SUPERSYMMETRIZATION OF $\zeta(3)(R_{\mu\nu\rho\sigma})^4$ IN SUPERSTRING THEORIES

B.E.W. Nilsson

CERN -- Geneva

and

A.K. Tollstén^{*)}

Institute for Theoretical Physics
S-41296 Göteborg, Sweden

ABSTRACT

We show that the $(R_{\mu\nu\rho\sigma})^4$ term that appears with a transcendental coefficient in the low-energy effective action for heterotic, type I and type II superstrings has a trivial extension to $d = 10$, $N = 1$ superspace. The simplicity of our construction is due to the fact that in ten dimensions, on-shell $N = 1$ supergravity is described by a scalar superfield.

^{*)} Address after 1 September 1986: Nordita, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.

Recently, much work has been devoted to the calculation of higher order corrections to the low energy effective action of various string theories. Of particular interest are supersymmetric string theories^{1,2)} both for phenomenological reasons and because they seem to provide us with a consistent quantum theory of gravity. It has been known for some time that to lowest order in number of derivatives the effective action corresponds to $d = 10$, $N = 2$ supergravity in the case of the type II superstring,³⁾ and to $d = 10$, $N = 1$ supergravity coupled to the supersymmetric Yang-Mills theory both in the case of the open type I superstring¹⁾ and in the case of the heterotic string²⁾. We will here confine our discussion to $N = 1$ superstrings. This is because, contrary to the $N = 2$ case, $N = 1$ superspace supergravity is quite well understood both on-shell⁴⁻¹⁰⁾ and off-shell^{11,5,12,13)}, and is simple enough to make possible a discussion, and in one case an explicit construction (see below), of manifestly supersymmetric higher order invariants.

Turning to the next to lowest order corrections, it was conjectured by Candelas et al.¹⁴⁾ guided by the results of Green and Schwarz in Ref. 15), that there must be a term of the form $\alpha'(\text{tr}F^2 - \text{tr}R^2)$ in the low-energy effective Lagrangian for the above-mentioned $N = 1$ superstrings. This was later verified in the case of the heterotic string in Refs. 2) and 16). Here of course the $\text{tr}F^2$ term is the one appearing already in the coupled supergravity-Yang-Mills theory. This theory has been constructed both in components¹⁷⁻¹⁹⁾ and in superspace^{20,6,7,9,10)}. What then remained to be done at this order in α' was the supersymmetrization of the $\alpha'\text{tr}R^2$ term. This was partly achieved in the usual formulation of supergravity, i.e., with a two-form potential, in Refs. 21) and 22) and carried a step further to completion using the dual six-index field formulation by the authors of Refs. 23) and 24). [The method of using superspace Bianchi identities in this context was first suggested in Ref. 20).] The same results were recently obtained using the group manifold approach²⁵⁾. The nature of the higher derivative terms in $d = 10$ has also been investigated by compactifying to $d = 4$ and employing $d = 4$ tensor calculus²⁶⁾.

The next term in the low energy effective action to be discovered was the $\zeta(3)\alpha'^3(R_{\mu\nu\rho\sigma})^4$ term. It was first found in a four-loop σ -model

calculation^{27)*}, and later read off from superstring four-point scattering amplitudes, calculated in Refs. 1) and 29), by the authors of Refs. 30) and 16). In Ref. 30) a very simple expression [see (1) below] for this otherwise rather complicated amplitude²⁹⁾ was presented. This simple expression is most easily understood by considering the one-loop amplitude instead of the tree amplitude. Doing so one finds³⁰⁾

$$Y = \int d\psi_L^\alpha d\psi_R^{\alpha'} \exp \left[\psi_L^\alpha \Gamma_{\alpha\beta}^{\mu\nu} \psi_L^\beta \psi_R^{\alpha'} \Gamma_{\alpha'\beta'}^{\rho\sigma} \psi_R^{\beta'} R_{\mu\nu\rho\sigma} \right], \quad (1)$$

