CERN-TH.4549/86

FROM POINTS TO GAUGE FIELDS

Y. Meurice^{*)}

CERN - Geneva

A B S T R A C T

We show that several classes of free field theories with local gauge invariance (e.g., the Yang-Mills, Einstein and p-forms linearized actions) can be constructed from classical actions for a finite number of points by applying the BRST-quantization initiated by W. Siegel. We briefly outline the generalization of this construction for strings.

^{*)} Address after 1 October 1986: H.E.P., Argonne National Laboratory, Argonne - IL 60439, U.S.A.

CERN-TH.4549/86

September 1986

1. Introduction

The classification of the unitary representations of the Poincaré group¹⁾ has played an important role in the development of relativistic quantum field theory. From this analysis we know all the possible Poincaré transformation laws for the physical states and their asymptotically related fields. For instance, a massless particle of helicity +1 or -1 might be described by a $(1,0) \oplus (0,1)$ Lorentz tensor $F_{\mu\nu}$. Using $F_{\mu\nu}$ and other fields with definite transformation properties we can construct a Poincaré-invariant S-matrix. However, a detailed analysis²⁾ shows that the momentum dependence of $F_{\mu\nu}$ is such that for instance we cannot reproduce the inverse-square law which is expected for electrostatic forces. Consequently, it is convenient to reexpress $F_{\mu\nu}$ in terms of the derivatives of the potentials A_μ . These potentials form a Lorentz vector only up to gauge-transformations but as an additional principle, we can require the gauge-invariance of the S-matrix.

More recently, Siegel³⁾ has proposed using the so-called BRST-charge associated with the constraint algebra of some classical mechanics system (point, string) as a tool to construct second-quantized theories. In contrast to the previous approach, the gauge-invariance appears here as a first principle which determines the Poincaré content of the theory. This procedure has been used with several variants and developments³⁾⁵⁾⁻⁸⁾ but there are still many possibilities to be explored⁹⁾.

In this letter, we investigate new applications of the procedure initiated by Siegel. We have mostly restricted ourselves to the case where we start with point mechanics. Our idea is to add an arbitrary number of coordinates $y_A^\mu(\tau)$ to the usual $x^\mu(\tau)$ necessary to describe a single particle with only momentum degrees of freedom. We then choose constraints among those coordinates and their canonical momenta in such a way that after using the usual Dirac covariant quantization we obtain a reasonable physical spectrum for a relativistic field

theory. The next step is to construct the BRST charge

$$Q = \eta^i \phi_i - \frac{1}{2} f_k^{ij} p^k \eta_i \eta_j \quad (1)$$

(see ref.4) for more details) and an "extended" vector space (including the ghosts) such that the condition

$$Q|\chi\rangle = 0 \quad (2)$$

together with the equivalence relation

$$|\chi\rangle \sim |\chi\rangle + Q|\Lambda\rangle \quad (3)$$

(or an appropriate restriction of these) correspond to the definition of the physical states in the usual covariant quantization. The ordering ambiguities which appear in the definition of the quantum BRST-charge will be fixed in some case by the nilpotency condition $Q^2 = 0$. The last step consists in constructing a classical field theory with the invariance (3) reproducing (2) as an equation of motion. Our main result is that the procedure described above yields several classes of field theories with local invariances (e.g. the Yang-Mills, Einstein and the p-forms linearized actions). Finally we briefly outline how this procedure can be generalized for strings.

2. Classical point dynamics

We start with a set of coordinates x^μ, y_A^μ ($A = 1, \dots, N; \mu = 1, \dots, D$) and their conjugate momenta P^μ, P_A^μ (resp.). We want P^μ to be the usual momentum while the new canonical variables x_A^μ and P_A^μ give additional degrees of freedom which we want to be discrete at the quantum level. We work in an arbitrary dimension D and with a metric left invariant by an $SO(m, n)$ group ($n+m = D$). In order to end up with a reasonable relativistic field theory we impose the following conditions on the constraints to be considered:

- I) The constraints are bilinear in the canonical variables and $SO(m,n)$ invariant. In other words the generators of $SO(m,n)$ have vanishing Poisson brackets with the constraints and are observable. Contraction over space-time indices will be implicit hereafter ($y_A P_B$ means $y_A^\mu P_{\mu B}$ etc...)
- II) There are no constraints made out of x^μ . Consequently P^μ is also observable.
- III) There exists at least one constraint of the form

$$\phi = \sum_{A=1}^N c_A \frac{1}{2} (P_A^2 + y_A^2) ; \quad c_A > 0 . \quad (4)$$

This implies a finite number of physical states at the quantum level.

