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FROM TWO TO THREE JETS IN HEAVY BOSON DECAYS:
AN ALGORITHMIC APPROACH

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ABSTRACT

The relation between two- and three-jet decays of vector particles is re-examined, with emphasis on the Monte Carlo simulation of multiboson production processes such as $e^+e^- \rightarrow W^+W^-$. An extremely simple algorithm is presented which allows one to include an extra gluon jet in two-jet decays while obtaining both momentum conservation and the correct angular correlations described by the complete vector boson production and three jet decay amplitude.

1. Introduction

Recently there has been considerable interest in processes in which two (or more) electroweak heavy vector particles are produced. In contrast to single-boson production, multi-boson production is described by the complete nonabelian Lagrangian of the Standard Model, or one of its alternatives: therefore, a process like $e^+e^- \rightarrow W^+W^-$ is one of the best probes of the $WW\gamma$ and WWZ vertices (see, for instance, the review in [1]) and boson-fusion processes like $WW \rightarrow WW$ and $WW \rightarrow ZZ$ can be used to elucidate the Higgs sector of the theory for large values of the Higgs mass [2]

Since the W 's and Z 's can only be observed through their decay products, any careful analysis or Monte Carlo simulation of multiboson production and decay has to take the decay product correlations properly into account. For the simplest decays, into a pair of leptons or light quarks, this problem has essentially been solved [3], [4]. However, even for the easiest multiboson case ($ee \rightarrow WW$) this entails the calculation of 6-point amplitudes usually consisting of more than one Feynman diagram. The computer time involved in evaluating these amplitudes tends to become very appreciable. The situation appears even more complicated if we consider three-jet decays of W or Z . It has been stressed recently [5] that these bosons will decay into three rather than two distinct jets about half the time: this implies that in $ee \rightarrow WW$ with hadronic decays a four-jet final state will actually be quite rare! The additional complications on e.g. the W mass determination arising from this effect are at present still under study [6]: it is clear, however, that

Monte Carlo simulations incorporating the complete, exact correlations in the three-jet decays will be of the essence. But, this implies that 7- and 8-point amplitudes will enter into the problem, with a corresponding increase in computational complexity¹.

In this paper I propose an alternative, simpler way to deal with the three-jet decays, which is applicable to any Monte Carlo simulation of W and Z decays into two light quark jets. It consists of two parts. The first part concerns the RELATIVE orientations of the q , q and g jets with respect to each other or, equivalently, their energies in the centre-of-mass frame of the boson V (denoting either W or Z). This part is already treated correctly in the existing calculations, as for instance in the relevant LUND Monte Carlo routines [7]: I discuss it here for completeness. The second part concerns the ABSOLUTE orientation of the three-jet configuration in the rest of the event, in other words the question: given a sample of two-jet decay configurations of V , which has the correct angular distribution, and an algorithm to transform each of these two-jet events into a three-jet event with the correct relative orientation, is there a way to relate the three-jet directions two the two-jet ones such that the correct, complete angular correlations predicted by the theory obtain? Clearly, if the gluon jet is infinitely soft, or collinear with q or $\bar{q}$, the quark jets can be chosen to retain their 'original' direction, and the correct correlations come out. In the case of actual interest, however, where the gluon is both hard and non-collinear, the situation is not so trivial. I will show that the problem can be solved by letting either the quark or the antiquark retain

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¹ In this paper, four-jet decays of a single boson are not considered; the 8-point functions are those having two 3-jet decays.

its direction, in a well-defined manner.

The lay-out of this paper is as follows. In section 2 the algorithm is derived. In section three the method for obtaining three-jet configurations with correct relative orientations is reviewed, while some numerical results are discussed in section 4.