where the spinors ψ and Γ - matrices are eight dimensional and subscripts L and R refer to left and right chirality. (1) can be evaluated and seen to give rise to the expression one naturally gets from the tree-level four-point scattering amplitude^{1,29,16)}. Explicitly, we get in this case

$$Y = t^{\mu_1 \dots \mu_8} t^{\nu_1 \dots \nu_8} R_{\mu_1 \mu_2 \nu_1 \nu_2} \dots R_{\mu_7 \mu_8 \nu_7 \nu_8} \quad (2)$$

where, using $\delta_{\gamma\delta}^{\alpha\beta} \equiv \frac{1}{2}(\delta_\gamma^\alpha \delta_\delta^\beta - \delta_\delta^\alpha \delta_\gamma^\beta)$,

$$\begin{aligned} t^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} = & -\frac{1}{2} \varepsilon^{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \mu_4 \nu_4} \\ & - 2 \left[\delta_{\mu_2 \nu_2}^{\mu_1 \nu_1} \delta_{\mu_4 \nu_4}^{\mu_3 \nu_3} + \delta_{\mu_3 \nu_3}^{\mu_2 \nu_2} \delta_{\mu_1 \nu_1}^{\mu_4 \nu_4} + \delta_{\mu_2 \nu_2}^{\mu_1 \nu_1} \delta_{\mu_4 \nu_4}^{\mu_3 \nu_3} \right] \\ & + \frac{1}{2} \left[\delta_{\mu_2 \nu_2}^{\nu_1} \delta_{\mu_3 \nu_3}^{\nu_2} \delta_{\mu_4 \nu_4}^{\nu_3} \delta_{\mu_1 \nu_1}^{\nu_4} + \delta_{\mu_3 \nu_3}^{\nu_1} \delta_{\mu_2 \nu_2}^{\nu_2} \delta_{\mu_4 \nu_4}^{\nu_3} \delta_{\mu_1 \nu_1}^{\nu_4} + \right. \\ & \left. + \delta_{\mu_3 \nu_3}^{\nu_1} \delta_{\mu_4 \nu_4}^{\nu_2} \delta_{\mu_2 \nu_2}^{\nu_3} \delta_{\mu_1 \nu_1}^{\nu_4} + 45 \text{ terms (antisymm. in pairs } (\mu_i \nu_i)) \right] \end{aligned} \quad (3)$$

The purpose of this note is to present the $d = 10$ covariant supersymmetrization of Y . This must be quite an arduous task if attempted with component techniques. Here we will instead use superfield methods. The answer is then very easy to obtain and extremely simple. We point out that our result also seems to be valid off-shell.

In fact, in order to construct the superspace version of Y , all we need to know is that there exists an on-shell formulation of $N = 1$ supergravity in superspace, and what kind of superfields are needed to describe the on-shell degrees of freedom. For the extension off-shell, we need the analogous information for the off-shell theory. Both these pieces of information are

* The fact that such a term is not ruled out by general σ - model arguments was pointed out in Ref. 28).

at hand: the on-shell scalar superfield $\Phi(Z)$ was discovered a few years ago⁴⁾ and shortly afterwards the linear off-shell formulation was developed¹¹⁾. Some recent work^{12,13)} has also established the connection between this linear off-shell formulation and the geometric quantities appearing in the Bianchi identities, i.e., the super torsion T_{AB}^C and the super curvature R_{ABC}^D .

We will now briefly review the off- and on-shell formulations of $N = 1$ supergravity, and then turn to the construction of the superspace invariant that supersymmetrizes the quantity Y in (1). [We will use the form of Y given by (1) for reasons that will become obvious later.]