Are there classical actions corresponding to any such algebra of constraints? We could try to obtain them by writing τ -reparametrization invariant actions like

$$S = \int d\tau (m_1^2 \sqrt{(\dot{x}y)^2 - \dot{x}^2 y^2} + m_2^2 \sqrt{(\dot{y}y)^2 - \dot{y}^2 y^2}) . \quad (5)$$

This particular action has a vanishing Hamiltonian and a set of primary plus secondary constraints satisfying the three conditions mentioned above. However, it is simpler to start with an arbitrary algebra of constraints ϕ_i satisfying those three conditions, then impose that the canonical Hamiltonian vanishes and write

$$S = \int d\tau (P\dot{x} + P_A \dot{y}_A - \frac{1}{2} g^i \phi_i) . \quad (6)$$

By condition I) this action is no more than quadratic in the momenta over which we can then integrate.

We will now try to obtain constraint algebras satisfying the three above conditions. If ϕ is an arbitrary constraint satisfying conditions I) and II), condition

III) and the fact that the constraints form an algebra tell us that

$$\mathcal{D}\phi \equiv \left\{ \sum_{A=1}^N c_A \frac{1}{2} (P_A^2 + y_A^2), \phi \right\} \quad (7)$$

is also a constraint. We can diagonalize $\mathcal{D}$ with the usual change of variables

$$P_A^\mu = \frac{1}{\sqrt{2}} (a_A^\mu + a_A^{\mu*}) ; \quad y_A^\mu = \frac{i}{\sqrt{2}} (a_A^\mu - a_A^{\mu*}) . \quad (8)$$

The general form of $\mathcal{D}\phi$ is then

$$\mathcal{D}\phi = i \sum_{c_n \in \{c_A, |c_A \pm c_B|\}} c_n (\phi^n - \phi^{n*}) . \quad (9)$$

Using now the fact that a finite polynomial in $\mathcal{D}$ applied on ϕ is a constraint we see that each

$$\phi^n + \phi^{n*} \text{ and } i(\phi^n - \phi^{n*}) \quad (10)$$

are constraints. The maximal algebra we can obtain is

$$P^2, P a_A, P a_A^* \text{ and } a_A^* a_B, a_A a_B, a_A^* a_B^* . \quad (11)$$

The three first terms form a (nilpotent) Heisenberg algebra denoted H_N and the three others an $SP(2N)$ algebra, the full algebra being the semi-direct product of these two subalgebras. Many subalgebras of this maximal algebra do not satisfy the third condition and can be ruled out with the above analysis. In the present context, we will only consider three examples for which conditions I), II) and III) are satisfied:

$$\begin{aligned} a) & \quad SP(2N) \oplus H_N , \\ b) & \quad U(N) \oplus H_N \quad ; \quad U(N) = \{a_A^\dagger a_B\} , \\ c) & \quad U(1) \oplus H_N \quad ; \quad U(1) = \sum_{A=1}^N a_A^\dagger a_A . \end{aligned} \quad (12)$$

We will see that these are sufficient to recover the usual field theories with a local gauge-invariance.

3. First quantization : the physical states

In order to perform a covariant (first) quantization, we set

$$[P^\mu, x_\nu] = i\delta_\nu^\mu \ ; \ [a_A^\mu, a_{\nu B}^\dagger] = \delta_{AB}\delta_\nu^\mu \ . \quad (13)$$

We consider a vector space where P^μ and $a_A^{\mu\dagger}a_{\mu A}$ (no sum) are diagonal. For a fixed eigenvalue of P^μ , the most general element of this vector space reads

$$|\psi(p)\rangle = \sum_{(\mu_1 \dots \mu_n)} A_{\mu_1 \dots \mu_n}^{A_1 \dots A_n}(p) a_{A_1}^{\mu_1\dagger} \dots a_{A_n}^{\mu_n\dagger} |0\rangle \ . \quad (14)$$

