SPIN-SPIN AND SPIN-ORBIT POTENTIALS FROM LATTICE GAUGE THEORY

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A B S T R A C T

We evaluate spin-spin and spin-orbit potentials from lattice gauge theory simulations for the pure gauge sector with both SU(2) and SU(3) colour groups. We pay particular attention to understanding the normalization of these results. The tensor, scalar and "opposite side" spin-orbit potentials are close to the results from perturbative gluon exchange and we discuss in detail this comparison on a lattice. The "same side" spin-orbit potential is substantial and long range - it can be associated with the confining component of the central potential if the confining mechanism is effectively Lorentz scalar. Our results for several β -values allow a discussion of scaling.

1. INTRODUCTION

For heavy quarks of mass m in Q.C.D., it is sensible to consider their interaction to order m^{-2} and this is described in terms of spin-spin and spin-orbit potentials¹⁻³

$$V = V_0(R) + \frac{\mathbf{S}_1 \cdot \mathbf{L}_1 + \mathbf{S}_2 \cdot \mathbf{L}_2}{2m^2 R} \left[\frac{dV_0}{dR} + 2 \frac{dV_1}{dR} \right] + \frac{\mathbf{S}_1 \cdot \mathbf{L}_2 + \mathbf{S}_2 \cdot \mathbf{L}_1}{m^2 R} \frac{dV_2}{dR} \\ + \frac{1}{m^2} \left[\mathbf{S}_1 \cdot \hat{\mathbf{R}} \mathbf{S}_2 \cdot \hat{\mathbf{R}} - \frac{1}{3} \mathbf{S}_1 \cdot \mathbf{S}_2 \right] V_3 + \frac{1}{3m^2} \mathbf{S}_1 \cdot \mathbf{S}_2 V_4 \quad (1.1)$$

These potentials can be related¹⁻³ to correlations of electric and magnetic colour fields in the presence of the heavy quark and anti-quark which act as colour sources. Taking the separation $\mathbf{R} = \mathbf{r}_2 - \mathbf{r}_1$ along the z -direction, this implies

$$V_1'(R) = g^2 \int_{-\infty}^{\infty} d\tau \tau \langle B_x(r_1, 0) E_y(r_1, \tau) \rangle \\ V_2'(R) = g^2 \int_{-\infty}^{\infty} d\tau \tau \langle B_x(r_2, 0) E_y(r_1, \tau) \rangle \\ V_3(R) = g^2 \int_{-\infty}^{\infty} d\tau \langle -B_x(r_1, 0) B_x(r_2, \tau) + B_z(r_1, 0) B_z(r_2, \tau) \rangle \\ V_4(R) = g^2 \int_{-\infty}^{\infty} d\tau \langle 2 B_x(r_1, 0) B_x(r_2, \tau) + B_z(r_1, 0) B_z(r_2, \tau) \rangle \quad (1.2)$$

These relations are valid non-perturbatively and thus a lattice gauge theory evaluation of these colour field correlations will lead directly to information about the spin structure of the heavy quark potential. This procedure has already been exploited⁴⁻⁷ to gain considerable understanding of this spin dependence. The salient points are that one gluon exchange is a good approximation to the spin-dependent potentials⁴ while the two spin-orbit potentials (V_2' and V_1') are respectively close to one gluon exchange⁵ and of long range corresponding to a Lorentz scalar confining mechanism⁶.

There remain unsettled questions which we address in this

paper. The absolute normalization of these potentials is as yet incompletely understood. The identity due to Gromes³

$$V_0'(R) = V_2'(R) - V_1'(R) \quad (1.3)$$

relates the central potential to the spin-orbit potentials and lattice determinations^{6,7} have reported factors of 3 - 4 adrift in normalization.

For both SU(2) and SU(3), we study the scaling (β -dependence) particularly for the long range spin-orbit to probe further its relationship to the confining component of the central potential. For SU(3), our results at $\beta = 5.7$ are compared with those of ref(7) at $\beta = 6.2$ to investigate scaling, while for SU(2) we have accurate results at $\beta = 2.4$ and $\beta = 2.575$.

Our technique allows us to probe the effective one gluon exchange in considerable detail and we compare with the perturbative expressions. Including order g^4 contributions gives a reasonable description of our results for the spin dependent potentials.

The next section presents our lattice evaluations of the colour field correlations and the resulting potentials. Section 3 discusses perturbative results for gluon exchange both in the continuum and on a lattice. This sheds light on the normalization and self-energy contributions and provides an indication of the nature of lattice artefacts. We then discuss the implications of our lattice results for continuum physics.

2. LATTICE EVALUATION OF COLOUR FIELD CORRELATIONS

In lattice evaluations, the presence of static colour sources at the origin and at $z = R$ is represented by a Wilson loop of size $R \times T$ where T is taken sufficiently large. Then the colour field $F_{\mu\nu}$ in the neighbourhood of one source can be evaluated by

considering the path ordered product $U_{\mu\nu}$ around a square of lattice spacing a in the μ - ν plane. Thus

$$U_{\mu\nu} = e^{ia^2 g F_{\mu\nu}} + O(a^4)$$

or $\frac{1}{2}(U_{\mu\nu} - U_{\mu\nu}^+)_{ij} = i a^2 g \lambda_{ij}^a F_{\mu\nu}^a + O(a^4)$ (2.1)

Then for lattice spacing a sufficiently small, one can evaluate the colour fields. Now corrections of order $g^2 a^4$ are not small at the g^2 -values we shall use - for instance the difference of the average plaquette action from 1 is just of this order. The prescription we use here is to remove the self-energy of the colour-field measuring loop by considering the ratio of $U_{\mu\nu} - U_{\mu\nu}^+$ to $U_{\mu\nu} + U_{\mu\nu}^+$. To order g this gives the same result as Eq. 2.1 but the self-energy terms are now removed - as we shall show in detail in section 3. In practice we evaluate the ratio of Wilson loops illustrated in fig. 1d.

