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A DEFINITE INTEGRAL OF A PRODUCT OF TWO POLYLOGARITHMS

V.S. Adamchik^{*)} and K.S. Kölbig

ABSTRACT

Residue techniques are used to evaluate a definite integral depending on several parameters and containing a product of two polylogarithm functions (or two logarithms in the degenerate case). The results are given in terms of infinite hypergeometric type series. Some special cases for which the infinite series can be expressed in closed form are considered.

^{*)} Belorussian State University, Minsk, USSR.

1. INTRODUCTION

It is the purpose of this paper to evaluate the integral

$$I_{n,m}(\alpha, \sigma, \omega, r) = \int_0^{\infty} x^{\alpha-1} \text{Li}_n(-\sigma x) \text{Li}_m(-\omega x^r) dx \quad (1.1)$$

for positive integers n, m , complex α, σ, ω , and real $r \neq 0$ such that the integral exists. $\text{Li}_k(z)$ is the polylogarithm function defined for $|z| \leq 1$ and $k \geq 2$ by the power series [5, p. 189]

$$\text{Li}_k(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^k} \quad (|z| \leq 1) \quad (1.2)$$

and for $|z| > 1$ by [5, p. 192] [4]

$$\begin{aligned} \text{Li}_k(z) &= (-1)^{k+1} \text{Li}_k\left(\frac{1}{z}\right) - \frac{1}{k!} \ln^k(-z) \\ &\quad - \sum_{j=0}^{k-2} \frac{1}{j!} (1 + (-1)^{k-j}) (1 - 2^{1-k+j}) \zeta(k-j) \ln^j(-z), \end{aligned} \quad (1.3)$$

where $\zeta(q)$ is the Riemann zeta function and where the logarithm is defined on its principal sheet. For $k = 1$ we obtain from (1.2)

$$\text{Li}_1(z) = -\ln(1 - z) \quad (1.4)$$

as a degenerate case. Interest in polylogarithms, which have been investigated in the past by many mathematicians, has revived in recent years on account of their applications in quantum electrodynamics, group theory, and geometry.

An integral representation of $\text{Li}_k(z)$ given recently by Marichev [8, p. 105], namely

$$\begin{aligned} \text{Li}_k(-z) &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(s)\Gamma(-s)}{(-s)^{k-1}} z^{-s} ds \\ &\quad (-1 < a < 0, |\arg z| < \pi, k = 0, 1, 2, \dots) \end{aligned} \quad (1.5)$$

allows us, by using the product theorem of the Mellin transform, and by applying the residue theorem, to evaluate the integral (1.1).

From (1.3), we find

$$\text{Li}_k(-z) = -\frac{1}{k!} \ln^k z + O(\ln^{k-2} z) \quad (z \rightarrow \infty) \quad (1.6)$$

which can be used, together with (1.2), to show that the integral (1.1) converges under the conditions:

$$\begin{aligned} -1 - r < \text{Re } \alpha < 0 & \quad \text{if } r > 0 \\ -1 < \text{Re } \alpha < -r & \quad \text{if } r < 0, \end{aligned} \quad (1.7)$$

and $|\arg \sigma| < \pi$, $|\arg \omega| < \pi$.

We may add here that, by making the substitution $x = \xi^{1/r}$, we obtain from (1.1) the 'symmetry' relation

$$I_{n,m}(\alpha, \sigma, \omega, r) = \frac{1}{|r|} I_{m,n}\left(\frac{\alpha}{r}, \omega, \sigma, \frac{1}{r}\right). \quad (1.8)$$

2. AN ALTERNATIVE INTEGRAL FOR $I_{n,m}(\alpha, \sigma, w, r)$

In this section, we derive a Mellin-Barnes integral for $I_{n,m}$. By introducing (1.5) into (1.1), we find with the (allowed) substitution $rs = -s'$,

$$I_{n,m} = \frac{1}{\sigma^\alpha |r|} \int_0^\infty \frac{du}{u} \left(u^\alpha \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\Gamma(s)\Gamma(-s)}{(-s)^{n-1}} u^{-s} ds \right) \left(\frac{1}{2\pi i} \int_{a'-i\infty}^{a'+i\infty} \frac{\Gamma\left(\frac{s}{r}\right)\Gamma\left(-\frac{s}{r}\right)}{\left(\frac{s}{r}\right)^{m-1}} \left(\frac{z}{u}\right)^{-s} ds \right)$$

where $z = \sigma w^{-1/r}$. Applying well-known theorems for the Mellin transform (see, e.g., Sneddon [12, pp. 262-297]), in particular the product theorem

$$\int_0^\infty f\left(\frac{z}{u}\right) g(u) \frac{du}{u} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f^*(s) g^*(s) z^{-s} ds,$$

we see that

$$g^*(s) = (-1)^{n+1} \Gamma(s + \alpha) \Gamma(-s - \alpha) (s + \alpha)^{1-n}$$

$$f^*(s) = \Gamma\left(\frac{s}{r}\right) \Gamma\left(-\frac{s}{r}\right) \left(\frac{s}{r}\right)^{1-m},$$

and that therefore

$$I_{n,m}(\alpha, \sigma, w, r) = \frac{(-1)^{n+1}}{\sigma^\alpha |r|} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s + \alpha) \Gamma(-s - \alpha) \Gamma\left(\frac{s}{r}\right) \Gamma\left(-\frac{s}{r}\right)}{(s + \alpha)^{n-1} \left(\frac{s}{r}\right)^{m-1}} z^{-s} ds, \quad (2.1)$$

where

$$\begin{aligned} \max(0, -1 - \operatorname{Re} \alpha) < \gamma = \operatorname{Re} s < \min(r, -\operatorname{Re} \alpha) & \quad (r > 0) \\ \max(r, -1 - \operatorname{Re} \alpha) < \gamma = \operatorname{Re} s < \min(0, -\operatorname{Re} \alpha) & \quad (r < 0). \end{aligned} \quad (2.2)$$

Using [2, No. 8.3343]

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z} \csc \pi z,$$

the integral (2.1) can be written as

$$I_{n,m}(\alpha, \sigma, \omega, r) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \varphi_{n,m}(s; \alpha, r, \zeta) ds \quad (2.3)$$

where

$$\varphi_{n,m}(s; \alpha, r, \zeta) = \frac{(-1)^{n+1} \omega^2}{\sigma^\alpha |r|} \csc[\omega(s + \alpha)] \csc\left(\frac{\omega s}{r}\right) (s + \alpha)^{-n} \left(\frac{s}{r}\right)^{-m} \zeta^{-s} \quad (2.4)$$

Formula (2.3) is more suitable than (2.1) for evaluation by the residue theorem.