2. From two to three jets

Consider the decay of a vector boson with momentum q into a massless quark-antiquark pair with momenta q_1 and q_2 , respectively:

$$V(q) \rightarrow q(q_1) \bar{q}(q_2) . \quad (1)$$

In the rest frame of V , the distribution of $\Omega_{1,2}$, the solid angle of q_1 or q_2 , reads

$$\frac{d\Gamma^{(2)}}{d\Omega_{1,2}} = \sum_{\lambda=\pm} |M_2(\lambda)|^2 , \quad (2)$$

where M_2 is the decay matrix element:

$$M_2(\lambda) = i \bar{u}_\lambda(q_1) \omega^\mu v_\lambda(q_2) \cdot J_\mu(q) . \quad (3)$$

Here, the current $J_\mu(q)$ denotes the rest of the amplitude (which possibly contains other decaying bosons as well), including the propagator of V and

phase space and color factors. The general vector/axial coupling is denoted by $\omega^\mu = (g_V + g_A \gamma^5) \gamma^\mu$, and λ labels the handedness of the fermions:

$$u_\lambda(q_i) \bar{u}_\lambda(q_i) = v_\lambda(q_i) \bar{v}_\lambda(q_i) = \omega_\lambda \not{q}_i, \quad \omega_\lambda = \frac{1 + \lambda \gamma^5}{2}. \quad (4)$$

The angular distribution is completely specified by M_2 , and the existence is assumed of an event generator yielding this correct distribution.

Now consider the three-jet decay:

$$V(q) \longrightarrow q(p_1) \bar{q}(p_2) g(k). \quad (5)$$

The corresponding (5-dimensional) decay distribution in the rest frame of V is

$$\frac{d\Gamma^{(3)}}{d\Omega_1 d\phi_2 dp_1^0 dp_2^0} = \frac{d\Gamma^{(3)}}{d\Omega_2 d\phi_1 dp_1^0 dp_2^0} = \frac{1}{6\pi^3} \sum_{\lambda, p=\pm} |M_3(\lambda, p)|^2, \quad (6)$$

where $\Omega_1(\Omega_2)$ is the solid angle of $q_1(q_2)$, and $\phi_2(\phi_1)$ is the azimuthal angle of $q_2(q_1)$ around $q_1(q_2)$. The gluon helicity is denoted by p , and the matrix element M_3 reads

$$M_3(\lambda, p) = i g_s \bar{u}_\lambda(p_1) \left\{ \omega^\mu \frac{\not{p}_2 + \not{k}}{2p_2 \cdot k} \not{\epsilon}_p - \not{\epsilon}_p \frac{\not{p}_1 + \not{k}}{2p_1 \cdot k} \omega^\mu \right\} v_\lambda(p_2) \cdot J_\mu(q). \quad (7)$$

with g_s the strong coupling constant, and ϵ_p the four-vector of the gluon polarization. It is important to note that $J_\mu(q)$ is exactly the same current as in Eq.(3). For $\not{\epsilon}_p$ one may choose the CALKUL representation [8]:

$$\not{\epsilon}_p = N [\omega_p \not{k} \not{p}_1 \not{p}_2 - \not{p}_1 \not{p}_2 \not{k} \omega_p] , \quad (8)$$

$$N = [4(p_1 \cdot k)(p_2 \cdot k)(p_1 \cdot p_2)]^{-1/2}$$

After some Dirac algebra the matrix element reduces to a simple form:

$$M_3(\lambda, \lambda) = -ig_s N \bar{u}_\lambda(p_1) \omega^\mu(\not{p}_2 + \not{k}) \not{p}_1 v_\lambda(p_2) \cdot J_\mu(q) , \quad (9)$$

$$M_3(\lambda, -\lambda) = -ig_s N \bar{u}_\lambda(p_1) \not{p}_2(\not{p}_1 + \not{k}) \omega^\mu v_\lambda(p_2) \cdot J_\mu(q) .$$

One can now proceed to write Eq.(9) in a form very similar to that of Eq.(3). First, define the following vectors:

$$\begin{aligned} r_1^\mu &= (1 + \alpha_1) p_1^\mu , \quad r_2^\mu = (1 + \alpha_2) p_2^\mu , \\ \bar{r}_1^\mu &= p_1^\mu + k^\mu - \alpha_2 p_2^\mu , \quad \bar{r}_2^\mu = p_2^\mu + k^\mu - \alpha_1 p_1^\mu , \\ \alpha_1 &= (p_2 \cdot k) / (p_1 \cdot p_2 + k) = (1 - x_1) / x_1 , \\ \alpha_2 &= (p_1 \cdot k) / (p_2 \cdot p_1 + k) = (1 - x_2) / x_2 , \end{aligned} \quad (10)$$

where x_1, x_2 denote the energy fractions $2(p_{1,2})^0 / q^0$ in the V rest frame. From the fact that $\bar{r}_1^2 = \bar{r}_2^2 = 0$, and the scaling property for lightlike p^μ :

