CERN-TH.4539/86

ANOMALIES, STRINGS AND ALGEBRAIC GEOMETRY

L. Alvarez Gaumé

CERN - Geneva

Lectures delivered at the
Erice School on "The Superworld"
August 1986

Part of these lectures were also presented at the
II Nobel Symposium, Marstrand, Sweden
June 1986

CERN-TH.4539/86

September 1986

ANOMALIES, STRINGS AND ALGEBRAIC GEOMETRY

Luis Alvarez-Gaumé

Theoretical Physics Division
CERN
1211 Geneva 23, Switzerland

1. - INTRODUCTION

In these lectures we would like to review some of the applications of global methods in geometry and topology to the quantum theory of fields and strings. The use of the theory of characteristic classes, index theory, algebraic geometry and differential topology is becoming common place in recent theoretical physics. It seems likely that these areas of mathematics will soon become part of the mathematical background of theoretical physicists, in the same way that Riemannian geometry, group theory, and complex analysis became part of our mathematical arsenal. These new techniques provide also rather powerful tools to obtain useful qualitative (and sometimes quantitative) information in quantum field theories with gauge and/or gravitational interactions, and in string theory.

In the first part of these lectures, we shall emphasize the use of global methods in the study of local and global anomalies in field theory; in the second part we shall explain some of the applications in string perturbation theory. Thus in Section 2 we will give a rather quick review of local and global anomalies; and in Section 3 we will present some recent results on multi-loop string diagrams and bosonization techniques on Riemann surfaces with many handles.

2. - ANOMALIES IN QUANTUM FIELD THEORY¹

We can divide the anomalies appearing in quantum field theory into two large categories: local and global. Local anomalies are those accessible by purely perturbative methods. After we have checked that a theory is free of local anomalies, we are still left with the problem of checking that it is also free of global anomalies. The latter measure the change of the effective action for a set of fields in the presence of gauge and gravitational backgrounds under gauge transformations or diffeomorphisms not connected with the identity. The study of these anomalies has been carried out by Witten², and we will review some of the basic results. In string theory, invariance under the group of disconnected diffeomorphisms on a Riemann surface plays a very important role in determining the consistency and the spectrum of the theory. Fortunately, the same methods developed in Ref. 2 can be applied to this

case³ to show that the set of fields needed in all known string theories to cancel the conformal anomalies in the string worldsheet guarantee that the theory can be written in a modular invariant way.

A) Local Anomalies

In its original form⁴ one considers a massless fermion triangle diagram with one axial current and two vector currents. Regulating the graph with the requirements of current conservation and Bose symmetry in the vector channels, one finds that the axial current is not conserved. This fact has had important phenomenological implications [decay of the π^0 ⁴, resolution of the U(1)-problem in QCD⁵]. By analogy, if instead of considering two vector currents in the triangle graph, one inserts two energy momentum tensors corresponding to the emission of two gravitons⁶, the same conclusion follows. The axial current is not conserved in the presence of external gravitational fields. If we now consider V-A currents coupled to gauge fields in the vertices of the triangle graph, the diagrammatic computation is essentially unchanged, but now the presence of the anomaly would imply the loss of gauge invariance and renormalizability. The cancellation of anomalies in this case provides some constraints on the fermion spectrum in chiral gauge theories. The best known example is probably the cancellation of anomalies between quarks and leptons in the standard $SU(3) \times SU(2) \times U(1)$ theory of strong weak and electromagnetic interactions. The representation content of a family of quarks and leptons is, in terms of left-handed Weyl fermions given by

$$(3,2)_{1/6} \oplus (\bar{3},1)_{-2/3} \oplus (1,2)_{-1/2} \oplus (\bar{3},1)_{1/3} \oplus (1,1)_1 \quad (1)$$

where the first entry indicates the $SU(3)$ representation, the second one gives the $SU(2)$ representation, and the third one is the value of the weak hypercharge. To check that the anomalies cancel, one has to make sure that the following condition is satisfied⁷:

$$\sum_L \text{tr} T_L^a \{T_L^b, T_L^c\} - \sum_R \text{tr} T_R^a \{T_R^b, T_R^c\} = 0 \quad (2)$$

where the T_L 's (resp. T_R 's) are the generators of the gauge group in the representation of the left- (resp. right-) handed fermions. The remarkable cancellation of anomalies between quark and lepton (which the reader is encouraged to check) lends substantial support to the family structure in the standard model. If one considers the computation in Ref. 6 for a V-A current, one finds that the triangle graph with one gauge field and two gravitons is proportional to the trace of the charge associated to the gauge field. Since the trace of a generator in a semi-simple group is always zero, this possible anomaly would only show up in gauge groups containing U(1) factors. It is easy to check that in the standard model the trace of the weak hypercharge generator vanishes in each family. The easiest way to show it is perhaps that the standard model can be embedded in an $SU(5)$ GUT theory without changing the number of fermions⁸.