We may note here that $I_{n,m}$ is a special case of a very general integral, namely

$$\frac{1}{2\pi i} \int_L \chi(s) z^s ds = H_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_1, A_1), (a_2, A_2), \dots, (a_p, A_p) \\ (b_1, B_1), (b_2, B_2), \dots, (b_q, B_q) \end{matrix} \right. \right] \quad (2.5)$$

where

$$\chi(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s) \prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s) \prod_{j=n+1}^p \Gamma(a_j - A_j s)},$$

and where L is a contour separating the poles of $\Gamma(b_j - B_j s)$ ($j = 1, 2, \dots, m$) from the poles of $\Gamma(1 - a_j + A_j s)$ ($j = 1, 2, \dots, n$). This integral is known as Fox's H -function (see Mathai and Saxena [10, p. 2]). By comparison with (2.1), it follows that

$$I_{n,m}(\alpha, \sigma, \omega, r) = \frac{1}{\sigma^\alpha |r|} H_{m+n, m+n}^{m+1, n+1} \left[\frac{\sigma}{\omega^{1/r}} \left| \begin{matrix} (1+\alpha, 1), \dots, (1+\alpha, 1), (1, 1/r), \dots, (1, 1/r) \\ (0, 1/r), \dots, (0, 1/r), (\alpha, 1), \dots, (\alpha, 1) \end{matrix} \right. \right] \quad (2.6)$$

with n repetitions of the brackets $(1+\alpha, 1)$, $(\alpha, 1)$ and m repetitions of $(1, 1/r)$, $(0, 1/r)$.

For rational $r = \pm p/q$, it is possible to express the integral (2.1) as a special case of Meijer's G-function in the logarithmic case. Substituting $\tilde{s} = s/p$ in (2.1), we obtain

$$I_{n,m} = \frac{(-1)^{n+1}}{\sigma^\alpha (\pm 1)^{m-1} p^{n-1} q^{m-2}} \frac{1}{2\pi i} \int_{\tilde{\gamma}-i\infty}^{\tilde{\gamma}+i\infty} \frac{\Gamma(p\tilde{s} + \alpha) \Gamma(-p\tilde{s} - \alpha) \Gamma(q\tilde{s}) \Gamma(-q\tilde{s})}{(\tilde{s} + \alpha/p)^{n-1} \tilde{s}^{m-1}} z^{-p\tilde{s}} d\tilde{s} \quad (2.7)$$

where $\tilde{\gamma} = \gamma/p$. Using the product theorem of the gamma function [2, No. 8.335], equation (2.7) can be written as

$$I_{n,m} = \frac{(-1)^{n-1} (2\pi)^{2-p-q}}{\sigma^\alpha (\pm 1)^{m-1} p^{n-1} q^{m-1}} \frac{1}{2\pi i} \int_{\tilde{\gamma}-i\infty}^{\tilde{\gamma}+i\infty} \frac{\Gamma^{n-1}(\tilde{s} + \alpha/p) \Gamma^{m-1}(\tilde{s})}{\Gamma^{n-1}(\tilde{s} + \alpha/p + 1) \Gamma^{m-1}(\tilde{s} + 1)} \times \\ \prod_{j=0}^{p-1} \Gamma\left(\tilde{s} + \frac{\alpha}{p} + \frac{j}{p}\right) \Gamma\left(-\tilde{s} - \frac{\alpha}{p} + \frac{j}{p}\right) \prod_{j=0}^{q-1} \Gamma\left(\tilde{s} + \frac{j}{q}\right) \Gamma\left(-\tilde{s} + \frac{j}{q}\right) z^{-p\tilde{s}} d\tilde{s} \quad (2.8)$$

By comparison with Meijer's G-function [9, p. 176] defined by

$$G_{p,q}^{m,n} \left(z \left| \begin{matrix} a_j \\ b_j \end{matrix} \right. \right) = \frac{1}{2\pi i} \int_{\tilde{\gamma}-i\infty}^{\tilde{\gamma}+i\infty} \frac{\prod_{j=1}^m \Gamma(b_j + s) \prod_{j=1}^n \Gamma(1 - a_j - s)}{\prod_{j=m+1}^p \Gamma(1 - b_j - s) \prod_{j=n+1}^q \Gamma(a_j + s)} z^s ds, \quad (2.9)$$

we see that

$$I_{n,m}(\alpha, \sigma, w, \pm \frac{p}{q}) = \frac{(-1)^{n+1} (2\pi)^{2-p-q}}{\sigma^\alpha (\pm 1)^{m-1} p^{n-1} q^{m-1}} G_{\tilde{p}, \tilde{q}}^{\tilde{m}, \tilde{n}} \left(z^{-p} \left| \begin{matrix} a_j \\ b_j \end{matrix} \right. \right), \quad (2.10)$$

where

$$\begin{aligned} \tilde{m} &= n + m + p + q - 2 \\ \tilde{n} &= p + q \\ \tilde{p} &= \tilde{q} = \tilde{m} \end{aligned} \quad (2.11)$$

and

$$a_j = \begin{cases} 1 + (1 + \alpha - j)/p & j = 1, \dots, p \\ 1 + (1 - j)/q & j = p+1, \dots, p+q \\ 1 + \alpha/p & j = p+q+1, \dots, p+q+n-1 \\ 1 & j = p+q+n, \dots, p+q+n+m-2 \end{cases} \quad (2.12)$$

$$b_j = \begin{cases} \alpha/p & j = 1, \dots, n-1 \\ 0 & j = n, \dots, n+m-2 \\ (\alpha + j - 1)/p & j = n+m-1, \dots, n+m+p-2 \\ (j - 1)/q & j = n+m+q-1, \dots, n+m+p+q-2 \end{cases}$$

From the general theory of the G-function we know [9, p. 16] that, for $\tilde{p} = \tilde{q}$, $G_{\tilde{p}, \tilde{q}}^{\tilde{m}, \tilde{n}}(z)$ has three singular points: at $z = 0$, $z = \infty$, and $z = (-1)^{\tilde{m} + \tilde{n} - \tilde{p}}$. Using results derived by Marichev [7], who investigated the behaviour of the G-function in the neighbourhood of its singularities, it follows that for $\tilde{m} + \tilde{n} - \tilde{p} > 1$, $G_{\tilde{p}, \tilde{q}}^{\tilde{m}, \tilde{n}}(z)$ is analytic and therefore continuous in the whole sector $|\arg z| < \pi(\tilde{m} + \tilde{n} - \tilde{p})$, and thus also at the singular point $(-1)^{\tilde{m} + \tilde{n} - \tilde{p}}$. From (2.11) we have $\tilde{m} + \tilde{n} - \tilde{p} = p + q > 1$ ($p, q \in \mathbb{N}$). It follows that, for rational r , the function (2.1) is analytic and continuous in the whole sector $|\arg z| < \pi(1 + 1/r)$. Using the analyticity of the integral (2.1) with respect to r , it follows that this property holds for any real r . We shall use this continuity property of (2.1) when considering the examples in Section 4.