It was shown by Howe et al.¹¹⁾ that the prepotential of (linear) $N = 1$ supergravity in ten dimensions is a superfield $V_{\alpha\beta\gamma}(Z)$ antisymmetric in its three vector indices. This superfield couples to the supercurrent of the super Yang-Mills theory, a fact from which one can deduce that $V_{\alpha\beta\gamma}$ has the following gauge transformation¹¹⁾:

$$\delta V_{\alpha\beta\gamma} = \bar{D}^a \Lambda_{\alpha\beta\gamma a} + \bar{D}^a (\gamma_{\alpha\beta} \gamma_a)_{\gamma}^b D_b L_{\gamma}^{\delta} + 28 D_{[\alpha} L_{\beta\gamma]}, \quad (4)$$

where the parameters $\Lambda_{\alpha\beta\gamma a}$ and $L_{\alpha\beta}$ transform as irreducible representations of $SO(1,9)$ and contain both Wess-Zumino components and the parameters of the physical symmetries. Introducing another unconstrained superfield, $\Phi(Z)$, one can construct the following superspace action which contains the linear Einstein-Hilbert action¹¹⁾:

$$S = \int d^{10}x d^{16}\theta \left[\frac{1}{2} V_{\alpha\beta\gamma} G_{\alpha\beta\gamma} - V_{\alpha\beta\gamma} \bar{D} \gamma_{\alpha\beta\gamma} D \Phi \right]. \quad (5)$$

Here $G_{\alpha\beta\gamma}$ is obtained from $V_{\alpha\beta\gamma}$ by applying 16 spinor derivatives, but is more easily derived starting from its lowest component¹¹⁾

$$G_{\alpha\beta\gamma}^{(0)} = \tilde{F}_{\alpha\beta\gamma} - \frac{8}{3} V_{\alpha\beta\gamma}^{(16)}, \quad (6)$$

where $\tilde{F}_{\alpha\beta\gamma}$ is the dual of the field strength of the six-form potential appearing at the 14th θ -level in $V_{\alpha\beta\gamma}$. The (linearized) equations of motion derived from (5) are

$$\bar{D} \gamma_{\alpha\beta\gamma} D V_{\alpha\beta\gamma} = 0 \quad , \quad (7)$$

$$G_{\alpha\beta\gamma} = \bar{D} \gamma_{\alpha\beta\gamma} D \Phi \quad . \quad (8)$$

This brings us to the on-shell theory. The on-shell $N = 1$ Poincaré supergravity theory is in superspace described completely by one scalar superfield $\Phi(Z)$. This was first shown by solving the relevant superspace Bianchi identities⁴⁾. Of course, from the equations of motion (7) and (8) one immediately arrives at the same conclusion. The on-shell superfield $\Phi(Z)$ contains in its components all the on-shell degrees of freedom. In fact, the beginning of its θ expansion reads [following the conventions of Refs. 5) and 6)]

$$\begin{aligned} \Phi|_0 &= \phi \quad , \quad D_a \Phi|_0 = i \lambda_a \quad , \\ D_{[a} D_{b]} \Phi|_0 &= -\frac{i}{72} (\gamma^{\alpha\beta\gamma})_{ab} H_{\alpha\beta\gamma} + \dots \quad , \\ D_{[a} D_b D_{c]} \Phi|_0 &= -\frac{1}{24} (\gamma^{\alpha\beta\gamma})_{[ab} T_{\gamma}{}^d (\gamma_d)_{c]} + \dots \quad , \\ D_{[a} D_b D_c D_{d]} \Phi|_0 &= -\frac{1}{96} (\gamma^{\alpha\beta\gamma\epsilon})_{[ab} (\gamma^{\delta\epsilon})_{cd]} R_{\alpha\beta\gamma\delta} + \dots \quad , \end{aligned} \quad (9)$$

where the fields ϕ , λ_a , $H_{\alpha\beta\gamma}$, $T_{\alpha\beta}{}^c$ and $R_{\alpha\beta\gamma\delta}$ are the dilaton, dilatino, three-form field strength, Rarita-Schwinger field strength and Riemann tensor respectively. [Note, however, that the antisymmetrization on the spinor indices in (9) picks out the representations 560 and 770 of $SO(1,9)$ at the third and fourth θ -level respectively.]