Now the path $U_{\mu\nu}$ effectively measures the colour field at the centre of the square, whereas the colour field at the source site is needed. This is accomplished by averaging over the four squares in the μ - ν plane with corners touching the source site. As discussed⁴, there are several alternative prescriptions for evaluating the correlation of the colour fields - some have better behaviour at small source separations R , some are faster to evaluate numerically, while some have better statistical precision when evaluated in lattice simulations. Here we take a prescription which is a compromise (it is the same as that used in ref.7). For B we take the sum of the four squares as discussed above, for E_x and E_y we take the sum of two squares with centre at a half-integral value of t/a , except that for the correlation of transverse B -fields the $R = 2a$ B -measuring loops would touch so we use instead $B_x(z=-a/2)B_x(z=R-a/2) + B_x(z=+a/2)B_x(z=R+a/2)$ where each value of B_x is here an average of two squares (at $y=\pm a/2$).

The generic Wilson loop to be evaluated is a $R \times T$ rectangle with two 'ears' corresponding to the $U_{\mu\nu}$ insertions - see Fig. 1. However, the results will only correspond to the colour field correlations in the **ground state** of the static quark potential at separation R if these 'ears' are each at a time separation from the end of the rectangle which is large enough to ensure that only the lowest energy eigenstate of the transfer matrix is contributing. This implies a time separation $t_E > R$ from each end and since the colour field correlation is needed for separation $At > R$ also, the total length T of the Wilson loop must be large ($> 3R$). Thus it will have a small numerical value and hence large statistical errors. To improve on this situation, one can take advantage of a variational ansatz for the path combinations at each end⁴. This already gives a good description of the ground state eigenmode for $t_E = a$ and the statistical errors are much reduced. In the present work we follow this variational approach and use the same two path combinations - a straight line and a sum of four straight lines translated by one transverse spacing at each end. The main difference in our present work is that we use multi-hit link evaluation to reduce the variance for the maximum of links - as illustrated in Fig. 1a by the thicker lines.

For $SU(2)$, we use 9 independent 16^4 lattices with periodic boundary conditions which are updated by 100 sweeps (after 500 sweeps from hot or cold starts to equilibrate) using a sweep of 4 Metropolis steps per link of the icosahedral subgroup of $SU(2)$ at $\beta = 2.4$ and $\beta = 2.575$. The multi-hit links are evaluated analytically using a look-up table. For $SU(3)$, we use 44 lattices at $\beta = 5.7$ of size 8^4 with periodic boundary conditions which are updated by 90 sweeps (after 540 sweeps from hot or cold starts) using a sweep of 3 Metropolis steps per link of the discrete

subset M(38880) of $SU(3)$ ^{8,9}. The multi-hit links are evaluated by averaging 100 Metropolis updates and these averages are stored. The β -value of 5.7 was chosen as the lowest value at which rotational invariance is restored¹⁰.

The potential itself has been measured on these lattices¹¹ and those results are used to determine the relative coefficient C of the two path combinations from the variational analysis of the $T = a$ to $T = 3a$ eigenmodes⁴. These potential measurements show that the statistical error on the Wilson loop is reasonably small for total time length $\leq 5a$. Our evaluations of the colour field correlations are thus performed with total time separation $T = 5a$, and for $SU(3)$ we also used $T = 4a$ to get smaller errors on some quantities. The source separation used was $R \geq 2a$ since our multi-hit implementation requires this. Since even the central potential shows large lattice artefacts at $R = a$, this is no restriction.

Each colour field correlation to be evaluated is in fact a sum of many Wilson loops- the measuring insertions are a sum over 8×8 terms (for B_z B_z correlations) and the variational paths involve a sum over 5×5 terms plus of course the effect of the multi-hit links which correspond to a sum over infinitely many samples of that link in the fixed presence of the other links. It is possible to carry these sums out before evaluating the Wilson loop in such a way that the total computing time required is quite modest (about 100 hours of IBM 3081 time).

Our results are contained in Table 1. The quantity tabulated is the ratio of the average of the Wilson loop with 'ears' to the average of the Wilson loop itself, normalized by dividing by the same ratio for each "ear" alone - see fig. 1d. These normalization factors are well determined numerically and are

insensitive to the t -value of the "ear". There are some symmetries in the evaluation of the "two-eared" Wilson loop which have been averaged over. For instance, correlations are obtained for field insertions at time values t and t' where $a \leq t, t' \leq T-a$, while the results should only depend on $\Delta t = t-t'$. This independence is a cross check of our assertion that we are able to separate the ground state eigenmode since for t, t' near the ends the relative amount of excited states will be enhanced. Within our statistical errors we see good overall evidence for this independence, and in the few cases where there is some sign of a dependence on $(t+t')$ we increase our error estimate to include this variation. For SU(3) a similar comparison can be made for the $T = 4a$ and $T = 5a$ results and the Table has errors derived from combining the information from these two evaluations.