3. THE EVALUATION OF THE INTEGRAL

Using techniques developed by Marichev [6, 8] and, for the logarithmic case of integrals (2.1), (2.5), by Adamchik and Marichev [1], and Mathai and Saxena [9, p. 157], [10, p. 70], it is possible to evaluate the integral (1.1) in terms of infinite (hypergeometric type) series. We thus obtain the following

Theorem

Let $r \neq 0$ be real, α, σ, w be complex with $|\arg \sigma| < \pi, |\arg w| < \pi$, and let $\zeta = \sigma w^{-1/r}$. Further, with $N = \{1, 2, 3, \dots\}$, let

$$h(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise,} \end{cases} \quad \bar{h}(x) = 1 - h(x),$$

and

$$H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Then, for $|\zeta| \neq 1$ and

$$\begin{aligned} -1 - r < \operatorname{Re} \alpha < 0 & \quad \text{if } r > 0 \\ -1 < \operatorname{Re} \alpha < -r & \quad \text{if } r < 0, \end{aligned}$$

the integral (1.1) can, for $\alpha \neq 0$, be expressed in the form

$$\int_0^\infty x^{\alpha-1} \operatorname{Li}_n(-\sigma x) \operatorname{Li}_m(-wx^r) dx =$$

$$\begin{aligned} & (-1)^n (\pm \operatorname{sgn} r)^{m+1} w \sigma^{-\alpha} \sum_k (-1)^k \csc [\pi(\alpha \pm |r|k)] (\alpha \pm |r|k)^{-n} k^{-m} \zeta^{\pm |r|k} \\ & + (\pm 1)^n (-1)^{m+n} \frac{w}{|r|} w^{-\alpha/r} \sum_\ell (-1)^\ell \csc \left(\pi \frac{\alpha + \ell}{r} \right) \left(\frac{\alpha + \ell}{r} \right)^{-m} \ell^{-n} \zeta^{\pm \ell} \\ & + (-1)^{n+1} (\pm \operatorname{sgn} r)^{m+1} \sigma^{-\alpha} \sum_K (-1)^{K+|r|K+\alpha} (\alpha \pm |r|K)^{-n} K^{-m} \zeta^{\pm |r|K} x \end{aligned}$$

$$\left(\pm \frac{m}{|r|K} + \frac{n}{\alpha \pm |r|K} + \ln z \right)$$

$$+ H(\pm r) (-1)^{n+1} \frac{\alpha}{\omega} \frac{m+1}{r} \frac{m-n}{\alpha} \left[h(\pm \alpha) (-1)^\alpha \sum_{m_1=0}^{m+1} \frac{1}{m_1!} \left(-\frac{1}{\omega} \ln z \right)^{m_1} \times \right.$$

$$\sum_{m_2=0}^{m+1-m_1} \binom{m_2+n-1}{n-1} (-\omega \alpha)^{-m_2} \sum_{m_3=0}^{m+1-m_1-m_2} r^{-m_3} B_{m_3}^* B_{m+1-m_1-m_2-m_3}^*$$

$$+ \bar{h}(\pm \alpha) \sum_{m_1=0}^m \frac{1}{m_1!} \left(-\frac{1}{\omega} \ln z \right)^{m_1} \times$$

$$\left. \sum_{m_2=0}^{m-m_1} \binom{m_2+n-1}{n-1} (-\omega \alpha)^{-m_2} \sum_{m_3=0}^{m-m_1-m_2} r^{-m_3} B_{m_3}^* C_{m-m_1-m_2-m_3}(\omega \alpha) \right\}$$

$$+ H(\pm 1) (-1)^m (\operatorname{sgn} r) \omega^{-\alpha/r} \frac{n+1}{r} r^{\frac{m}{\alpha}-m}$$

$$\left[h\left(-\frac{\alpha}{|r|}\right) (-1)^{n+\alpha/r} \sum_{n_1=0}^{n+1} \frac{1}{n_1!} \left(-\frac{1}{\omega} \ln z \right)^{n_1} \times \right.$$

$$\sum_{n_2=0}^{n+1-n_1} \binom{n_2+m-1}{m-1} (\omega \alpha)^{-n_2} \sum_{n_3=0}^{n+1-n_1-n_2} r^{-n_3} B_{n_3}^* B_{n+1-n_1-n_2-n_3}^*$$

$$- \bar{h}\left(-\frac{\alpha}{|r|}\right) r^{-n-1} \sum_{n_1=0}^n \frac{1}{n_1!} \left(\frac{r}{\omega} \ln z \right)^{n_1} \times$$

$$\left. \sum_{n_2=0}^{n-n_1} (-1)^{n_2} \binom{n_2+m-1}{m-1} \left(\frac{\alpha}{\omega r} \right)^{-n_2} \sum_{n_3=0}^{n-n_1-n_2} (-r)^{n_3} B_{n_3}^* C_{n-n_1-n_2-n_3} \left(\frac{\alpha}{\omega r} \right) \right\}, \quad (3.1)$$

and for $\alpha = 0$ (which implies $r < 0$), in the form

$$\int_0^{\infty} \text{Li}_n(-\alpha x) \text{Li}_m(-\omega x^r) \frac{dx}{x} =$$

$$-(\bar{r}+1)^{m+n} \left[|r|^{-n} \sum_k (-1)^k \csc(\pi|r|k) k^{-(m+n)} \bar{z}^{|r|k} \right.$$

$$+ |r|^{m-1} \sum_\ell (-1)^\ell \csc\left(\pi \frac{\ell}{|r|}\right) \ell^{-(m+n)} \bar{z}^{\ell} \left. \right]$$