$$u_\lambda(Cp) \stackrel{\Omega}{=} \sqrt{C} u_\lambda(p) \quad (11)$$

for any C (where $\stackrel{\Omega}{=}$ stands for an equality modulo an overall complex phase), it is easily seen that Eq.(9) can be written as

$$\begin{aligned} M_3(\lambda, \lambda) &\stackrel{\Omega}{=} C_+ \bar{u}_\lambda(r_1) \omega^\mu v_\lambda(\bar{r}_2) \cdot J_\mu(q) , \\ M_3(\lambda, -\lambda) &\stackrel{\Omega}{=} C_- \bar{u}_\lambda(\bar{r}_1) \omega^\mu v_\lambda(r_2) \cdot J_\mu(q) . \end{aligned} \quad (12)$$

The coefficients $C_\pm$ are given by

$$\begin{aligned}
C_+ &= g_s N \bar{v}_\lambda(\bar{r}_2) \not{p}_1 v_\lambda(p_2) / \sqrt{1+\alpha_1}, \\
C_- &= g_s N \bar{u}_\lambda(p_1) \not{p}_2 u_\lambda(\bar{r}_1) / \sqrt{1+\alpha_2}.
\end{aligned} \tag{13}$$

It is instructive to examine the vectors r_i^μ and $\bar{r}_i^\mu$ more closely. They have the following properties:

$$\begin{aligned}
r_1^\mu + \bar{r}_2^\mu &= \bar{r}_1^\mu + r_2^\mu = q^\mu, \\
r_1^0 &= r_2^0 = q^0/2, \\
\vec{r}_1 &\parallel \vec{p}_1, \quad \vec{r}_2 \parallel \vec{p}_2.
\end{aligned} \tag{14}$$

All this implies that the 3-jet width $\Gamma^{(3)}$ can be related to the 2-jet width $\Gamma^{(2)}$ as follows:

$$\begin{aligned}
\frac{d\Gamma^{(3)}}{dp_1^0 dp_2^0 d\Omega_{1,2} d\phi_{2,1}} &= \frac{1}{6\pi^3} \left\{ |C_+|^2 \left[\frac{d\Gamma^{(2)}}{d\Omega_{1,2}} \right] (q_1, q_2) \rightarrow (r_1, \bar{r}_2) \right. \\
&\quad \left. + |C_-|^2 \left[\frac{d\Gamma^{(2)}}{d\Omega_{1,2}} \right] (q_1, q_2) \rightarrow (\bar{r}_1, r_2) \right\},
\end{aligned} \tag{15}$$

where replacement of the variables q_1, q_2 is indicated, and the two terms correspond to the two gluon helicities. Owing to the properties of the vectors r , Eq.(15) holds irrespective of the actual functional form in which $\Gamma^{(2)}$ is expressed.

The above result can be formulated in a way that leads to a Monte Carlo algorithm, as follows: if the gluon helicity p equals the quark helicity λ , the vector r_1^μ defined for the 3-jet decay plays the same role as the momentum q_1^μ of the 2-jet decay: for $p=-\lambda$, it is r_2^μ that is associated with the momentum q_2^μ . Since $\vec{r}_1$ is parallel to $\vec{p}_1$ one arrives at the following al-

gorithm for associating a given 3-jet decay to a given 2-jet decay: if $p=\lambda$ then $\vec{p}_1$ should be parallel to $\vec{q}_1$ (i.e. the QUARK should retain its 'original' direction); if $p=-\lambda$ then $\vec{p}_2$ should be parallel to $\vec{q}_2$ (i.e. the ANTIQUARK keeps its direction). A sample of 3-jet decays oriented according to this algorithm is guaranteed to have the correct angular and energy distributions, provided the 2-jet sample had the correct distributions.