To evaluate the potentials one should sum over $t-t'$ as indicated in Eq 1.2. For the spin-spin potentials this sum converges rapidly - for the extrapolated component we use as upper error estimate the t -dependence of the perturbative one gluon exchange⁴ and as lower estimate we truncate the $t-t'$ sum at the measured values. For the spin-orbit potentials, the extra factor of $(t-t')$ in the sum causes a slower convergence. To control our estimate of the tail, we use the theoretical fact that the slowest fall-off at large $t-t'$ of the E.B correlation corresponds to the largest E_u representation (of D_{4h}) eigenstate of the transfer matrix⁴. This has been measured¹¹ using variational techniques and so provides an upper bound on the extrapolation - see Fig. 3. An underestimate is to truncate the $t-t'$ sum at the largest value measured. As a best estimate we use the rate of fall-off observed from the largest two $t-t'$ values measured and extrapolate at that rate (this is equivalent to estimating the effective E_u eigenvalue

from this data). Table 1 contains the results of these extrapolated $t-t'$ sums.

3. PERTURBATIVE GLUON CONTRIBUTIONS AND NORMALIZATION

To interpret the lattice results on the spin dependent potentials, it is worthwhile to compare them with the perturbative results. We briefly review the continuum results to leading order in g^2 and then discuss the lattice evaluation to the same order which throws light on the normalization. Finally we discuss the effect of order g^4 terms.

To order g^2 , the continuum perturbation theory results, expressed as integrals over the gluon 3-momentum q , are

$$\begin{aligned}
 V_0(R) &= - \frac{N^2-1}{2N} g^2 \int \frac{dq}{2\pi^2} \frac{\sin qR}{qR} &= - \frac{N^2-1}{2N} \frac{g^2}{4\pi R} \\
 V_4(R) &= \frac{N^2-1}{2N} g^2 \int \frac{dq}{2\pi^2} \frac{\sin qR}{qR} q^2 &= 2 \frac{N^2-1}{2N} g^2 \delta^3(R) \\
 V_9(R) &= - \frac{N^2-1}{2N} g^2 \int \frac{dq}{2\pi^2} j_2(qR) q^2 &= 3 \frac{N^2-1}{2N} \frac{g^2}{4\pi R^3} \\
 V_2'(R) &= \frac{N^2-1}{2N} g^2 \int \frac{dq}{2\pi^2} j_1(qR) q^2 R &= \frac{N^2-1}{2N} \frac{g^2}{4\pi R^2} \\
 V_1'(R) &= 0
 \end{aligned} \tag{3.1}$$

For $q^2 \gg \Lambda^2$, asymptotic freedom applies and g^2 is sufficiently small to justify a perturbative treatment (albeit with some resummation which changes $g^2/4\pi$ into α_s at a scale given by R^{-12} as discussed later). Eq. 3.1 shows that the central potential V_0 has important contributions from small values of q^2 for any value of R and so there will be a component not well approximated by lowest order perturbation theory. For the spin dependent potentials, however, the integrals over q^2 are suppressed by factors of q^2 to q^4 at low q^2 so that the lowest order result is more reliable. The generalization to a vector exchange of the form $G(q)\{g_{\mu\nu} - q_\mu q_\nu/q^2\}/q^2 + \xi q_\mu q_\nu/q^4$ is that the integrands of

Eq. 3.1 are multiplied by $G(q)$. For such a generalized vector exchange the potentials are related since³

$$V_4 = 2\nabla^2 V_0, \quad V_3 = -(V_0'' - V_0'/R), \quad V_2' = V_0', \quad V_1' = 0. \quad (3.2)$$

As well as these potentials themselves, our method of evaluation gives information on the t -integrand of Eq. 1.2. This can be evaluated easily for one gluon exchange yielding

$$\begin{aligned} v_0(R, t) &= -\frac{e}{\pi} \frac{1}{R^2 + t^2}, & v_4(R, t) &= \frac{e}{\pi} \frac{12t^2 - 4R^2}{(R^2 + t^2)^3} \\ v_3(R, t) &= \frac{e}{\pi} \frac{8R^2}{(R^2 + t^2)^3}, & \tilde{v}_2(R, t) &= \frac{e}{\pi} \frac{8tR}{(R^2 + t^2)^3} \end{aligned} \quad (3.3)$$

$$\text{where } V(r) = \int dt \, v(R, t), \quad V_2'(R) = \int dt \, t \, \tilde{v}_2(R, t)$$

$$\text{and } e = \frac{N-1}{2N} \frac{g^2}{4\pi}.$$

This shows that the scalar potential $V_4(R)$ is zero at $R \neq 0$ because of a cancellation in the t -integral and so can be studied in more depth by investigating the t -integrand itself.

Now on a lattice one can also use weak coupling perturbation theory to evaluate Wilson loops¹³. In order to show how certain self-energy contributions come about, we shall present this for those Wilson loops involved in our spin-dependent potential evaluations. Working to order g^2

$$\langle WL \rangle = \frac{1}{N} \text{Tr} \langle \prod U \rangle = \frac{1}{N} \text{Tr} \langle \prod e^{\pm i a g \lambda^b A_\mu^b(x)} \rangle = \frac{1}{N} \text{Tr} \langle e^{i \sum J_\mu^b(x) A_\mu^b(x)} \rangle,$$

so that a Wilson loop acts like a sum of sources of strength $\pm a g \lambda^b$ to this order in g^2 . The sign depends on whether the link is directed in the μ or $-\mu$ direction. Also to this order, the gluon propagator is like a photon propagator for each colour and one can choose a gauge in which it has a Lorentz structure of $\delta_{\mu\nu}$. Then the gaussian integral over the A -fields yields