We have now to impose that the constraints have vanishing matrix elements between physical states. The algebras (12) have three different types of 3-dimensional subalgebras. For instance, $a_1^\dagger a_1$, $a_1 a_1$ and $a_1^\dagger a_1^\dagger$ form an $SU(1,1)$ algebra while Pa_1 , $Pa_1^\dagger$ and $a_1^\dagger a_1$ (at $p^2 = 0$) form an E_2 algebra. These are non-compact and $|\psi\rangle$ contains infinite (unitary) representations bounded from below. Consequently, we have to impose

$$\begin{aligned} Pa_A |\psi\rangle &= 0 \\ a_A a_B |\psi\rangle &= 0 \end{aligned} \quad (15)$$

as the conditions defining the physical states. On the other hand, $a_1^\dagger a_2$, $a_2^\dagger a_1$ and $\frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$ form an $SU(2)$ algebra and $|\psi\rangle$ is an infinite direct sum of finite representations. We have in any case

$$(a_A^\dagger a_A - \alpha) |\psi\rangle = 0 \quad (\text{no sum}) \ . \quad (16)$$

For the other generators, it is sufficient to impose

$$a_A^\dagger a_B |\psi\rangle = 0 \ ; \ B > A \quad (17)$$

but now we have finite representations bounded from below and above, so we

could also impose the strongest conditions

$$\begin{aligned} (a_A^\dagger a_B + a_B^\dagger a_A) | \psi \rangle &= 0 \\ i(a_A^\dagger a_B - a_B^\dagger a_A) | \psi \rangle &= 0 \end{aligned} \quad (18)$$

for all $A \neq B$. Condition (17) will be called the "highest weight option" and condition (18) the "symmetric option".

We can see that a certain amount of arbitrariness is left in the usual covariant quantization. We can now use the condition $Q^2 = 0$ in order to see if we get additional conditions. The main difficulty of this procedure is to determine which ghosts are creators and annihilators⁶⁾⁸⁾. The generic form of the quadratic terms of the BRST-charge is

$$\eta_1(\phi + \phi^\dagger) - \eta_2 i(\phi - \phi^\dagger) \quad (19)$$

The prescription we use is that if

$$\phi | \psi \rangle = 0 \text{ and } \phi^\dagger | \psi \rangle = 0 \quad (20)$$

then η_1 and η_2 are creators. These ghosts will be called real ghosts hereafter. Obviously, if $\phi = \phi^\dagger$, η_2 does not appear. On the other hand if

$$\phi | \psi \rangle = 0 \text{ (only)} \quad (21)$$

it is more convenient to write

$$(\eta_1 - i\eta_2)\phi + (\eta_1 + i\eta_2)\phi^\dagger \quad (21)$$

and $\eta_1 - i\eta_2$ is a creator while its hermitean conjugate is an annihilator. We call these ghosts complex ghosts. We have worked out the conditions to have $Q^2 = 0$ (see ref.9) for a general formulation) for the three algebras a), b) and c) given

in (12) and with the two different options (17) and (18). In case c), we do not obtain any new conditions. In cases a) and b) with the highest weight option we get

$$\begin{aligned} a) \alpha_A &= \frac{4A + 2 - D}{2}, \\ b) \alpha_A &= 2A + C \end{aligned} \quad (23)$$

where C is an arbitrary constant independent of A. With the symmetric option the results are

$$\begin{aligned} a) \alpha_A &= \frac{2(N + 2) - D}{2}, \\ b) \alpha_A &= C. \end{aligned} \quad (24)$$