As a last step: the relative probabilities to have $p=\lambda$ or $p=-\lambda$ are given by $|C_+|^2$ and $|C_-|^2$, respectively. Using standard techniques, these values are easily found to be

$$\begin{aligned} |C_+|^2 &= \frac{8\pi\alpha_s}{q^2} \frac{x_1^2}{(1-x_1)(1-x_2)}, \\ |C_-|^2 &= \frac{8\pi\alpha_s}{q^2} \frac{x_2^2}{(1-x_1)(1-x_2)}. \end{aligned} \quad (16)$$

Therefore, the relative probabilities of having the quark or the antiquark retain its direction are in the ratio of their respective energies squared. This accords with the intuitive idea that the fastest parton is somehow the 'least likely' to have radiated the gluon, and therefore the 'most likely' to follow the original q-q direction. Of course this notion is not gauge-invariant: therefore the algorithm is only valid on a statistical basis, with a finite probability even for the fastest parton to 'have radiated' the gluon.

3. Generating three jets

In this section I briefly discuss an algorithm for obtaining 3-jet decay samples with the correct relative orientations, or equivalently the correct x_1, x_2 distributions. From the previous section follows the well-known relation

$$\frac{1}{\Gamma^{(2)}} \frac{d\Gamma^{(3)}}{dx_1 dx_2} = \frac{2\alpha_s}{3\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \quad , \quad (17)$$

which, when integrated over a suitably defined phase space, gives the fraction of 3-jet decays in the total decay rate. It is customary (e.g. in the LUND program [7]) to try, in the definition of this phase space, to approximate as much as possible the experimental distinction between two- and three-jet decays, for instance using cuts on the relative p_T between jets. Here I would like to propose a different approach: one can define a 3-jet phase space as follows:

$$\begin{aligned} \gamma \leq x_1 \leq 1-\gamma \quad , \quad 1-x_1 \leq x_2 \leq 1-\gamma, \\ 0 < \gamma \leq 1/2 \quad . \end{aligned} \quad (18)$$

The allowed region in the x_1, x_2 plane is depicted in fig.1. One can tune γ such that the integral $\Gamma^{(3)}/\Gamma^{(2)}$ comes out to precisely $1+\alpha_s/\pi$, the famous value of the total one-loop QCD correction to the total width. Then EVERY event will be a three-jet event, sometimes with an awfully soft or collinear gluon jet. Subsequently, these events can be tested on whether or not three distinct jets would actually be seen in the experiment. In this way the real decay process, in which always some gluon radiation takes place [9], is better approximated, and the experimental 2/3 jet distinction can be implemented in as much detail as desired.

Performing the integral (17) over the phase space defined by Eq.(18) one finds

$$\frac{1}{\Gamma^{(2)}} \int dx_1 dx_2 \frac{d\Gamma^{(3)}}{dx_1 dx_2} = \frac{2\alpha_s}{3\pi} G(\gamma),$$

$$G(\gamma) = 2 \ln^2 \gamma - (3-\gamma)(1-\gamma) \ln \gamma$$

$$+ 4 Li_2(\gamma) - \frac{\pi^2}{3} + \frac{5}{2} - 5\gamma$$
(19)

The above-mentioned tuning of $\Gamma^{(3)}/\Gamma^{(2)}$ requires solving the equation

$$G(\gamma) = \frac{3}{2} (1 + \pi/\alpha_s)$$
(20)

For realistic values of α_s , γ is to a very good approximation given by

$$\gamma \sim \exp\left[-\frac{1}{4}\left(3 + \sqrt{1 + \frac{8\pi^2}{3} + 12 \frac{\pi}{\alpha_s}}\right)\right]$$
(21)

The relation between α_s and γ is depicted in fig.2: for α_s between 0.1 and 0.2, γ varies between 0.003 and 0.012. It is now possible to formulate a complete algorithm for obtaining from a Monte Carlo V-decay event with two jets one with 3 jets, as follows:

1. Go to the centre-of-mass frame of the 2-jet system, and calculate the invariant mass $\sqrt{q^2}$ (not necessarily equal to the mass of V, if V has a finite width);
2. compute $\alpha_s(q^2)$, and γ , using Eq.(21);

3. generate two values x_1 and x_2 according to distribution (17), in the phase space of Eq.(18). This can be done by a few simple mapping-and-rejection procedures [10]. The other variables, Ω and ϕ , are trivially distributed;
4. using a random number, choose between the quark or the antiquark, using x_1^2 and x_2^2 as relative probabilities, respectively;
5. if the quark was chosen, construct the three parton momenta, fixing the quark direction to be the one of the original two-jet event; if the antiquark was chosen, the direction of the antiquark should be equal to the original one.

