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ACCIDENTAL DEGENERACIES IN STRING COMPACTIFICATION

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A B S T R A C T

The equivalence of the torus and group manifold compactification of strings is established. Accidental degeneracies are shown to occur for a large class of compactifications. This way many examples are obtained in which modular invariance does not uniquely fix the representation content of the spectrum.

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In this paper we investigate the relation between various conformally invariant two-dimensional theories which have a current algebra structure, i.e. models whose dynamics is governed by a Kac-Moody-Virasoro (KMV) algebra $\hat{G}$. Throughout we assume the stress energy tensor L to be of the Sugawara type. Theories of this type include the WZW-models, (free) fermion models and are of vital importance in the study of string compactification both over tori (orbifolds) and group manifolds.

We will first analyse the KMV representation content of the torus compactification of the bosonic string, and show that for the closed string it coincides with the group manifold compactification, assuming of course G to be simply laced and the level (central extension of the KM algebra) to be $k = 1$. Then we turn to an analysis of the "subgroup truncations", which were proposed in reference 1 and classified in reference 2. In this context one considers KMV-subalgebras based on subalgebras $\bar{G} \subset G$, which satisfy the constraint that the central charge $D(k)$ in the Virasoro algebras of $\hat{G}$ and $\hat{\bar{G}}$ is the same. In that case the irreducible representations of the G Kac-Moody algebra are reducible under the Virasoro algebra. This leads to various interesting observations concerning the Hilbert space of the KMV algebra $\hat{G}$.

We show that for the cases under consideration the $\hat{\bar{G}}$ invariant spectrum has accidental degeneracies which enlarge the underlying symmetry group from $\bar{G}$ to G . Furthermore it may happen that, because of this accidental degeneracy, there is not a unique modular invariant spectrum. Explicit examples will be given towards the end of the paper.

For an open bosonic string compactified on the r -dimensional torus of a simply laced group G ($r = \text{rank } G$), the contribution of the compactified dimensions to L_0 is

$$L_0^{(c)} = \frac{1}{2} \sum_{I=1}^r P_I^I P_I + N(\alpha^I) \quad (1)$$

where $N(\alpha^I)$ is the total number operator for the oscillators in the compactified dimensions, and $\vec{P}$ the compactified momentum, quantized on the weight lattice Λ^* of G . The Frenkel-Kac construction [3], shows that this model possesses a $k = 1$ KMV algebra based on the nonabelian algebra G rather than the abelian algebra $(U_1)^r$. Our first objective is to determine the KMV representation content of the spectrum (1). Let a weight λ of $\hat{G}$ be denoted as $\lambda = (\bar{\lambda}, k, n)$ where $\bar{\lambda}$ is a weight of a representation of the finite dimensional algebra G , k is the level and n the grade. For a highest weight representation $\hat{R}$ of $\hat{G}$, the weights can be obtained by acting with a suitable sequence of current operators J_m^a on a highest weight vector $\Lambda = (\bar{\Lambda}, k, 0)$ where $\bar{\Lambda}$ is a highest weight vector for some representation R of G . To compute the weight multiplicities $m(\lambda)$ of a weight $\lambda \in \hat{R}$, it is most convenient to use the generalization of the recursive Freudenthal formula:

$$\begin{aligned} m(\lambda) \{(\bar{\Lambda} + \bar{\rho}, \bar{\Lambda} + \bar{\rho}) - (\bar{\lambda} + \bar{\rho}, \bar{\lambda} + \bar{\rho}) - 2n(g + k)\} = \\ = 2 \sum_{\alpha > 0} \sum_{\ell=1}^{\infty} m(\lambda') \{(\bar{\lambda} + \ell\bar{\alpha}, \bar{\alpha}) + k\ell\} \end{aligned} \quad (2)$$

The highest weight $\bar{\Lambda}$ has unit multiplicity. $m(\lambda)$ is then obtained from the multiplicities of weights $\lambda' = (\bar{\lambda} + \ell\bar{\alpha}, k, n + \ell m)$ which differ from λ by a positive root. The vector $\rho = (\bar{\rho}, g, 0)$ is defined by $\bar{\rho} = \frac{1}{2} \sum_{\alpha > 0} \bar{\alpha}$ and the dual Coxeter number g of G . Using the Sugawara form of the stress energy tensor one finds for $L_0^{(c)}$

$$L_o^{(c)} = \frac{C_R}{k+C_A} + N(J_m^a) \quad (3)$$

where C_A and C_R are the quadratic Casimirs of the adjoint representation and the representation R . $N(J_m^a)$ is minus the total grade of the string of current operators needed to go from $\bar{\Lambda}$ to λ , i.e. $N = |\mathbf{n}|$. As we will see the representation theory of a KM algebra is greatly facilitated by extending it to a KMV algebra and thereby including the quantum number (3) corresponding to $L_o^{(c)}$.