From (23) one can check explicitly that there are no physical states at all with the highest weight option. In order to understand this result, we have computed the nilpotency condition for a single SU(2) algebra. Defining first the physical states by

$$(T_3 - \alpha) | \psi \rangle = T_+ | \psi \rangle = 0 \quad (25)$$

(the highest weight option) we obtain $\alpha = -1$ i.e. the wrong sign for a finite dimensional (unitary) representation. Replacing the second condition in (25) by

$$T_1 | \psi \rangle = T_2 | \psi \rangle = 0 \quad (26)$$

(the symmetric option) yields consistently $\alpha = 0$ and the physical states are SU(2)-invariant. This analysis can be generalized for any compact group by using the results of ref. 9) together with the theorem on the Killing form of the compact groups. In our case, the fact that only the symmetric option is consistent is due to the compactness of SU(N) and we will only consider this option hereafter. We now determine the subspace of physical states in each cases considered. This subspace has the form (14) with appropriate restrictions on the $A_{\mu_1 \dots \mu_n}^{A_1 \dots A_n}$. The conditions of transversality and masslessness occur in all cases. In cases a) and b), the internal indices are totally antisymmetrized by groups of

N and can be dropped (remember we only consider the symmetric option) and we are left with a space-time tensor corresponding to a rectangular Young tableau with N rows and α columns (α is given by (24)). This tensor is traceless in case a) only. In case c), we only have to impose that the number of space-time indices (which is also the number of internal indices) is equal to a given integer (level) that can be chosen arbitrarily.

4. Gauge-invariant actions

We will now construct explicitly field theory actions corresponding to the two particular cases $U(1) \oplus H_N$ and $U(2) \oplus H_2$. In the first case, the BRST-charge reads

$$Q = \eta_0 P^2 + \eta(\mathcal{N} - \alpha) + p_0 T_+ + \Omega \quad (27)$$

where

$$\begin{aligned} \mathcal{N} &= \sum_{A=1}^N (a_A^\dagger a_A + p_A^\dagger \eta_A + \eta_A^\dagger p_A) \\ T_+ &= - \sum_{A=1}^N \eta_A^\dagger \eta_A \\ \Omega &= \sum_{A=1}^N (\eta_A^\dagger P a_A + \eta_A P a_A^\dagger) \end{aligned} \quad (28)$$

The anticommutation relations for the complex ghosts are

$$\{\eta_A^\dagger, p_B\} = \{\eta_A, p_A^\dagger\} = \delta_{AB} \quad (29)$$

On the other hand, η_0 and η are real ghosts and

$$\{\eta_0, p_0\} = \{\eta, p\} = 1 \quad (30)$$

Extending the argument of ref. 10), M real ghosts produce 2^M -degenerate vacua

and in our particular case we can expand

$$\begin{aligned} |\chi\rangle = & |\chi_1\rangle \otimes |--\rangle + |\chi_2\rangle \otimes |+-\rangle \\ & + |\chi_3\rangle \otimes |-+\rangle + |\chi_4\rangle \otimes |++\rangle \end{aligned} \quad (31)$$

with

$$\begin{aligned} \eta_0 |--\rangle &= |+-\rangle \\ \eta |--\rangle &= |-+\rangle \\ \eta_0 \eta |--\rangle &= |++\rangle \\ \langle -- | \eta_0 \eta |--\rangle &= 1 \end{aligned} \quad (32)$$

(the $|\chi_i\rangle$ are independent of the real ghosts). The conditions defining the physical states in the usual covariant quantization are

$$P^2 |\psi\rangle = P a_A |\psi\rangle = \left(\sum_{A=1}^N a_A^\dagger a_A - \alpha \right) |\psi\rangle = 0 \quad (33)$$

while $P a_A^\dagger$ generate the states orthogonal to the physical states (spurious states). A detailed analysis shows that eqs. (2) and (3) reproduce these conditions but for two different types of states. A similar situation appears for the closed bosonic string³⁾⁵⁾⁻⁷⁾. A correct definition of the physical states might be instead :

$$Q\Gamma |\chi\rangle = 0 \quad (34)$$

We can keep the equivalence relation (3) provided

$$Q[\Gamma, Q]_\pm |\Lambda\rangle = 0 \quad (35)$$

We have considered the cases where Γ is made out of $\mathcal{P}$ and $\mathcal{P}_0$. This guarantees that we still have the quadratic parts of Q which are necessary to reproduce (33). In order to keep the gauge-invariance corresponding to the spurious states

of the covariant formulation, we need to require that eq. (35) has algebraic solutions. This selects $\Gamma = \mathcal{P}$ because

