and the expansion of this determinant in a power series in α' is responsible for the difference between beta functions (or any other quantities) calculated in these two quantization schemes.

If, furthermore, one wishes to describe the right-moving fermions of the heterotic string in terms of the $SO(8)$ spinors introduced by Green and Schwarz¹⁹⁾ in the light-cone gauge, these are related to $\hat{\lambda}^\alpha$ by²⁰⁾

$$S^r = \rho_s(X) \gamma_{\alpha}^{rs} \hat{\lambda}^{\alpha} \quad (3.11)$$

where γ_{α}^{rs} are the $SO(8)$ gamma matrices (α runs from 1 to 8), and $\rho_s(X)$ are covariantly constant spinors and consequently X^i -dependent on a curved manifold. Thus, although the classical Green-Schwarz action is equivalent to the NSR spinning string formalism and the quantum theories are also equivalent in flat space (they give the same spectrum once the GSO projection has been imposed), if in the case of a curved manifold we quantize the model by defining a path integral in which we integrate over all spinor fields $d[S^r]$, we are ignoring a further X^i -dependence (and hence α' -dependence) in the comparison between the higher order effective actions for the two formalisms.

We close this section by commenting on the effect of the statement by Hull¹³⁾ that if one replaces the gauge field by $\omega_i^{(+)\alpha}$ (the metric spin connection plus the torsion), the action for the heterotic string reduces to that of the type II superstring. Once again this involves the assumption that λ^α (or S^r) are treated in an analogous way to the fermions ϕ^α which couple to the gauge field, so that the theory is quantized using $d[\hat{\lambda}^\alpha]$ or $d[S^r]$ as the measure in the path integral. If one does this, then the contribution from the diagram of Fig. 1a plus its counterpart for the right-moving fermions to β_{ij}^G is

$$-\frac{\alpha'}{4\pi} \left(R^{(+)\alpha}{}_{ki} R^{(+)\alpha,k}{}_j + R^{(-)\alpha}{}_{ki} R^{(-)\alpha,k}{}_j \right) \quad (3.12)$$

which should cancel the contribution from the purely bosonic diagrams, since β vanishes at two loops for the (1,1) supersymmetric sigma model²¹⁾. However, the anomalous contributions do not quite cancel, since the left- and right-moving fermions give anomalous contributions related to the Chern-Simons forms calculated from $\omega_3^{(+)\text{L}}$ and $\omega_3^{(-)\text{L}}$ respectively, so that redefined torsion has an exterior derivative

$$\partial_\mu H_{ijk} = \frac{3\alpha'}{8} \left(R^{(+)\alpha}{}_{[ki} R^{(+)\alpha}{}_{j]k} - R^{(-)\alpha}{}_{[ki} R^{(-)\alpha}{}_{j]k} \right) \quad (3.13)$$

in agreement with Ref. 22). Since the coupling of torsion to fermions is an axial coupling, then one would not expect the anomalies to cancel in the sigma model for the superstring with non-zero torsion. Any theory for which fermions have both vector and axial vector couplings will in general be anomalous. It has been suggested by Townsend and Hull⁷⁾ that this anomaly can be cancelled by adding the local counterterm

$$\frac{1}{4\pi} \int d^2x \sqrt{g} \Gamma^{(+)k}{}_{li} H_{jk}{}^l \varepsilon^{\mu\nu} \partial_\mu X^i \partial_\nu X^j + \text{supersymmetry completion} \quad (3.14)$$

to the action. The introduction of such a counterterm would certainly affect the order α' beta functions, since one would have to add to them the one-loop contributions obtained from the expansion of Eq. (3.14) about the background string.

SUMMARY

In this paper, a regularization scheme was discussed in which left-moving fermions were continued away from two dimensions while maintaining their helicity properties. It was shown that using this regularization scheme, the covariant

derivative of the gauge current gave the required anomaly term by explicit calculation. It was then shown by expanding about a background string up to quadratic order in the quantum corrections and calculating the contributions to beta functions from the resulting diagrams that both β^G and β^B acquire anomalous terms which are cancelled by adding the full gauge Chern-Simons form to the torsion and substituting the redefined torsion into the one-loop expressions for the beta functions. This underwrites the statement that the anomalies are cancelled by the introduction of such Chern-Simons terms. It has been suggested⁷⁾ that this cancellation should work to all orders in the expansion parameter α' . The cancellation found in the two-loop beta functions does not occur in any obvious way, but it may have been expected since two loops is the lowest order at which the beta functions acquire contributions from fermion-loop anomalies. Although the Adler-Bardeen theorem tells us that one should expect no further corrections to the axial anomalies, the introduction of explicit gauge dependence into the antisymmetric tensor field B_{ij} results in a classical action which is not gauge invariant. In general, the introduction of gauge-dependent counterterms into a field theory generates gauge-dependent S-matrix elements as one goes to higher order in perturbation theory^{*)}. It is therefore quite possible that the expansion of these counterterms about a background string will give rise to higher order interactions which will lead to gauge dependence of the effective action (or other S-matrix elements), which are not absorbed by the anomalous contributions from fermion loops, thereby necessitating the introduction of higher-order gauge-dependent counterterms in order to render the quantum theory gauge invariant. Explicit higher-order calculations are required to determine the presence or absence of such extra gauge dependent counterterms.

For the Lorentz sector it was pointed out that the interaction term in which the spin connection couples to the divergence of the fermion current can never appear in the normal co-ordinate expansion, since such a co-ordinate system sets the spin connection locally to zero. It was further pointed out that in a model with Lorentz anomalies the normal co-ordinate expansion is not valid, since this automatically leads to gauge invariant interactions only. One is therefore obliged to perform a linear expansion in this sector, and in general the expansion of the metric multiplying the fermionic kinetic term leads (through the anomalies) to contributions to beta functions which are not of the form that can

*)

This is in contrast to the introduction of gauge dependent counterterms in an effective field theory, whose action is to be treated as the generating functional of all Green functions and not used to generate higher loops.

be absorbed by Chern-Simons terms. If, however, one chooses to quantize the theory by defining the quantum fermion field to be the fermion field referred to the tangent frame, then the action expanded about a background is analogous to that obtained in the gauge sector and so the anomalous contributions to the beta functions are cancelled by substituting the torsion term by a redefined torsion with an analogous Chern-Simons term, defined by replacing the gauge field by $\omega_i^{(-)}$, subtracted. Such a choice of quantization breaks the original world-sheet supersymmetry between the right-moving bosonic and fermionic co-ordinates and would thus be expected in general to give different results in higher orders from calculations performed using manifest world-sheet supersymmetry. On the other hand, such a choice should be equivalent to the formulation of the heterotic string in terms of the Green-Schwarz $SO(8)$ spinors which are related to the NSR fermions only after they have been referred to the tangent frame.

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FIGURE CAPTIONS

Fig. 1 : Diagrams contributing to the gauge part of β^G at two loops. The continuous line is a fermion field, ϕ^α , the wavy line is a boson quantum co-ordinate ξ^i and the broken line is the insertion of $\partial_\mu X_\mu^i$. A blob represents the interaction with the background field and a cross indicates an interaction involving the covariant divergence of the fermion current.

Fig. 2 : Two-loop diagrams giving rise to the gauge field contribution to β^B (same notation as Fig. 1).

- Figure 1 -

- Figure 2 -