The above five points completely specify the algorithm for going from a two-jet decay Monte Carlo to a three-jet decay one while ensuring the correctness of the resulting distributions.

4. Discussion

To appreciate the above algorithm it is instructive to examine a few three-jet distributions in detail. Since the relative orientation of the three jets can be assumed to be treated correctly, quantities like jet energies and opening angles (in the V rest frame) are not interesting at this point. Instead one has to concentrate on absolute quantities like angles or transverse momenta with respect to a given axis. In order not to obscure the

discussion I have chosen this axis to be the 'original' $q\bar{q}$ direction; this way the results below become independent of the actual production mechanism of V (note that, if the decay $V \rightarrow q\bar{q}$ was isotropic, so would be the decay $V \rightarrow q\bar{q}g$, and no interesting discussion would be possible). In order to study the importance of the precise form of the algorithm for the absolute orientation, I have compared a few alternative, semi-reasonable-looking algorithms. These are:

- (a) the correct algorithm, i.e. the one described in section 2: choose q or $\bar{q}$ with relative probabilities E_q^2 and $E_{\bar{q}}^2$, respectively;
- (b) choose q or $\bar{q}$ with equal probability;
- (c) choose $q(\bar{q})$ if E_q is larger(smaller) than $E_{\bar{q}}$;
- (d) always choose q .

Actually, possibility (d) appears not very attractive: I have included it because it has actually been used in recent simulations [11]. I have generated a sample of 100,000 Monte Carlo events for each of the alternatives, with $\sqrt{s} = 80$ GeV. Fig 3. shows the angular distribution of the gluon with respect to the 'original' quark direction, which I have fixed to be the positive z axis. In all cases, the distributions peak at 0° and 180° , as expected; however, their 'dip ratios' (defined as the ratio of the contents of the two central bins over that of the two outer ones) are quite different, as indicated in the table. The asymmetry in case (d) is, of course, due to the asymmetry in the algorithm. The same holds, although to a lesser extent, for the angular distributions of all partons, depicted in fig.4. These are steeper at the endpoints since the q and $\bar{q}$ are usually - but not always! - moving in the forward and backward direction, respectively. The dip ratios are also given in the table, and seen to be different.

Angular distributions are not directly measurable, however, since arbitrarily soft and collinear partons enter as well as hard, widely separated ones. A slightly more realistic distribution is the one of fig.5 where I have plotted the distribution of the p_T of each parton with respect to the original qq axis. In all cases, the distribution peaks at zero, as expected; however, the tails are quite different. In particular, the high- p_T fractions, defined as the fraction of all entries with p_T larger than 20 GeV, vary dramatically between different algorithms: they are also given in the table.

Upon inspection of the table, algorithms (b) and (d) are seen to be quite similar (as they should be, since the above quantities are symmetric under $q \rightarrow -q$). The difference between (a) and (c) appears quite striking, since these algorithms look very much alike: in fact they are only similar if x_1 and x_2 are of different magnitude, while in many decays x_1 and x_2 are more or less equal. It is interesting to note that in case (c) the p_T distribution cuts off around 24 GeV. In fact, for this algorithm there is a rigorous kinematical limit of $\sqrt{(q^2/8)}$ for the p_T , a good indication of the necessity of having a correct algorithm.

From the above considerations it appears that the precise form of the algorithm for the absolute orientation of 3-jet events may have appreciable phenomenological implications. It seems clear that the approach described above is necessary to obtain reliable Monte Carlo predictions.

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Table

algorithm	dip ratio (gluons)	dip ratio (all partons)	high- p_T fraction
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a	36.9	6.45	1.40
b	43.9	6.91	3.26
c	23.6	4.58	0.53
d	44.2	6.95	3.34

Dip ratios of the angular distributions of the gluons and of all partons, and the high- p_T -fraction of the transverse momentum distribution, for the four different Monte Carlo algorithms (all entries are in percents).

Figure captions

fig.1

The integration region for the variables x_1 and x_2 (shaded area).

fig.2

The value of γ as a function of α_s .

fig.3

Angular distributions of the gluon momentum in samples of 100,000 Monte Carlo events, each generated according to one of the alternative algorithms (a) through (d), discussed in the text.

fig.4

Angular distributions of all parton momenta for the four different algorithms.

fig.5

Transverse momentum distributions of all partons for the four different algorithms.

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5