$$\{Q, \mathcal{P}\} = \mathcal{N} - \alpha \quad (36)$$

is the only algebraic operator we can get within this procedure (no P^2). We can now construct directly⁶⁾⁷⁾ a gauge-invariant action

$$\begin{aligned} S &= -\frac{1}{2} \langle \chi | Q \mathcal{P} | \chi \rangle \\ &= \frac{1}{2} [\langle \chi_3 | P^2 | \chi_3 \rangle + \langle \chi_3 | \Omega | \chi_4 \rangle + \langle \chi_4 | \Omega | \chi_3 \rangle \\ &\quad - \langle \chi_4 | T_+ | \chi_4 \rangle - \langle \chi_1 | (\mathcal{N} - \alpha) | \chi_4 \rangle - \langle \chi_2 | (\mathcal{N} - \alpha) | \chi_3 \rangle] \end{aligned} \quad (37)$$

where $|\chi\rangle$ is restricted to its ghost-number zero ($Q\mathcal{P}$ has ghost-number 0). Equivalently, we could start from a BRST gauge-fixed action³⁾, shift the auxiliary fields and drop the ghost part. Note that $|\chi_1\rangle$ and $|\chi_2\rangle$ are just Lagrange multipliers restricting all other fields to the sector where $\mathcal{N} = \alpha$. Keeping only this sector, we have an unrestricted gauge-invariance according to (35) and (36). We can now expand in components³⁾ for different values of α and N (which are arbitrary in this particular case). For $\alpha = 1$, we get

$$S = \int d^D x \sum_{A=1}^N \left[-\frac{1}{2} A_\mu^A \square A^{\mu A} - \phi^A \partial^\mu A_\mu^A + \frac{1}{2} \phi^A \phi^A \right] \quad (38)$$

which is invariant under

$$\begin{aligned} \delta A_\mu^A &= \partial_\mu \Lambda^A \\ \delta \phi^A &= \square \Lambda^A \end{aligned} \quad (39)$$

Integrating ϕ^A , we recover the linearized Yang-Mills action (i.e. N abelian $F^{\mu\nu} F_{\mu\nu}$ terms). In the case $\alpha = 2$, $N=1$ the action and its gauge-invariance

read

$$\begin{aligned}
 S &= \int d^D x \left[-h_{\mu\nu} \square h^{\mu\nu} + \frac{1}{2} A \square A + 2h_{\mu\nu} \partial^\mu \phi^\nu \right. \\
 &\quad \left. - A \partial^\mu \phi_\mu + \frac{1}{2} \phi^\mu \phi_\mu \right] \\
 \delta h_{\mu\nu} &= \frac{1}{2} (\partial_\mu \Lambda_\nu + \partial_\nu \Lambda_\mu) ; \quad h_{\mu\nu} = h_{\nu\mu} \\
 \delta \phi_\mu &= \square \Lambda_\mu \\
 \delta A &= \partial^\mu \Lambda_\mu .
 \end{aligned} \tag{40}$$

After integrating over ϕ^μ and performing a field redefinition³⁾ we recover the Einstein action plus a massless scalar. If we had followed the same procedure for $SP(2) \oplus H_1$ we would have obtained the additional condition

$$\alpha = \frac{6-D}{2} \tag{41}$$

i.e. $D=2$ for $\alpha = 2$. This has a simple explanation : the Einstein action is invariant under

$$\delta h_{\mu\nu} = g_{\mu\nu} \Lambda \tag{42}$$

only for $D=2$.

In the case $U(2) \oplus H_2$, we can obtain Q by adding to (27) the piece

$$\eta_i T_i - i \epsilon_{ijk} \eta_i \eta_j P_k \tag{43}$$

where

$$T_i = a^\dagger \tau_i a + \eta^\dagger \tau_i P + P^\dagger \tau_i \eta \tag{44}$$

(the spinor indices are implicit). Repeating the procedure described in the previous case, we obtain

$$S = -\frac{1}{2} \langle \chi | Q P \epsilon_{ijk} P_i P_j P_k | \chi \rangle . \tag{45}$$