We now turn to the supersymmetrization of Y [interpreted in ten-dimensional Minkowski space^{1,27,28)}]. The similarities between the way the Riemann tensor appears in (9) and the exponent in (1) immediately suggests that the $d = 10$ superspace invariant containing Y is (at the linear level)

$$L^{(4)} \approx \Phi^4(z) \quad . \quad (10)$$

It is then clear that

$$S^{(4)} \approx (\alpha')^3 \zeta(3) \int d^{10}x d^{16}\theta E e^{-6\Phi} \quad , \quad (11)$$

where E is the superfield whose 0th component is $\text{sdet}(E_M^A)^{11}$, generates upon performing the θ -integrations a term of the form

$$S_Y \approx (\alpha')^3 \zeta(3) \int d^{10}x \sqrt{-g} e^{-6\phi} Y \quad . \quad (12)$$

The reason the factor $e^{-6\phi}$ appears in the non-linear result in (12) is that the component fields used in (9) are the manifestly scale invariant ones^{20,6)}. In terms of these fields the x-space Lagrangian takes the general form (at the string tree level)

$$L \sim e^{-6\phi} (\text{scale invariant}) \quad . \quad (13)$$

To show that (10) contains a term of the form (1) one proceeds as follows. The first step is to perform a reduction to the light-cone of the ten-dimensional expression $\theta_Y^{\alpha\beta\epsilon} \theta_Y^{\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}$. By dropping all terms where at least one of the indices on the Riemann tensor equals + or -^{1,27,28,16)} (recall that in light-cone co-ordinates we have $x^\alpha x_\alpha = x^i x^i - 2x^+ x^-$) one directly arrives at the expression appearing in the exponent of Y in (1). Thus we conclude that at the linear level (10) must be the covariant supersymmetrization of Y on-shell.

In fact, as far as supersymmetry is concerned, (11) seems to make sense also off-shell. If so, by adding the superspace kinetic term of $d = 10, N = 1$ supergravity recently conjectured by Howe and Umerski¹³⁾ the non-linear off-shell superspace action corresponding to $R + Y$ becomes

$$S = \int d^{10}x d^{16}\theta E e^{-6\Phi} [W + c \zeta(3)] \quad , \quad (14)$$

where c is some normalization constant and W is a scalar superfield derived from the prepotential¹³⁾. By varying with respect to $\Phi(Z)$ we obtain one of

the two superspace equations of motion

$$W + c \, \zeta(3) = 0 \quad . \quad (15)$$

This shows that the action (14) vanishes on-shell and that the lowest component of the auxiliary superfield W acquires a constant non-zero value. It is not yet possible to derive the second field equation from (14) since the exact definitions of E and W in terms of the prepotential are not known at the non-linear level. [For the linear relations see 13).]

Finally, we would like to make some comments about the supersymmetrization of $(R_{\alpha\beta\gamma\delta})^2$ terms and higher derivative terms beyond $(R_{\alpha\beta\gamma\delta})^4$. In the former case, under the proviso that there exists an on-shell superspace invariant, one is faced with a new problem since it appears to be necessary to integrate over only part of superspace, i.e., to use measures like $d^n\theta$ where $n < 16^*$. Such constructions have previously been considered in $d = 6^{31)}$ and in $d = 4^{32)}$. As regards the latter kind of terms $[(R_{\alpha\beta\gamma\delta})^n, n > 4]$, they should be more readily constructed using explicit spinor derivatives. For instance, at the linear level, $\int d^{10}x (R_{\alpha\beta\gamma\delta})^8$ is most likely contained in a superspace invariant of the form

$$\int d^{10}x \, d^{16}\theta \, (D^2 \Phi)^8 \quad . \quad (16)$$

ACKNOWLEDGEMENT

We are grateful to I. Bengtsson, L. Brink, M.J. Duff, S. Ferrara, M. Green, P. Howe, and A. Neveu for interesting and useful discussions.