$$\begin{aligned} \langle WL \rangle &= \frac{1}{N} \text{Tr} e^{-\frac{1}{2} \sum J_\mu^b G_{\mu\nu}^{bc} J_\nu^c} \\ &= e^{-\frac{1}{2} a^2 g^2 C_F \sum j_\mu(x) G(x-x') j_\mu(x')} \end{aligned} \quad (3.4)$$

where since $(\lambda^b \lambda^b)_{ij} = C_F \delta_{ij} = (N^2-1)\delta_{ij}/(2N)$, the colour matrix in the exponent of Eq. 3.4 is diagonal. Here j_μ is a unit source (± 1) for each link of the Wilson loop and G is the lattice gluon propagator which for one colour and an infinite lattice of spacing a is

$$G(R,t) = a^2 \int_{-\pi/a}^{\pi/a} d^3 q \, dq_4 \frac{e^{i q \cdot R} e^{i q_4 t}}{(2\pi)^4} \frac{1}{\sum_i 4 \sin^2 q_i a/2} \quad (3.5)$$

This is approximately $\{4\pi^2(R^2+t^2)\}^{-1}$ for large R or t . For rectangular Wilson loops these expressions give

$$\langle WL \rangle = \exp \left[- \frac{N^2-1}{2N} g^2 a^2 \sum_i^T \sum_i^T \{ - G(R, t-t') + G(0, t-t') \} \right]$$

keeping the terms extensive in T only. Then since this expression at large T is $\exp(-V_0(R) T)$, one has

$$V_0(R) = - \frac{N^2-1}{2N} g^2 \left[G(R) - G(0) \right] \quad (3.6)$$

where $G(R) = a \sum_i^T G(R, t)$ and behaves as $(4\pi R)^{-1}$ at large R and $aG(0) = 0.254$. This expression, Eq. 3.6, is very well known. The extension to the more complicated shapes of Wilson loop we need involves only a little more work because of the $\delta_{\mu\nu}$ form of the propagator. The extra terms for large T are those linking the insertions, those self-energy contributions to an insertion and those linking an insertion to the rectangular Wilson loop itself (this case occurs only for electric insertions which have time-like links). Then for such a Wilson loop one has schematically

$$g^2 a^4 \langle B_z(0, t) B_z(R, t') WL \rangle = \exp \left[- \frac{N^2-1}{2N} g^2 a^2 \{ - G(R)T + G(0)T - \square_{xy}(R, t-t') + \square_{xy}(0, 0) \} \right]$$

where $\square_{xy}(R, t) = 4G(R, t) - G(R-ax, t) - G(R+ax, t) - G(R-ay, t) - G(R+ay, t)$ for a typical B-measuring loop as shown in fig. 1a. Now linear

combinations of Wilson loops are actually evaluated ($\frac{1}{2}U - \frac{1}{2}U^+$ at each side) and for them the sign of the term $\square_{xy}(R, t-t')$ changes so that the result (after dividing by the rectangular loop itself) is

$$g^2 a^4 \langle B_z(0, t) B_z(R, t') \rangle = \sinh \{ C_F g^2 a^2 \square_{xy}(R, t-t') \} \cdot \exp \{ - C_F g^2 a^2 \square_{xy}(0, 0) \}$$

Now $C_F g^2 a^2 \square_{xy}(2a, 0) \approx e/(4\pi)$ which is small in practice so the sinh can be approximated by its argument for $R \geq 2a$. On the other hand, $C_F g^2 a^2 \square_{xy}(0, 0) = 2\pi e = (N^2 - 1)/(2\beta)$ which is comparatively large. This latter contribution, $\exp(-2\pi e)$, corresponds to the self energy of the two unit squares and it has the same value as the square of the plaquette action to this order in g^2 . This contribution has been neglected previously as an order g^4 contribution but it seems consistent to keep all terms of order g^2 in the exponent of Eq. 3.4 in which case it should be included.

Another subtle point concerns the electric field measuring insertions which have time-directed links. Basically the two configurations illustrated in fig. 1c have different self energies. To order g^2 , these self energy contributions to a Wilson loop are given by factors of

$$e^{g^2 C_F \left[-\frac{1}{2} G(0)T - \frac{1}{2} a^2 \square_{yt}(0, 0) \pm a \{ G(0) - G(1) \} \right]}$$

where $\square_{yt}(0, 0) = 2\pi e$ and is the self energy of the unit square, the first term is the self energy of the line of length T and the third term is different in the two cases. Now $ag^2 C_F \{G(0) - G(1)\} = 2\pi e/3$ which is quite large. The net effect for $U - U^+$ is relatively small though because after summing over the different Wilson loops involved in the $B_x E_y$ evaluation, a factor of $\cosh(2\pi e/3)$ is present and this is quite close to 1.

This evaluation of the self-energy effects suggests that a

suitable ratio of Wilson loops can be devised in which these effects cancel. The simplest such ratio is just to divide the "two-eared" Wilson loop by the product of the average of the Wilson loop without "ears" and the square of the average plaquette action as suggested in ref.5. This does not take account of the self energy of the electric insertions even to order g^2 , so we prefer to use the more complete cancellation produced by the ratio illustrated in fig. 1d. In practice these two prescriptions are rather close numerically.

Empirical support for our normalization prescription comes from the results comparing $a \times a$ measuring loops with $a \times 2a$ loops. The latter were found to be suppressed⁴ by 70% which agrees with the ratio of loop self-energies expected. So our present prescription would give the same potential for either case.