This action has the same form as (37) except that there are additional Lagrange multipliers enforcing the $SU(2)$ -invariance of $|\chi_3\rangle$ and $|\chi_4\rangle$ while the corre-

sponding condition holds for the gauge parameters. Expanding as previously for $\alpha_1 = \alpha_2 = 1$ we get

$$\begin{aligned}
 S &= \int d^D x \frac{1}{2} \left[-A_{\mu\nu} \square A^{\mu\nu} - \frac{1}{2} \phi \square \phi + \phi \partial^\mu \phi_\mu \right. \\
 &\quad \left. + 2A^{\mu\nu} \partial_\mu \phi_\nu + \frac{1}{2} \phi^\mu \phi_\mu \right] \\
 \delta A_{\mu\nu} &= \frac{1}{2} (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu) \quad ; \quad A_{\mu\nu} = -A_{\nu\mu} \\
 \delta \phi_\mu &= \square \Lambda_\mu \\
 \delta \phi &= \partial^\mu \Lambda_\mu \quad .
 \end{aligned} \tag{46}$$

After the ϕ^μ integration, ϕ disappears and we recover the usual free action for an antisymmetric tensor of rank 2.

5. Generalization to strings

It is tempting to apply this procedure for strings. As a first step in that direction, we introduce in addition to the usual canonical variables $x^\mu(\sigma)$ and $P^\mu(\sigma)$, two sets $y_A^\mu(\sigma)$, $P_A^\mu(\sigma)$ and $y_{\bar{A}}^\mu(\sigma)$, $P_{\bar{A}}^\mu$ with an arbitrary range for A and $\bar{A}$. The constraints (omitting σ)

$$\begin{aligned}
 &(P \pm x')^2 \\
 &(P + x')P_A; (P + x')y_A; P_A y_B; y_A y_B; P_A P_B \\
 &(P - x')P_{\bar{A}}; (P - x')y_{\bar{A}}; P_{\bar{A}} y_{\bar{B}}; y_{\bar{A}} y_{\bar{B}}; P_{\bar{A}} P_{\bar{B}}
 \end{aligned} \tag{47}$$

close under Poisson brackets. The quantum consistency of those theories is presently under investigation.

ACKNOWLEDGEMENTS

We thank L.Alvarez-Gaumé, I.Bars, A.Neveu, H.B.Nielsen, R.Stora,
G.Veneziano and S.Yankielowicz for useful conversations and encouragements.

REFERENCES

1. E.P.Wigner - Ann. Math. 40 (1936) 149.
2. S.Weinberg - in Brandeis Summer Institute 1964 vol.2, eds Deser and Ford (Englewood Cliffs, N.J, 1965).
3. W.Siegel - Phys. Lett. 151B (1985) 391, 396; Nucl. Phys. B263 (1986) 93;
W.Siegel and B.Zwiebach - Nucl. Phys. B263 (1986) 105.
4. E.Fradkin and G.Vilkovisky - Phys. Lett. 55B (1975) 224.
5. K.Itoh, T.Kugo, H.Kunimoto and H. Ooguri - Prog. in Th. Phys. 75 (1986) 1986.
H.Terao and S. Uehara - Phys. Lett. 168B (1986) 70.
T.Banks, M.Peskin, C.R.Preitschopf, D.Friedan and E.Martinec - Nucl. Phys. B274 (1986) 71.
P.Ramond - UFTP preprint 85-18 (1985).
H.Hata, K.Itoh, T.Kugo, H.Kunimoto and K.Ogawa - Phys. Lett. 172B (1986) 186 and 195, 175B (1986) 138 and their recent review RIFP-660 (1986) for a more complete list of reference.
6. E.Witten - Nucl. Phys. B268 (1986) 79 and B276 (1986) 325.
7. A.Neveu, H.Nicolai and P.West - Phys. Lett. 167B (1986) 307
A.Neveu and P.West - Phys. Lett. 168B (1986) 192 and CERN preprint TH 4358 (1986).
Y.Kazama, A.Neveu, H.Nicolai and P.West - Nucl. Phys. B276 (1986) 366.
8. D.Friedan, E. Martinec and S. Shenker - Nucl. Phys. B271 (1986) 93.
9. I.Bars and S.Yankielowicz - USC preprint 86-10 (to appear).
10. M.Kato and K.Ogawa - Nucl. Phys. B212 (1983) 443.