Because the theory is of the Sugawara type, we may compare the expressions (1) and (3) with $k = 1$, to obtain that the only highest weight representations that can be contained in (1) have to satisfy the relation

$$\frac{C_R}{1+C_A} = \frac{1}{2} (\bar{\Lambda}, \bar{\Lambda}) \quad (4)$$

Using that $C_R = \frac{1}{2}(\bar{\Lambda} + 2\bar{\rho}, \bar{\Lambda})$ and $C_A = (\bar{\theta} + 2\bar{\rho}, \bar{\theta})$ (where $\bar{\theta}$ is the highest root of G , normalized such that $\bar{\theta}^2 = 2$), it is easy to show that (4) is equivalent to

$$(2\bar{\rho}, \bar{\Lambda}) = g(\bar{\Lambda}, \bar{\Lambda}) \quad (5)$$

It is then straightforward to prove that for a simply laced algebra (5) is satisfied if and only if $\bar{\Lambda} = 0$ or $(\bar{\Lambda}, \bar{\theta}) = 1$, i.e. $\bar{\Lambda}$ is the singlet or a fundamental dominant weight. Therefore the torus compactification of the open string contains each KMV representation $\hat{R}$ for which $1 - (\bar{\Lambda}, \bar{\theta}) > 0$ exactly once. These are by definition the so-called integrable representations of the $k = 1$ algebra $\hat{G}$, for which the algebra can be exponentiated [4], to a group.

Consider as an example the integrable $\hat{SU}_2$ representation of level $k = 1$, given by $\bar{\Lambda} = 0$ and $\bar{\Lambda} = \frac{1}{2} \bar{\theta}$. We denote them by $\hat{1}$ and $\hat{2}$. For the first one $C_R = 0$ so that in view of (1) and (3) one should take $\vec{P} = r\theta$, $r \in \mathbb{Z}$ i.e. $\vec{P}$ is on the root lattice of SU_2 . For the second representation $\hat{2}$, $C_R/(1+C_A) = \frac{1}{2}$ and we have to take $\vec{P} = \frac{1}{2}(2r+1)\bar{\theta}$, $r \in \mathbb{Z}$. It is interesting to note that the equivalence of (1) and (3) provides a very efficient method to compute the multiplicities of weights $m(\lambda)$ explicitly. For the $\hat{2}$ representation we have

$$n = r^2 + r + n(\alpha^I) \quad (6)$$

where n is the eigenvalue of $N(J_m^a)$ or minus the grade and $n(\alpha^I)$ the eigenvalue of $N(\alpha^I)$. At fixed grade, one can solve (6) for $r \in \mathbb{Z}$, and $n(\alpha^I)$ the eigenvalue of $N(\alpha^I)$. To each value of $n(\alpha^I)$ correspond $p_n(1)$ states, with $p_n(1)$ the partition function of $n(\alpha^I)$ with only one type of oscillator for SU_2 .

Let us now turn to the torus compactification of closed strings, where one obtains a $\hat{G} \times \hat{G}$ KMV algebra corresponding to the left and right moving sectors. The $L_o^{(c)}$ operator becomes

$$L_o^{(c)} = \frac{\vec{W}^2}{2} + N(\alpha^I) + \frac{\tilde{\vec{W}}^2}{2} + \tilde{N}(\tilde{\alpha}^I) \quad (7)$$

where $\vec{W}$ and $\tilde{\vec{W}}$ are on the weight lattice Λ^* of G , and are furthermore constrained such that their difference is on the root lattice [5].