For $\beta = 2.575$ with SU(2) these various prescriptions for the self energy normalization factor give (i) 1.79 from the order g^2 term in the exponent ($e^{+1.5/\beta}$) (ii) 2.26 from $\langle \text{plaquette} \rangle^{-2}$ as measured by Monte Carlo, and (iii) 2.24 and 2.21 from the measured Wilson loop ratios using the method of fig. 1d for B-B and B-E respectively (using the average of the two insertions shown in fig. 1c for the E-measurer). As discussed above, we use (iii).

As well as the self-energy contribution, lattice perturbation theory gives predictions for the observables of interest to order g^2 . For the measuring insertions we use, the potentials are given by

$$\begin{aligned}
 V_2'(R) &= \frac{1}{4} \left[V_0(R+a,0) - V_0(R-a,0) + V_0(R+a,a) - V_0(R-a,a) \right] \\
 V_3(R) &= \frac{1}{2} \left[-\Delta^2 V_0(R,0) - \Delta^2 V_0(R,a) + V_0(R,2a,a) - V_0(R,0,a) \right] \\
 V_4(R) &= \Delta^2 V_0(R,0) + \Delta^2 V_0(R,a) + \frac{1}{2} \left[V_0(R,2a,a) - V_0(R,0,a) \right] \\
 &\quad + \frac{3}{2} \left[V_0(R,2a,0) - V_0(R,0,0) \right] \tag{3.7}
 \end{aligned}$$

where $\Delta^2 V_0(R,x) = V_0(R+a,x) + V_0(R-a,x) - 2 V_0(R,x)$

This is the lattice equivalent of eq. 3.2 and relates these potentials to the central potential $V_0(z,x,y)$ as given in eq. 3.6 for vector exchange. To evaluate the gluon exchange contributions, we use the formalism developed above and carry out the momentum sums in Eq. 3.5 by explicitly summing over the discrete momenta on a 32^4 lattice with periodic boundary conditions (excluding the zero mode with $q = 0$ and $q_4 = 0$).

Results are shown in fig. 2 for the t -dependence of the order g^2 lattice gluon exchange contributions to the potentials at $\beta = 2.575$ for $SU(2)$. This t -dependence agrees well with the measured Wilson loop ratios and so provides a useful guide in completing the t -sums to obtain the potentials. The differences between lattice perturbation results and those from the continuum (again to order g^2) are shown for the R -dependences in fig. 3. The main difference is for V_4 which is non-zero on a lattice - the δ -function spills out to $R \neq 0$. The contribution to V_1' is zero from one gluon exchange and we return to a discussion of our non-zero results in the next section. Here we are trying to understand the normalization of those potentials that do get one gluon exchange pieces. Fig. 3 shows that the order g^2 gluon exchange gives too small a contribution.

To get more confidence in the normalization it is necessary to estimate the contributions from the next order in g^2 . For the central potential V_0 the required lattice calculations have been performed¹³. Basically the next order contributes a factor $1 + g^2 b_0 \ln(RM_T)^2$ where M_T is known¹³. Typically the correction is $\approx 50\%$ and it improves the agreement - for example for $V_0(R=a)$ at $\beta = 2.7$, the order g^4 calculation gives a value 0.82 of that from the lattice Wilson loop extraction¹⁴ of this potential (including the self-energy part). One can use renormalization

group arguments to re-express this order g^4 result as¹²

$$V_o'(R) = - \frac{C_F}{4\pi R^2 (b_o \ln(RA_R)^{-2} + (b_1/b_o) \ln \ln(RA_R)^{-2})} \quad (3.8)$$

For SU(2) at $\beta = 2.7$, this expression is found to agree perfectly with the lattice result¹⁴ for $R = a$ to $2a$. This is somewhat fortuitous since the order g^4 terms are relatively large so that the uncertainty at order g^6 must remain significant. In summary the central potential V_o measured by lattice simulation has a normalization about twice the naive order g^2 value ($-e/R$) at small R , the order g^4 terms add 50% while the resummation of eq. 3.8 can increase this to 100% to make the agreement even better.

For the spin dependence the picture is less complete. The order g^4 terms have been calculated in the continuum limit¹⁵. At this order other exchanges than vector also contribute. There are now two dimensionful parameters - R and m . This dependence on the quark mass m is irrelevant for the central potential to order m^{-2} , but it is essential for the spin dependence which is precisely an order m^{-2} effect. Now the order g^4 terms (for the Fourier transform of the spin dependent potential) include $\ln m^2$ factors. These arise from QCD radiative corrections which imply that the potentials as defined in eq. 1.1 must be m -dependent¹⁵. For $m \gg R^{-1}$, the $\ln m$ terms can be renormalization group improved into an anomalous dimension of the colour magnetic moment. Thus the naive magnetic moment $g/(2m)$ becomes⁴ at scale a

$$\mu_a = \frac{g_m}{2m} \left[\frac{g_a}{g_m} \right]^{5/11} \quad (3.9)$$

Physically this relation can be thought of as giving a reduced effective colour magnetic moment when sampled at a smaller scale a . This is relevant for comparing lattice results at different β -values although the small value of the exponent makes this

correction very small (circa 8% for our range of β -values). The treatment of the $\ln q$ terms at order g^4 is less straightforward than in the central potential case since two scales (m and R) are present. Thus a fully resummed expression like Eq. 3.8 cannot be readily obtained - so the overall normalization is harder to establish accurately.