$$- |r|^{-n} \sum_K (-1)^{K(1+r)} K^{-(m+n)} \bar{z}^{|r|K} \left(\frac{m+n}{|r|K} \pm \ln \bar{z} \right) \left. \right]$$

$$+ H(\pm 1) (-1)^{n+1} \pi^{m+n+1} r^m x$$

$$\sum_{m_1=0}^{m+n+1} \frac{1}{m_1!} \left(-\frac{1}{\pi} \ln \bar{z} \right)^{m_1} \sum_{m_2=0}^{m+n+1-m_1} r^{-m_2} B_{m_2}^* B_{m+n+1-m_1-m_2}^* \quad (3.2)$$

where

$$B_j^* = \frac{|2^j - 2|}{j!} |B_j|, \quad (3.3)$$

B_j are the Bernoulli numbers, where

$$C_j(x) = \frac{1}{j!} \frac{d^j}{ds^j} \csc(s+x) \Big|_{s=0}, \quad (3.4)$$

and where the logarithm is defined on its principal sheet.

The summation indices k, ℓ for the sums I_k, I_ℓ respectively are restricted to those values $k, \ell \in \mathbb{N}$ for which

$$|r|k \pm \alpha \notin \mathbb{N}, \quad (\ell \mp \alpha)/|r| \notin \mathbb{N}. \quad (3.5)$$

The sum $\sum_K$ is to be taken over those values of $K \in \mathbb{N}$ for which

$$|r|K \pm \alpha \in \mathbb{N}. \quad (3.6)$$

In all the expressions above, the upper sign corresponds to $|\zeta| > 1$, the lower sign to $|\zeta| < 1$.

Proof:

Using the theory of the H-function [10], we find that the poles of the function $\psi_{n,m}(s; \alpha, r, \zeta)$, or equivalently, of

$$\tilde{\psi}_{n,m}(s; \alpha, r) = \csc[\pi(s+\alpha)] \csc\left(\pi \frac{s}{r}\right) (s+\alpha)^{-n} \left(\frac{s}{r}\right)^{-m}, \quad (3.7)$$

which need to be taken into account when evaluating the integral (1.1), are those which, for $|\zeta| > 1$ lie on the right, and for $|\zeta| < 1$ on the left, of the line $\operatorname{Re} s = \gamma$. In the first case the values of the corresponding residues must be multiplied by a factor -1 . The relevant poles are:

$|\zeta| > 1$

- i) At $s = 0$ if $r < 0$. Pole of order $m+2$ if $\alpha \in \mathbb{N}$, of order $m+n+2$ if $\alpha = 0$, of order $m+1$ if $\alpha \notin \{0, \mathbb{N}\}$.
- ii) At $s = -\alpha$. If $r < 0$: pole of order $n+2$ if $\alpha/r \in \mathbb{N}$, of order $n+1$ if $\alpha/r \notin \{0, \mathbb{N}\}$. If $r > 0$: pole of order $n+2$ if $-\alpha/r \in \mathbb{N}$, of order $n+1$ if $-\alpha/r \notin \mathbb{N}$.
- iii) At $s = |r|k$ and at $s = -\alpha + \ell$ for $k, \ell \in \mathbb{N}$. Both poles of order 1 if $|r|k \neq -\alpha + \ell$, single pole of order 2 if $|r|k = -\alpha + \ell$.

$|\zeta| < 1$

- i) At $s = 0$ if $r > 0$. Pole of order $m+2$ if $-\alpha \in \mathbb{N}$, of order $m+1$ if $-\alpha \notin \mathbb{N}$.
- ii) At $s = -|r|k$ and at $s = -\alpha - \ell$ for $k, \ell \in \mathbb{N}$. Both poles of order 1 if $|r|k \neq \alpha + \ell$, single pole of order 2 if $|r|k = \alpha + \ell$.

The residues corresponding to these poles can be evaluated by standard methods, making use of the Leibniz formula for the m th derivative of a product of k functions of one variable, namely

$$\begin{aligned}
 \left(\frac{d}{dx}\right)^m f_1(x) f_2(x) \dots f_k(x) = \\
 \sum_{m_1=0}^m \binom{m}{m_1} f_1^{(m_1)}(x) \sum_{m_2=0}^{m-m_1} \binom{m-m_1}{m_2} f_2^{(m_2)}(x) \times \\
 \dots \sum_{m_{k-1}=0}^{m-m_1-\dots-m_{k-2}} \binom{m-m_1-\dots-m_{k-2}}{m_{k-1}} f_{k-1}^{(m_{k-1})}(x) f_k^{(m-m_1-\dots-m_{k-1})}(x),
 \end{aligned} \quad (3.8)$$

together with elementary properties of the cosecant and cotangent functions, and the power series expansion

$$\pi x \operatorname{csc} \pi x = \sum_{j=0}^{\infty} \frac{|2^j - 2|}{j!} |B_j| (\pi x)^j \quad (|x| < 1). \quad (3.9)$$

By collecting the computed residues appropriately one obtains formulae (3.1) and (3.2) of the Theorem.

We here list, for convenience, the first six functions $C_j(x)$ defined by (3.4):

$$C_j(x) = \frac{1}{j!} \frac{d^j}{ds^j} \operatorname{csc}(s+x) \Big|_{s=0} = \frac{1}{j!} \operatorname{csc}^{j+1} x \tilde{C}_j(x) \quad (3.10)$$

where

$$\tilde{C}_0(x) = 1$$

$$\tilde{C}_1(x) = -\cos x$$

$$\tilde{C}_2(x) = 2 - \sin^2 x$$

$$\tilde{C}_3(x) = \cos x (\sin^2 x - 6)$$

$$\tilde{C}_4(x) = \sin^4 x - 20 \sin^2 x + 24$$

$$\tilde{C}_5(x) = \cos x (-\sin^4 x + 60 \sin^2 x - 120).$$

We may add here that it is not difficult to evaluate the finite sums in (3.1) and (3.2) by a symbolic algebra system such as REDUCE [3].

4. SPECIAL CASES

In this section, we consider some special cases of (3.1) and (3.2). For notational purposes, we make use of the Lerch function [2, No. 9.550]

$$\Phi(z, u, v) = \sum_{k=0}^{\infty} \frac{z^k}{(k+v)^u} \quad (|z| < 1, -v \notin \{0, N\}), \quad (4.1)$$

which is a generalization of the polylogarithm function (1.2). In particular, for $u = 2$, $v = 1/2$, we have

$$\Phi\left(-z, 2, \frac{1}{2}\right) = \frac{4}{\sqrt{z}} \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{z})^{2k+1}}{(2k+1)^2} = \frac{4}{\sqrt{z}} \text{Ti}_2(\sqrt{z}) \quad (4.2)$$

where

$$\text{Ti}_2(z) = \int_0^z t^{-1} \arctan t \, dt \quad (4.3)$$

is the arctangent integral.