$$\vec{W} - \tilde{\vec{W}} \in \Lambda \quad (8)$$

For a highest weight representation of $\hat{G} \times \hat{G}$ we have on the other hand (3)

$$L_o^{(c)} = \frac{C_R}{k+C_A} + N(J_m^a) + \frac{C_{\tilde{R}}}{k+C_A} + \tilde{N}(\tilde{J}_m^a). \quad (9)$$

It then follows that if $k = 1$, both $\hat{\Lambda}$ and $\tilde{\Lambda}$ have to satisfy $1 - (\bar{\Lambda}, \theta) > 0$, which combined with (8) yields that in fact $\bar{\Lambda} = \tilde{\Lambda}$. In other words the torus compactification of a closed string yields a spectrum $\sum_R (\hat{R}, \hat{R})$ where the sum extends over all integrable representations of $\hat{G}$. This result coincides with that of Gepner and Witten [6], who considered the compactification of the closed string over a group manifold G , and it establishes the expected equivalence of torus and group manifold compactifications.

Let us now turn to the problem of KMV subalgebras $\hat{\bar{G}} \subset \hat{G}$. We are in particular interested in those subalgebras, for which the Sugawara tensors L and $\bar{L}$ are the same [1]. In that case one has to demand

$$\bar{D}(\bar{k}) = D(k) \quad (10)$$

where $D(k)$ is the central charge of the Virasoro algebra

$$D(k) \equiv \frac{k \dim G}{k+C_A} \quad (11)$$

The embedding of $\hat{\bar{G}} \subset \hat{G}$ is induced by some embedding $\bar{G} \subset G$. The central charges are related through $\bar{k} = jk$ where j is the Dynkin index of the embedding $\bar{G} \subset G$ [1]. It turns out that equation (10) has a large number of solutions but only for $k = 1$, hence our analysis is in particular relevant for the torus compactifications discussed before. We refer to reference 2 for a discussion of the characteristic properties of the embeddings. To determine the decomposition of the Hilbert space

under the subalgebra $\hat{\bar{G}}$, one has to find the branching rules for integrable representations of KMV algebras. For the cases we are interested in where $L_0^{(c)} = \bar{L}_0^{(c)}$ this problem becomes very simple. Suppose we want the branching of some irreducible integrable $\hat{G}$ representation $\hat{R}$, into a sum of irreducible (also integrable) $\hat{\bar{G}}$ representations $\hat{\bar{R}}_i$:

$$\hat{R} = \sum_i \hat{\bar{R}}_i, \quad (12)$$

then the only representation $\hat{\bar{R}}_i$ that can appear on the right-hand-side must have the property that their $L_0^{(c)}$ eigenvalues differ from those of $\hat{R}$ by integers:

$$\frac{C_R}{1+C_A} = \frac{\bar{C}_{\bar{R}_i}}{\bar{R}+\bar{C}_A} + m. \quad (13)$$

This is an extremely restrictive criterion, greatly simplifying the determination of branching rules for embedding satisfying (10).

We give an important class of examples to (13). Most solutions to (10) correspond to isotropy irreducible spaces $G/\bar{G}$, i.e. one has that $Ad = \bar{Ad} + \bar{R}[+\bar{R}^*]$, where $\bar{R}$ is irreducible. The term in brackets is only present if the space is real but not complex isotropy irreducible. For these embeddings there is a simple expression for $\bar{k}$:

$$\frac{\bar{k}}{k} = j \equiv \frac{\lambda(\bar{A}) + \lambda(\bar{R}) [+ \lambda(\bar{R}^*)]}{\lambda(A)} \quad (14)$$

where the index $\lambda(R)$ is defined as

$$\lambda(R) = \frac{1}{2} \frac{\dim R}{\dim G} C_R \quad (15)$$

It is then simple to show, using (10), that for the coset representation $C_{\bar{R}}/(\bar{k} + C_{\bar{A}}) = 1$, so that the identity representation $\hat{1}$ of $\hat{G}$ and the coset representation $\hat{R} [\hat{R}^*]$ have degeneracies indeed. In general one expects the sum in (12) to be infinite [7], and the branching to contain highest weight vectors $\bar{\Lambda}$ at an infinity of different grades. However, for the class of embeddings satisfying (10), one obtains the important mathematical result that the integrable representation $\hat{R}$ has a finite branching into integrable representations $\hat{R}_i$. A proof that condition (10) is necessary and sufficient for this to occur has been given [8].