An alternative route to obtain such resummed expressions for the spin-dependent potentials is to use eq. 3.2 to obtain them from derivatives of $V'_0(R)$ which is itself given absolutely by eq. 3.8. This gives relations for the vector exchange component

$$\begin{aligned} R^2 V'_0(R) &\equiv \frac{C_F g_{eff}^2(R)}{4\pi} \\ R^2 V'_2(R) &= \frac{C_F g_{eff}^2(R)}{4\pi} \\ R^3 V_3(R) &= \frac{C_F g_{eff}^2(R)}{4\pi} \left(3 - \frac{2}{\ln(\Lambda_R R)^{-2}} \right) \\ R^3 V_4(R) &= \frac{C_F g_{eff}^2(R)}{4\pi} \left(\frac{4}{\ln(\Lambda_R R)^{-2}} \right) \end{aligned} \quad (3.10)$$

Now eq. 3.8 implies that g_{eff}^2 is about twice g^2 and that it grows at large R . Then since $\ln(\Lambda_R R)^{-2}$ is about 4 (eg. for SU(2) at $\beta = 2.575$ and $R = 2a$), the effective strengths of the potentials will be about 2.0, 5.0 and 2.0 respectively rather than 1.0, 3.0 and 0.0 as at order g^2 . This estimate is closer to the observed results but it is important to take lattice effects into account.

This can be achieved by using eq. 3.7 to obtain predictions for our lattice observables from those for the central potential which has been studied to order g^4 on a lattice¹³. For a large lattice the momentum components q are continuous within the range $-\pi/a$ to π/a . This makes it easier to transcribe continuum results to lattice observables in momentum space. To satisfy the periodicity of the boundary conditions, aq_x is replaced by $2 \sin(aq_x/2)$, etc in this conversion. We have checked that the lattice results¹³ to order g^4 for the central potential can be

reproduced starting from the continuum result to this order in momentum space. Thus we modify the lattice gluon propagator of eq. 3.5 by including an extra factor

$$G(q) = 1 - b_0 g^2 \ln(a^2 q^2 / F^2) \quad (3.11)$$

in the integrand to take account of order g^4 effects. Now renormalization group invariance implies¹² that $F = \exp(31/66) (\Lambda_{\text{MSB}}/\Lambda_L)$. The resulting absolute prediction from the lattice Fourier transform is shown in fig. 3 where it is seen to be a big improvement on the order g^2 result. A resummation of this correction to give $G(q) = 1/b_0 \ln(q^2/\Lambda_q^2)$ with $\Lambda_q = F \Lambda_L$ leads to an even larger result, somewhat too large. Since the measured lattice $a(\beta)$ does not agree with the perturbative β -function, one should not expect this resummation to be very reliable.

4. DISCUSSION

Our method of taking account of the self-energy effects due to the colour-field measuring insertions gives the results shown in the Table and figures. The normalization is now such that Gromes' identity

$$V_0'(R) = V_2'(R) - V_1'(R)$$

is indeed satisfied as shown in fig. 3. Thus the central potential $V_0(R)$ obtained from lattice measurements of rectangular Wilson loops, etc is related to the difference of the spin-orbit potentials measured by using the colour field correlations. The difference of the continuous curves plotted in fig. 3 is taken from the fit to the measured central potential¹¹. For both SU(2) and SU(3) the agreement is acceptable although for SU(3) the spin-orbit difference is a little smaller than V_0' .

Furthermore the fact that V_2' has vector exchange contributions while V_1' has Lorentz scalar contributions enables

the nature of the confining component of the central potential to be studied. As already discovered⁶, the lattice evaluation suggests that the confining component is wholly scalar. This can be quantified by taking a model of the central potential

$$V_0(R) = S(R) + V(R), \quad \text{with } S(R) = \eta KR$$

$$\text{and } V(R) = (1-\eta)KR - e/R$$

so that $V_1'(R) = -\eta K$, and $V_2'(R) = (1-\eta)K + e/R^2$. Then decreasing η from 1 corresponds to moving both the solid curves up by the same amount in fig. 3. There is excellent agreement with $\eta = 1$ and a limit of $\eta > 0.7$ can be set.

This conclusion is in complete agreement with analysis of experimental results on the splitting of the χ_b levels¹⁶. Our overall results for the potentials also lend support to the assumptions used in extracting these conclusions from the spectrum.

We have discussed Lorentz vector and scalar exchange contributions but one should check whether other contributions are present. One way to do this is to study the combination $R(2V_3 + V_4) - 6V_2'$ which is zero if there are only vector or scalar exchanges. For $R > 2a$ (to avoid lattice effects) this combination is consistent with zero.

Another topic of interest is whether the confining potential has a component $\pi/(12R)$ arising from the fluctuations of the string¹⁷. Such a component has the same R -dependence as the gluon exchange piece although it is applicable at large R where a string picture is valid while the gluon exchange applies at small R where perturbative methods are valid. Even so there has been some confusion as to whether the lattice results at moderate R corresponded to the one or the other. One way to illuminate this is to study the spin orbit potentials to see whether the $-e/R$

component observed in the central potential corresponds to V_1 (i.e. scalar) or V_2 (i.e. vector). Our results show that V_1' has a flat R -dependence and so has no room for a $-\pi/(12R^2)$ component from string fluctuation while V_2' is consistent with gluon exchange. This rules out a string fluctuation component of Lorentz scalar exchange nature at these R -values. Of course our results are for $R < 3 \text{ GeV}^{-1}$ which means that a string would be of a length not greater than its width so that the universal result¹⁷ need not apply.