4.1 $r = -2$, $\alpha = 0$

In this case conditions (3.5), (3.6) become

$$2k \notin N, \quad \frac{1}{2}l \notin N, \quad 2K \in N,$$

which exclude all $k \in N$ (hence $L_k = 0$), but permit all odd values of $l \in N$, and all $K \in N$. Thus, from (3.2) with $\zeta = \sigma/\omega$, and using (1.2),

$$\begin{aligned}
\int_0^{\infty} \text{Li}_n(-\sigma x) \text{Li}_m(-\omega x^{-2}) \frac{dx}{x} &= (\pm)^{m+n} \left[2^{-(n+1)} \pi \zeta^{\pm 1} \Phi\left(-\zeta^{\pm 2}, m+n, \frac{1}{2}\right) \right. \\
&\quad \left. + 2^{-(n+1)} (m+n) \text{Li}_{m+n+1}(-\zeta^{\pm 2}) \pm 2^{-n} \ln \zeta \text{Li}_{m+n}(-\zeta^{\pm 2}) \right] \\
&\quad + H(\pm 1) (-\pi)^{m+n+1} 2^m \sum_{m_1=0}^{\infty} \frac{1}{m_1!} \left(-\frac{1}{\pi} \ln \zeta\right)^{m_1} \sum_{m_2=0}^{\infty} (-2)^{-m_2} B_{m_2}^* B_{m+n+1-m_1-m_2}^* \\
&\quad (|\zeta| > 1, m, n \in \mathbb{N}, |\arg \sigma| < \pi, |\arg \omega| < \pi). \quad (4.4)
\end{aligned}$$

In particular, for $n = m = 1$, using (1.4):

$$\begin{aligned}
\int_0^{\infty} \ln(1 + \sigma x) \ln(1 + \omega x^{-2}) \frac{dx}{x} &= \\
&= \pi \text{Li}_2(\zeta^{\pm 1}) + \frac{1}{2} \text{Li}_3(-\zeta^{\pm 2}) \pm \frac{1}{2} \ln \zeta \text{Li}_2(-\zeta^{\pm 2}) + \begin{cases} \frac{1}{3} \ln^3 \zeta + \frac{5}{12} \pi^2 \ln \zeta & (|\zeta| > 1) \\ 0 & (|\zeta| < 1) \end{cases} \quad (4.5)
\end{aligned}$$

For $n = 1, m = 2$:

$$\begin{aligned}
\int_0^{\infty} \ln(1 + \sigma x) \text{Li}_2(-\omega x^{-2}) \frac{dx}{x} &= \\
&= \pm \frac{1}{4} \pi \zeta^{\pm 1} \Phi\left(-\zeta^{\pm 2}, 3, \frac{1}{2}\right) \pm \frac{3}{4} \text{Li}_4(-\zeta^{\pm 2}) + \frac{1}{2} \ln \zeta \text{Li}_3(-\zeta^{\pm 2}) \\
&\quad - \begin{cases} \frac{1}{6} \ln^4 \zeta + \frac{5}{12} \pi^2 \ln^2 \zeta + \frac{53}{480} \pi^4 & (|\zeta| > 1) \\ 0 & (|\zeta| < 1) \end{cases} \quad (4.6)
\end{aligned}$$

For $n = 2, m = 1$:

$$\int_0^{\infty} \text{Li}_2(-\sigma x) \ln(1 + \omega x^{-2}) \frac{dx}{x} = \frac{1}{2} \int_0^{\infty} \ln(1 + \sigma x) \text{Li}_2(-\omega x^{-2}) \frac{dx}{x}, \quad (4.7)$$

For $n = m = 2$:

$$\int_0^{\infty} \text{Li}_2(-\sigma x) \text{Li}_2(-\omega x^{-2}) \frac{dx}{x} =$$

$$\frac{1}{8} \pi \zeta^{\bar{1}} \phi\left(-\zeta^{\bar{2}}, 4, \frac{1}{2}\right) + \frac{1}{2} \text{Li}_5(-\zeta^{\bar{2}}) \pm \frac{1}{4} \ln \zeta \text{Li}_4(-\zeta^{\bar{2}})$$

$$- \begin{cases} \frac{1}{30} \ln^5 \zeta + \frac{5}{36} \pi^2 \ln^3 \zeta + \frac{159}{1440} \pi^4 \ln \zeta & (|\zeta| > 1) \\ 0 & (|\zeta| < 1) \end{cases} \quad (4.8)$$

Using the fact that the integral is continuous for $|\arg \sigma/\omega| < \pi/2$, we can set $\sigma = \omega = 1$ and obtain from (4.4):

$$\int_0^{\infty} \text{Li}_n(-x) \text{Li}_m(-x^{-2}) \frac{dx}{x} = 2^{-(n+1)} \left[\pi \phi\left(-1, n+m, \frac{1}{2}\right) + (n+m) \text{Li}_{n+m+1}(-1) \right]$$

$$= 2^{-(n+1)} \left[\pi 2^{n+m} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{n+m}} - (n+m) \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{n+m+1}} \right]. \quad (4.9)$$

In the case of $n+m$ odd, these series are well known (e.g. [11, No. 5.1.3.3, 5.1.4.2]). We thus obtain

$$\int_0^{\infty} \text{Li}_n(-x) \text{Li}_m(-x^{-2}) \frac{dx}{x} = 2^{-(n+2)} \frac{\pi^{n+m+1}}{(n+m-1)!} \left[|E_{n+m-1}| - \frac{2^{n+m+1} - 2}{n+m+1} |B_{n+m+1}| \right].$$

$$(n, m \in \mathbb{N}, n+m \text{ odd}) \quad (4.10)$$

This formula is remarkable in so far as it contains both the Bernoulli and Euler numbers. For $n = m = 1$, formula (4.9) reduces to

$$\int_0^{\infty} \ln(1+x) \ln(1+x^{-2}) \frac{dx}{x} = \pi G - \frac{3}{8} \zeta(3), \quad (4.11)$$

where G is Catalan's constant.