*This may be necessary even for the R^4 term that is part of $(\text{tr}R^2 - \text{tr}F^2)^2$.

REFERENCES

- 1) M.B. Green and J.H. Schwarz - Nucl. Phys. B181 (1981) 502 and B198 (1982) 252.
- 2) D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm - Nucl. Phys. B256 (1985) 253 and B267 (1986) 75.
- 3) M.B. Green and J.H. Schwarz - Phys. Lett. 109B (1982) 444.
- 4) B.E.W. Nilsson - Nucl. Phys. B188 (1981) 176.
- 5) B.E.W. Nilsson and A.K. Tollstén - Phys. Lett. 169B (1986) 369.
- 6) B.E.W. Nilsson and A.K. Tollstén - Phys. Lett. 171B (1986) 212 and Goteborg preprint 86-8 (1986).
- 7) J.J. Atick, A. Dhar, and B. Ratra - Phys. Rev. D33 (1986) 2824.
- 8) E. Witten - Nucl. Phys. B266 (1986) 245.
- 9) S.J. Gates and H. Nishino - Phys. Lett. 173B (1986) 46.
- 10) S.W. MacDowell and M.T. Rakowski - Yale preprint YTP 86-01 (1986).
- 11) P.S. Howe, H. Nicolai and A. van Proeyen - Phys. Lett. 112B (1982) 446.
- 12) B.E.W. Nilsson - Phys. Lett. 175B (1986) 319.
- 13) P.S. Howe and A. Umerski - King's College preprint (1986).
- 14) P. Candelas, G. Horowitz, A. Strominger and E. Witten - Nucl. Phys. B258 (1985) 46.
- 15) M.B. Green and J.H. Schwarz - Phys. Lett. 149B (1984) 117.
- 16) Y. Cai and C.A. Núñez - Texas University preprint, UTTG-13-86 (1986)
- 17) A.H. Chamseddine - Phys. Rev. D24 (1981) 3065.
- 18) E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen - Nucl. Phys. B195 (1982) 97.
- 19) G.F. Chapline and N.S. Manton - Phys. Lett. 120B (1983) 105.
- 20) R.E. Kallosh and B.E.W. Nilsson - Phys. Lett. 167B (1986) 320.
- 21) L.J. Romans and N.S. Warner - Nucl. Phys. B273 (1986) 320.
- 22) S.K. Han, J.K. Kim, I.G. Koh and Y. Tanii - Preprint KAIST 85/22 and TIT-HEP (1985).

- 23) S.J. Gates and H. Nishino - Phys. Lett. 173B (1986) 52.
- 24) A.H. Chamseddine and P. Nath - Beirut (American University) preprint AUB - 86:1 (1986).
- 25) S. Ferrara, P. Fré and M. Porrati - preprint UCLA/86/TEP/20 (June 1986).
- 26) S. Cecotti, S. Ferrara, L. Girardello and M. Porrati - Phys. Lett. 164B (1985) 46;
S. Ferrara, in Proceedings of the Second Nobel Symposium on Elementary Particle Physics, Marstrand, Sweden, (June 1986), (CERN preprint TH.4497/86).
- 27) D. Zanon - Harvard preprint HUTP - 85/AO64 (1985);
M.T. Grisaru, A.E.M. van de Ven and D. Zanon - Harvard preprints HUTP - 86/AO20 (BRX - TH - 196) and HUTP - 86/AO27 (BRX - TH - 199) (1986).
- 28) C.N. Pope, M.F. Sohnius and K.S. Stelle - Imperial College preprint TP/85-86/16 (March 1986).
- 29) J.H. Schwarz - Phys. Rep. 89 (1982) 223.
- 30) D.J. Gross and E. Witten - Princeton University preprint (February 1986).
- 31) B.E.W. Nilsson - Nucl. Phys. B174 (1980) 335.
- 32) P.S. Howe, K.S. Stelle and P.K. Townsend - Nucl. Phys. B191 (1981) 445.