Passing now to gluon exchange, after taking account of lattice artefacts, the naive one gluon exchange contribution gives results roughly $1/1.5$ of V_3 and $1/2$ to $1/3$ of V_2' and V_0 . As discussed in section 3, we expect the tensor potential V_3 to be relatively closer to one gluon exchange since it gets less contributions from small q^2 . The contributions from the next order of g^2 can be estimated as discussed at the end of section 3. This yields an increase in V_3 of 50%, bringing it much closer to the measured result as shown in fig. 3. For the scalar potential V_4 the order g^4 estimate is close to the naive order g^2 result and agrees satisfactorily. The spin-orbit potential V_2' is also increased by about 50% which is insufficient to obtain agreement. Attempts to resum the perturbation series can increase this 50% substantially but are not very well founded quantitatively.

For our results to be used as continuum potentials in heavy quark spectroscopy, we should discuss lattice artefacts. Eq. 3.7 shows that the $R = 2a$ potentials depend on the interaction between links of measuring loops separated by only $R = a$. Thus, as shown from our gluon exchange estimates, there are still substantial corrections to the potentials at $R = 2a$ although these are qualitatively understood.

We put physical units on our results using the string tension values measured from lattice simulations and using $\sqrt{K} = 0.44$ GeV. The fact that our results at different β -values are in agreement is support that scaling (rather than asymptotic scaling) applies for these β -values and so ratios of physical quantities are independent of β .

In conclusion, we find a picture of the interquark potential as a sum of a perturbative gluon exchange contribution (with inclusion of order g^4 terms giving the bulk of the measured contribution) and a confining contribution which is approximately linear in R and which has a Lorentz scalar nature.

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FIGURE CAPTIONS

1. (a) A typical Wilson loop evaluated to study $\langle B_z(0,t)B_z(R,t) \rangle$. The thick lines refer to multi-hit links in the evaluation. The dashed lines refer to variational path combinations as defined in (b). (c) Insertions used to evaluate $E_y(0,t)$. (d) Ratio of Wilson loops which is evaluated to cancel out the self-energies of the insertions.
2. Contributions to the Δt (or τ) sum of eq. 1.2 measured on a lattice at $\beta = 2.575$ using SU(2) pure gauge fields. The dashed curves represent lattice one gluon exchange to order g^2 with a normalization increased by a factor of 1.5. For the spin-orbit potentials the rate of decrease with Δt corresponding to the largest E_L eigenmode is also illustrated by the continuous curve.
3. (a),(b) Results for SU(2) gauge fields. The potentials at $\beta = 2.4$ and 2.575 are plotted using $a(2.4) = 1.74$ $a(2.575)$ from ref. 11. The continuous curves for the spin-orbit potentials V_1' and V_2' correspond to the scalar ($-KR$) and vector (e/R^2) components of the measured central potential¹¹ which was parametrized as $KR - e/R$. The lines corresponding to gluon exchange are (i) dotted for continuum order g^2 , (ii) dash-dotted for lattice order g^2 and (iii) dashed for lattice order g^4 as given by eq. 3.11. Where these gluon exchange predictions are different for different β -values, they are drawn between the R -values appropriate to that β -value.

(c),(d) Results for SU(3) gauge fields. Our results at $\beta = 5.7$ and those of ref. 7 at $\beta = 6.2$ are plotted using the measured string tension values which imply $a(5.7) = 0.86 \text{ GeV}^{-1}$ and $a(6.2) = 0.37 \text{ GeV}^{-1}$. For $\beta = 6.2$, we use $n = 2.655$ as a

normalization factor (from $\langle \text{plaq.} \rangle^{-2}$). Otherwise as (a), (b).

TABLE CAPTION

(a) The spin-orbit potentials V_1' and V_2' and (b) the spin-dependent potentials V_3 and V_4 . Lattice units with $a(\beta) = 1$ are used. The normalization factor n comes from the "one-eared" diagrams of fig. 1d. The coefficient C is used to make the variational path combinations of fig. 1b. The contribution from each $\Delta t/a$ value is shown, together with their sum (see eq. 2.1). The estimated value and maximum value include the remainder of the Δt -sum calculated as described in the text. The potential values are multiplied by 10^5 in the Tables.