4.2 $r = 2, \alpha = -2$

In this case conditions (3.5), (3.6) become

$$2k + 2 \notin \mathbb{N}, \quad \frac{1}{2} \ell \pm 1 \notin \mathbb{N}, \quad 2K + 2 \in \mathbb{N},$$

which exclude all $k \in \mathbb{N}$ (hence $\Gamma_k = 0$), but permit all odd values of $\ell \in \mathbb{N}$, all $K \in \mathbb{N} \setminus \{1\}$ if $|\zeta| > 1$, and all $K \in \mathbb{N}$ if $|\zeta| < 1$. Thus, from (3.1) with $\zeta = \sigma/\sqrt{\omega}$

$$\int_0^\infty x^{-3} \text{Li}_n(-\sigma x) \text{Li}_m(-\omega x^2) dx =$$

$$\begin{aligned} & (-1)^n \left[2^{m-1} \frac{\omega^{3/2}}{\sigma} \sum_{\ell=0}^\infty \frac{(-\zeta^{-2})^\ell}{(2\ell+1)^n (2\ell+3)^m} \right. \\ & + 2^{-n-1} \omega \left[m \sum_{k=1}^\infty \frac{(-\zeta^{-2})^k}{k^n (k+1)^{m+1}} + n \sum_{k=1}^\infty \frac{(-\zeta^{-2})^k}{k^{n+1} (k+1)^m} + 2 \ln \zeta \sum_{k=1}^\infty \frac{(-\zeta)^{-2}}{k^n (k+1)^m} \right] \\ & - \omega^{n+1} \sum_{n_1=0}^{n+1} \frac{1}{n_1!} \left(-\frac{1}{\omega} \ln \zeta \right)^{n_1} \times \\ & \left. \sum_{n_2=0}^{n+1-n_1} \binom{n_2+m-1}{m-1} (-2\omega)^{-n_2} \sum_{n_3=0}^{n+1-n_1-n_2} 2^{-n_3} B_{n_3}^* B_{n+1-n_1-n_2-n_3}^* \right] \quad (4.12a) \end{aligned}$$

$$(|\zeta| > 1, m, n \in \mathbb{N}, |\arg \sigma| < \pi, |\arg \omega| < \pi)$$

$$\int_0^{\infty} x^{-3} \text{Li}_n(-\sigma x) \text{Li}_m(-\omega x^2) dx =$$

$$(-1)^m \left[2^{m-1} \sigma \omega \sum_{\ell=0}^{\infty} \frac{(-z^2)^{\ell}}{(2\ell+1)^n (2\ell-1)^m} \right. \\ \left. - 2^{-n-1} \sigma^2 \left[m \sum_{k=1}^{\infty} \frac{(-z^2)^k}{k^{m+1} (k+1)^n} + n \sum_{k=1}^{\infty} \frac{(-z^2)^k}{k^m (k+1)^{n+1}} - 2 \ln z \sum_{k=1}^{\infty} \frac{(-z^2)^k}{k^m (k+1)^n} \right] \right] \\ - \pi^{m+1} 2^{m-n} \sigma^2 \sum_{m_1=0}^{m+1} \frac{1}{m_1!} \left(-\frac{1}{\pi} \ln z \right)^{m_1} x \\ \sum_{m_2=0}^{m+1-m_1} \binom{m_2-n-1}{n-1} (2\pi)^{-m_2} \sum_{m_3=0}^{m+1-m_1-m_2} 2^{-m_3} B_{m_3}^* B_{m+1-m_1-m_2-m_3}^* \quad (4.12b)$$

$$(|z| < 1, m, n \in \mathbb{N}, |\arg \sigma| < \pi, |\arg \omega| < \pi).$$

Using the relations [11, No. 5.2.5.5, 15, 5.2.6.1, 2]

$$\sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} x^{2\ell+3}}{(2\ell+1)(2\ell+3)} = \frac{1}{2} (1+x^2) \arctan x - \frac{1}{2} x, \\ \sum_{k=1}^{\infty} \frac{x^{k+1}}{k(k+1)^2} = 2x + (1-x) \ln(1-x) - \text{Li}_2(x), \\ \sum_{k=1}^{\infty} \frac{x^{k+1}}{k^2(k+1)} = (x-1) \ln(1-x) + x(\text{Li}_2(x) - 1), \\ \sum_{k=1}^{\infty} \frac{x^{k+1}}{k(k+1)} = x + (1-x) \ln(1-x), \quad (4.13)$$

We find for $n = m = 1$, after some calculation, (4.13) can be written as

$$\int_0^{\infty} x^{-3} \ln(1+ox) \ln(1+wx^2) dx =$$

$$= \begin{cases} \frac{1}{2} \left\{ (\sigma^2 + \omega) \left[\ln \left(\frac{\sigma}{\sqrt{\omega}} \right) \ln \left(1 + \frac{\omega}{\sigma^2} \right) - \frac{1}{2} \text{Li}_2 \left(-\frac{\omega}{\sigma^2} \right) - \pi \arctan \left(\frac{\sqrt{\omega}}{\sigma} \right) \right] \right. \\ \left. + \omega \left[\pi \frac{\sigma}{\sqrt{\omega}} + \ln^2 \left(\frac{\sigma}{\sqrt{\omega}} \right) + \frac{5}{12} \pi^2 \right] \right\} & (|\sigma/\sqrt{\omega}| > 1) \\ \frac{1}{2} \left\{ (\sigma^2 + \omega) \left[\ln \left(\frac{\sigma}{\sqrt{\omega}} \right) \ln \left(1 + \frac{\sigma^2}{\omega} \right) + \frac{1}{2} \text{Li}_2 \left(-\frac{\sigma^2}{\omega} \right) + \pi \arctan \left(\frac{\sigma}{\sqrt{\omega}} \right) \right] \right. \\ \left. + \sigma^2 \left[\pi \frac{\sqrt{\omega}}{\sigma} - \ln^2 \left(\frac{\sigma}{\sqrt{\omega}} \right) - \frac{5}{12} \pi^2 \right] \right\} & (|\sigma/\sqrt{\omega}| < 1) \end{cases} \quad (4.14)$$