In the remainder of the paper we work out some representative examples and discuss their physical implications. We first consider the special SO_3 embedding in SU_3 defined by the branching $3 = 3$ ($8 = 3 + 5$); which has index $j = 4$. The branching rules for the integrable $k = 1$ $\hat{SU}_3$ representations are $\hat{1} = \hat{1} + \hat{5}$, $\hat{3} = \hat{3}$ and $\hat{3}^* = \hat{3}$. Consequently the $\hat{SU}_3 \times \hat{SU}_3$ closed string spectrum $(\hat{1}, \hat{1}) + (\hat{3}, \hat{3}) + (\hat{3}^*, \hat{3}^*)$ decomposes under $\hat{SO} \times \hat{SO}_3$ as $(\hat{1}, \hat{1}) + 2(\hat{3}, \hat{3}) + (\hat{5}, \hat{5}) + (\hat{1}, \hat{5}) + (\hat{5}, \hat{1})$. This is exactly the modular invariant $\bar{k} = 4$ spectrum of the closed string compactification over the (not simply connected) SO_3 manifold [6]. So, what we have demonstrated here is that this SO_3 compactification with $\bar{k} = 4$ has accidental degeneracies which enlarge the symmetry in fact to SU_3 . In other words, the $\bar{k} = 4$ SO_3 and $k = 1$ SU_3 WZW models are equivalent as far as their spectrum is concerned. A similar situation arises for example for the SU_3/Z_3 theory with $\bar{k} = 3$, which has in fact a Spin 8 invariance. This is easily verified using the branching rules $\hat{1} = \hat{1} + \hat{10} + \hat{10}^*$ and $\hat{8}_{v,c,s} = \hat{8}$.

There is another type of situation which can occur as for example in the case where $\hat{G} = \hat{SU}_6$ and $\bar{\hat{G}} = \hat{SU}_3$ with $\bar{k} = 5$. The modular invariant $\hat{SU}_6$ spectrum with $k = 1$ is $(\hat{1}, \hat{1}) + (\hat{6}, \hat{6}) + (\hat{6}^*, \hat{6}^*) + (\hat{15}, \hat{15}) + (\hat{15}^*, \hat{15}^*) + (20, 20)$. Using the branching rules $\hat{1} = \hat{1} + \hat{27}$, $\hat{6} = \hat{6}$, $\hat{15} = \hat{15}$ and $\hat{20} = \hat{10} + \hat{10}^*$, we obtain a modular invariant $\hat{SU}_3$ spectrum with $\bar{k} = 5$. This spectrum is however notably different from the standard one obtained in reference 6, which contains representations up to $(\hat{42}, \hat{42})$. A similar situation arises for the case $\hat{E}_6 \supset \hat{Sp}_8$ with $k = \bar{k} = 1$. The $\hat{E}_6$ spectrum is simply $(\hat{1}, \hat{1}) + (\hat{27}, \hat{27}) + (\hat{27}^*, \hat{27}^*)$. The corresponding $\hat{Sp}_8$ spectrum becomes $(\hat{1}, \hat{1}) + (\hat{27}, \hat{27}) + (\hat{42}, \hat{42}) + (\hat{42}, \hat{1}) + (\hat{1}, \hat{42})$ whereas the canonical solution yields $(\hat{1}, \hat{1}) + (\hat{8}, \hat{8}) + (\hat{27}, \hat{27}) + (\hat{42}, \hat{42}) + (\hat{48}, \hat{48})$. The lesson to be drawn from these examples is that because of the occurrence of accidental degeneracies, there are different modular invariant $\hat{G} \times \bar{\hat{G}}$ spectra possible. We observe that in these cases the $\hat{G}$ spectrum is not a truncation of the standard $\bar{\hat{G}}$ spectrum because of the cross terms. Nor is the converse correct, which implies that our original motivation, to obtain the $\bar{\hat{G}}$ spectrum as a truncation of the $\hat{G}$ spectrum, cannot be maintained if one insists on modular invariance. Then our original interpretation should be replaced by the one given in this paper.

In the same sense the generalization of Witten's non abelian bosonisation [9], as advocated in reference 10, does not hold in general. This is clear from the examples given above, i.e. the $\bar{\hat{G}}$ spectrum obtained from a G WZW model does not always correspond to the spectrum of $\bar{G}$ WZW model.

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