	R	C	$-V_1'(R, 1/2)$	$-V_1'(R, 3/2)$	$-V_1'(R, 5/2)$	$-V_1'(R)_{sum}$	$-V_1'(R)_{est}$	$-V_1'(R)_{max}$
$\beta=2.575$	2	1.90	682(29)	242(5)	82(2)	1818(22)	2152	2264
	3	2.92	551(36)	206(6)	88(5)	1612(46)	2171	2291
	4	3.70	492(37)	193(6)	85(6)	1496(41)	2064	2853
	5	3.92	474(39)	186(10)	78(6)	1423(60)	1902	2877
$n=2.21$			$V_2'(R, 1/2)$	$V_2'(R, 3/2)$	$V_2'(R, 5/2)$	$V_2'(R)_{sum}$		$V_2'(R)_{max}$
	2		1100(13)	593(6)	251(5)	4134(20)		5502
	3		194(16)	205(10)	133(9)	1473(31)		2496
	4		40(18)	73(12)	65(11)	581(20)		1616
	5		1(19)	31(23)	32(25)	255(57)		859
	R	C	$-V_1'(R, 1/2)$	$-V_1'(R, 3/2)$	$-V_1'(R, 5/2)$	$-V_1'(R)_{sum}$	$-V_1'(R)_{est}$	$-V_1'(R)_{max}$
$\beta=2.4$	2	1.17	1435(49)	549(16)	226(8)	4214(105)	5548	5968
	3	1.88	1278(51)	523(24)	247(12)	4085(121)	6036	6517
	4	2.34	1197(51)	499(25)	248(20)	3933(154)	6133	7299
	5	2.46	1171(60)	541(24)	249(34)	4042(175)	5899	9584
$n=2.45$			$V_2'(R, 1/2)$	$V_2'(R, 3/2)$	$V_2'(R, 5/2)$	$V_2'(R)_{sum}$		$V_2'(R)_{max}$
	2		1315(13)	720(9)	312(4)	5034(14)		7453
	3		242(19)	276(23)	161(23)	1873(88)		3452
	4		59(24)	104(42)	65(46)	697(150)		1581
	5		30(53)	43(23)	11(29)	217(192)		467
	R	C	$-V_1'(R, 1/2)$	$-V_1'(R, 3/2)$	$-V_1'(R, 5/2)$	$-V_1'(R)_{sum}$	$-V_1'(R)_{est}$	$-V_1'(R)_{max}$
$\beta=5.7$	2	1.00	2727(63)	1049(22)	378(30)	7765(211)		9390
	3	1.33	2536(64)	1046(46)	542(105)	8363(586)	10358	11848
	4	1.51	2646(105)	1150(106)	495(366)	8570(943)	11871	13171
			$V_2'(R, 1/2)$	$V_2'(R, 3/2)$	$V_2'(R, 5/2)$	$V_2'(R)_{sum}$		$V_2'(R)_{max}$
$n=3.22$	2		1762(33)	988(42)	404(31)	6747(176)		8484
	3		451(130)	511(147)	327(125)	3621(506)		5725
	4		-41(75)	120(90)		321(29)		1062

Table 1a : Spin-Orbit potentials

	R	C	$V_3(R,0)$	$V_3(R,1)$	$V_3(R,2)$	$V_3(R,3)$	$V_3(R)_{sum}$	$V_3(R)_{max}$
$\beta=2.575$	2	1.90	2024(9)	792(8)	243(8)	75(4)	4244(35)	4317
	3	2.92	519(20)	335(11)	164(8)	94(10)	1704(43)	1840
	4	3.70	181(20)	143(11)	104(15)	61(10)	797(61)	904
	5	3.92	66(25)	85(16)	57(14)	56(21)	463(72)	557
$n=2.24$			$-V_4(R,0)$	$-V_4(R,1)$	$-V_4(R,2)$	$-V_4(R,3)$	$-V_4(R)_{sum}$	$-V_4(R)_{max}$
	2		1703(24)	179(13)	-187(14)	-164(9)	1357(60)	847
	3		308(33)	74(22)	-30(17)	-45(14)	302(82)	9
	4		104(38)	41(25)	10(18)	2(26)	208(105)	-11
	5		22(40)	44(35)	44(29)	39(33)	278(140)	143
	R	C	$V_3(R,0)$	$V_3(R,1)$	$V_3(R,2)$	$V_3(R,3)$	$V_3(R)_{sum}$	$V_3(R)_{max}$
$\beta=2.4$	2	1.17	2189(21)	818(15)	218(17)	35(7)	4330(79)	4391
	3	1.88	534(41)	329(26)	161(20)	50(21)	1615(120)	1709
	4	2.34	123(36)	70(44)	16(34)	20(37)	335(207)	375
	5	2.46	93(57)	66(51)	3(38)	40(69)	312(254)	395
$n=2.49$			$-V_4(R,0)$	$-V_4(R,1)$	$-V_4(R,2)$	$-V_4(R,3)$	$-V_4(R)_{sum}$	$-V_4(R)_{max}$
	2		1673(47)	-6(17)	-259(22)	-237(21)	667(101)	245
	3		202(54)	23(29)	-73(40)	-31(54)	39(174)	-258
	4		35(75)	-21(55)	-63(66)	40(81)	-54(318)	-164
	5		72(114)	5(78)	-53(83)	109(100)	194(447)	117
	R	C	$V_3(R,0)$	$V_3(R,1)$	$V_3(R,2)$	$V_3(R,3)$	$V_3(R)_{sum}$	$V_3(R)_{max}$
$\beta=5.7$	2	1.00	2577(93)	948(47)	240(35)	109(55)	5280(190)	5385
	3	1.33	675(82)	265(80)	122(96)		1449(351)	1665
	4	1.51	-51(186)	-98(152)	-182(220)		-611(617)	
			$-V_4(R,0)$	$-V_4(R,1)$	$-V_4(R,2)$	$-V_4(R,3)$	$-V_4(R)_{sum}$	$-V_4(R)_{max}$
$n=3.32$	2		1653(132)	-228(50)	-404(63)	-443(98)	-606(328)	-1094
	3		272(133)	-132(132)	-236(161)		-463(544)	-907
	4		133(270)	-316(259)	-206(319)		-910(937)	

Table 1b : Spin-dependent potentials

(a)

$$- \text{dashed line} = \text{solid line} + c \Sigma_1^L$$

Diagram (b) shows a path equation. A dashed line with an arrow pointing right is equal to a solid line with an arrow pointing right, plus a term $c \Sigma_1^L$. The parallelogram Σ_1^L has an arrow pointing right along its top edge.

(b)

(c)

(d)

Figure 1

Figure 3d