For $\sigma \rightarrow \sqrt{\omega}$, we obtain in the limit, using $\text{Li}_2(-1) = -\pi^2/12$,

$$\int_0^{\infty} x^{-3} \ln(1+x) \ln(1+x^2) dx = \frac{\pi}{2}. \quad (4.15)$$

Using the fact that the integral is continuous at $\sigma = \omega = 1$, we obtain

$$\begin{aligned} \int_0^{\infty} x^{-3} \text{Li}_n(-x) \text{Li}_m(-x^2) dx &= (-1)^n \left[2^{m-1} \pi \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)^n (2\ell+3)^m} \right. \\ &\quad \left. + 2^{-n-1} \left[m \sum_{k=1}^{\infty} \frac{(-1)^k}{k^n (k+1)^{m+1}} + n \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{n+1} (k+1)^m} \right] \right. \\ &\quad \left. - \pi^{n+1} \sum_{n_2=0}^{n+1} \binom{n_2+m-1}{m-1} (-2\pi)^{-n_2} \sum_{n_3=0}^{n+1-n_2} 2^{-n_3} B_{n_3}^* B_{n+1-n_2-n_3}^* \right]. \end{aligned} \quad (4.16)$$

From [11, No. 5.1.24.14] we see that the first sum can in general be expressed only in terms of generalized zeta functions $\zeta(j, 1/2)$ and $\zeta(j, 1/4)$. For $n = m$, however, using the relations [11, No. 5.1.24.10, 5.1.24.15]

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^m (k+1)^n} &= (-1)^{m-1} \sum_{k=0}^{n-2} \binom{m+k-1}{k} (1 - 2^{1-n+k}) \zeta(n-k) \\ &+ \sum_{k=0}^{m-2} (-1)^k \binom{n+k-1}{k} (1 - 2^{1-m+k}) \zeta(m-k) \\ &+ (-1)^m \binom{m+n-1}{n-1} + (-1)^{m-1} \binom{m+n-2}{m-1} 2 \ln 2^* \end{aligned} \quad (4.17)$$

($m, n \in \mathbb{N}$)

and

$$\sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{(2\ell+1)^n (2\ell+3)^n} = (-1)^n \left[\frac{1}{2} - \frac{1}{2^{2n} (n-1)!} \sum_{k=0}^{[(n-1)/2]} \frac{(2n-2k-2)!}{(n-2k-1)! (2k)!} 2^{2k} |E_{2k}| \right]^{**} \quad (4.18)$$

($n \in \mathbb{N}$)

*) Note the misprint in [11, No. 5.1.24.10], where the factor $(-1)^m$ reads $(-1)^n$.

**) Note that [11, No. 5.1.24.15] is erroneous: $|E_{2k}|$ there reads E_{2k} and

$\frac{(2n-2k-2)!}{(n-2k-1)!}$ reads $\frac{(2n-k-2)!}{(n-k-1)!}$.

where E_{2k} are the Euler numbers, we find that

$$\begin{aligned}
 \int_0^{\infty} x^{-3} \text{Li}_n(-x) \text{Li}_n(-x^2) dx = & \\
 & 2^{n-2} \pi - \frac{2^{-n-1}}{(n-1)!} \left[\pi^2 \sum_{k=0}^{[(n-1)/2]} \frac{(2n-2k-2)!}{(n-2k-1)!(2k)!} \pi^{2k} |E_{2k}| \right. \\
 & + (-1)^{n+1} \left[\sum_{k=0}^{n-1} [(-1)^n - (-1)^k] (k-n) \frac{(n+k-1)!}{k!} (1-2^{k-n}) \zeta(n-k+1) \right. \\
 & \left. \left. - (2\pi)^{n+1} \sum_{k=0}^n (-1)^k \frac{(k+n-1)!}{k!} (2\pi)^{-k} \sum_{j=0}^{n+1-k} 2^{-j} B_j^* B_{n+1-k-j}^* \right] \right] . \quad (4.19)
 \end{aligned}$$

Because of the fact that only even integers occur as arguments for the zeta function, expression (4.19) represents a polynomial in π with rational coefficients. However, when computing (with REDUCE) the right-hand side of (4.19) for small values of n , it turns out that only the first term remains. We therefore conjecture that, for all $n \in \mathbb{N}$,

$$\int_0^{\infty} x^{-3} \text{Li}_n(-x) \text{Li}_n(-x^2) dx = 2^{n-2} \pi . \quad (4.20)$$

This is equivalent to the statement that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
& (-1)^n \sum_{k=0}^{[(n-1)/2]} \frac{(2n-2k-2)!}{(n-2k-1)!(2k)!} 2^{2k+2} |E_{2k}| \\
& + \sum_{k=0}^{n-1} [(-1)^n - (-1)^k] (k-n) \frac{(n+k-1)!}{k!} (1-2^{k-n}) \zeta(n-k+1) \\
& + \sum_{k=0}^n (-1)^k \frac{(n+k-1)!}{k!} (2\pi)^{n-k+1} \times \\
& \sum_{j=0}^{n-k+1} 2^{-j} \frac{|2^j-2| |2^{n+1-k-j}-2|}{j!(n+1-k-j)!} |B_j| |B_{n+1-k-j}| = 0. \quad (4.21)
\end{aligned}$$

However, we have not been able to prove this relation. We note that, for $n = 1$, equation (4.20) agrees with (4.15).

4.3 $r = 1, g = -1/2$

In this case conditions (3.5), (3.6) become

$$k + \frac{1}{2} \notin \mathbb{N}, \quad \ell \pm \frac{1}{2} \notin \mathbb{N}, \quad K + \frac{1}{2} \in \mathbb{N}$$

which permit all $k, \ell \in \mathbb{N}$ and exclude all $K \in \mathbb{N}$ (hence $\Gamma_K = 0$). Thus from (3.1)

with $\zeta = \sigma/\omega$

$$\int_0^{\infty} x^{-3/2} \text{Li}_n(-\sigma x) \text{Li}_m(-\omega x) dx =$$

$$-(\pm 1)^{m+1} 2^{n/\sigma} \sqrt{\sigma} \sum_{k=1}^{\infty} \frac{z^{\bar{k}}}{k^m (1 \pm 2k)^n} \pm (\mp 1)^{n+1} 2^{m/\omega} \sqrt{\omega} \sum_{\ell=1}^{\infty} \frac{z^{\bar{\ell}}}{(1 \pm 2\ell)^m \ell^n}$$

$$- H(\mp 1) 2^{n/m+1} \sqrt{\sigma} \sum_{m_1=0}^m \frac{1}{m_1!} \left(-\frac{1}{\sigma} \ln z\right)^{m_1} \times$$

$$\sum_{m_2=0}^{m-m_1} \binom{m_2+n-1}{n-1} \left(\frac{2}{\sigma}\right)^{m_2} \sum_{m_3=0}^{m-m_1-m_2} B_{m_3}^* C_{m-m_1-m_2-m_3} \left(-\frac{\sigma}{2}\right)$$

$$- H(\pm 1) 2^{m/n+1} \sqrt{\omega} \sum_{n_1=0}^n \frac{1}{n_1!} \left(-\frac{1}{\omega} \ln z\right)^{n_1} \times$$

$$\sum_{n_2=0}^{n-n_1} \binom{m_2+n-1}{n-1} \left(\frac{2}{\omega}\right)^{n_2} \sum_{n_3=0}^{n-n_1-n_2} (-1)^{n_3} B_{n_3}^* C_{n-n_1-n_2-n_3} \left(-\frac{\omega}{2}\right) \quad (4.22)$$

$$(|z| \geq 1, m, n \in \mathbb{N}, |\arg \sigma| < \pi, |\arg \omega| < \pi).$$

Using the relations

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{k(2k+1)} = 2 - \frac{2}{x} \text{Arth } x - \ln(1-x^2) \quad *)$$

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{k(2k-1)} = 2x \text{Arth } x + \ln(1-x^2) \quad (4.23) \text{ cont.}$$

*) Note the misprint in [11, No. 5.2.5.8].

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{k(2k+1)^2} = -\ln(1-x^2) - \frac{2}{x} \operatorname{Arth} x - \frac{1}{x} [\operatorname{Li}_2(x) - \operatorname{Li}_2(-x)] + 4$$

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{k(2k-1)^2} = -\ln(1-x^2) - 2x \operatorname{Arth} x + x [\operatorname{Li}_2(x) - \operatorname{Li}_2(-x)]$$

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2(2k+1)} = \operatorname{Li}_2(x^2) + 2 \ln(1-x^2) + \frac{4}{x} \operatorname{Arth} x - 4 \quad (4.23)$$

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2(2k-1)} = -\operatorname{Li}_2(x^2) + 2 \ln(1-x^2) + 4x \operatorname{Arth} x$$

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2(2k+1)^2} = \operatorname{Li}_2(x^2) + 4 \ln(1-x^2) + \frac{8}{x} \operatorname{Arth} x + \frac{2}{x} [\operatorname{Li}_2(x) - \operatorname{Li}_2(-x)] - 12$$

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{k^2(2k-1)^2} = \operatorname{Li}_2(x^2) - 4 \ln(1-x^2) - 8x \operatorname{Arth} x + 2x [\operatorname{Li}_2(x) - \operatorname{Li}_2(-x)]$$

and [5, p. 6]

$$\operatorname{Li}_2(x^2) = 2 [\operatorname{Li}_2(x) + \operatorname{Li}_2(-x)] , \quad (4.24)$$

we can express (4.22) in closed form for $n = 1, 2$ and $m = 1, 2$. After some calculation with REDUCE we obtain for $n = m = 1$:

$$\int_0^{\infty} x^{-3/2} \ln(1+ox) \ln(1+wx) dx =$$

$$4\pi \left[\sqrt{o} \ln \left(1 + \sqrt{\frac{w}{o}} \right) + \sqrt{w} \left[\ln \left(1 + \sqrt{\frac{w}{o}} \right) - \ln \sqrt{\frac{w}{o}} \right] \right] , \quad (4.25)$$

where, because of the symmetry of the integral with respect to σ and ω , we can assume $|\sigma/\omega| > 1$.

For $n = 1$, $m = 2$, formula (4.22) reduces to

$$\int_0^{\infty} x^{-3/2} \ln(1 + \sigma x) \text{Li}_2(-\omega x) dx$$

$$= \begin{cases} 4\omega \left[2\sqrt{\sigma} \left[\text{Li}_2\left(-\sqrt{\frac{\omega}{\sigma}}\right) - \ln\left(1 + \sqrt{\frac{\omega}{\sigma}}\right) \right] - \sqrt{\omega} \left[2 \ln\left(1 + \sqrt{\frac{\omega}{\sigma}}\right) - \ln \sqrt{\frac{\omega}{\sigma}} \right] \right] & (|\sigma/\omega| > 1) \\ -\omega \left[8\sqrt{\omega} \ln\left(1 + \sqrt{\frac{\sigma}{\omega}}\right) + \sqrt{\sigma} \left[8 \ln\left(1 + \sqrt{\frac{\sigma}{\omega}}\right) + \ln^2 \frac{\sigma}{\omega} - 4 \ln \frac{\sigma}{\omega} + 8 \text{Li}_2\left(-\sqrt{\frac{\sigma}{\omega}}\right) + \frac{4}{3} \pi^2 \right] \right] & (|\sigma/\omega| < 1) \end{cases} \quad (4.26a)$$

$$(4.26b)$$

The same expression, with σ and ω interchanged, holds for $n = 2$, $m = 1$.

For $n = m = 2$, we obtain from (4.22)

$$\int_0^{\infty} x^{-3/2} \text{Li}_2(-\sigma x) \text{Li}_2(-\omega x) dx =$$

$$2\omega \left[8\sqrt{\sigma} \left[2 \ln\left(1 + \sqrt{\frac{\omega}{\sigma}}\right) - \text{Li}_2\left(-\sqrt{\frac{\omega}{\sigma}}\right) \right] \right.$$

$$\left. + \sqrt{\omega} \left[16 \ln\left(1 + \sqrt{\frac{\omega}{\sigma}}\right) + \ln^2 \frac{\omega}{\sigma} - 8 \ln \frac{\omega}{\sigma} + 8 \text{Li}_2\left(-\sqrt{\frac{\omega}{\sigma}}\right) + \frac{4}{3} \pi^2 \right] \right], \quad (4.27)$$

where, as for (4.25), we can assume $|\sigma/\omega| > 1$.

For $\sigma = w = 1$, the above results lead to

$$\int_0^{\infty} x^{-3/2} \ln^2(1+x) dx = 8\pi \ln 2$$

$$\int_0^{\infty} x^{-3/2} \ln(1+x) \operatorname{Li}_2(-x) dx = -\frac{2}{3} \pi(w^2 + 24 \ln 2) \quad (4.28)$$

$$\int_0^{\infty} x^{-3/2} [\operatorname{Li}_2(-x)]^2 dx = \frac{8}{3} \pi(w^2 + 24 \ln 2).$$

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