For theories in more than four dimensions, one finds a potentially anomalous diagram analogous to the triangle graph in any even number of dimensions. In $d = 2n$ dimensions, the first potentially dangerous graph is a polygon with $n+1$ vertices. The anomaly can be evaluated by diagrammatic techniques as in four dimensions⁹, and the group theory factor in front of the kinematical form of the anomaly is

$$\sum_L \text{str} T_L^{a_1} \dots T_L^{a_{n+1}} - \sum_R \text{str} T_R^{a_1} \dots T_R^{a_{n+1}} = 0 \quad (3)$$

Str means the symmetrized trace which follows from the Bose symmetry and we use the same notation as in (2). A more concise form of writing (3) is to consider an arbitrary element of the Lie algebra of the gauge group X. Then the cancellation of (3) is equivalent to the cancellation of

$$\sum_L \text{Tr} X_L^{n+1} - \sum_R \text{Tr} X_R^{n+1} = 0 \quad (4)$$

An important difference between $d = 4k$ and $d = 4k+2$ can already be noticed in (4). In $4k$ dimensions we can have purely left-handed anomaly free theories. This is because in this case we have to compute $\text{tr} X^{2k+1}$ (an odd power). Thus, for example, if the representation is real or pseudoreal, we know that $\text{tr} X^{2k+1} = 0$, for in this case ${}^t X = -SXS^{-1}$, and we can conclude in four dimensions that $Sp(n)$, $SO(2n+1)$, $SO(4n)$, G_2 , F_4 , E_7 and E_8 are free of anomalies in any $4k$ -dimensions. Anomalies would only appear for groups $SO(4n+2)$, $U(n)$ and E_6 . Instead, if we consider $d = 4k+2$, then the anomaly for a given representation is proportional to $\text{tr} X^{2k+2}$. Since X is Hermitian, this trace is always positive and it is not possible to have a purely left-handed anomaly free theory in $d = 4k+2$. The reason for this difference (which is also at the origin of gravitational anomalies) is the interplay between chirality and charge conjugation. In $4k+2$ (resp. $4k$) dimensions, charge conjugation commutes (resp. anticommutes) with Γ (the higher dimensional analogue of γ_5). One should recall that, for example in $d = 2$, both particles and antiparticles have the same helicity, while in $d = 4$ they have opposite helicity. In particular, the purely gravitational coupling of a Weyl fermion in $d = 4k+2$ is genuinely chiral, and the theory may be afflicted with anomalies in the energy-momentum tensor¹⁰. If at the same time the fermions are also coupled to external gauge fields, we can in general have three types of anomalies: purely gravitational, purely gauge, or mixed anomalies depending on whether we have only gravitons or only gauge fields or both on the external legs of the $(n+1)$ polygon. In particular, in $d = 10$, the first potentially anomalous graph is the hexagon, and in the external lines we can have: (i) six gravitons, (ii) six gauge fields; (iii) two gravitons and four gauge fields, or (iv) four gravitons and two gauge fields. The constraints imposed on a higher-dimensional chiral theory by the cancellation of anomalies are highly restrictive, and for a while the only supersymmetric theory known to have a non-trivial cancellation of anomalies was the chiral $N = 2$ supergravity theory in ten dimensions¹⁰ (or its compactifications). The whole picture changed drastically with the discovery of a new anomaly cancellation mechanism by Green and Schwarz¹¹. They showed that in $d = 10$, the $N = 1$ supergravity multiplet coupled to the $N = 1$ super-Yang-Mills theory is anomaly free provided that the gauge group is either $SO(32)$ or $E_8 \times E_8$. Soon after the discovery of the anomaly cancellation, the heterotic string was constructed¹² (see Prof. Gross' lectures for more details), and the string Big Bang begun. Before explaining in more detail the Green-Schwarz mechanism, we would like to review the general prescription for the computation of anomalies in any number of dimensions.

The general prescription for the computation of local anomalies can be given in a very elegant way in terms of the Atiyah-Singer index theorem¹³. For the case of global gauge singlet currents, such as the axial current in QCD, it was shown some years ago¹⁴ that the tensor structure and normalization of the anomaly follow from the classical index theorem applied to the Dirac operator in the presence of gauge and gravitational fields. The topological interpretation of the anomaly in gauge currents requires more powerful machinery. In this case, one has to appeal to the more general form of the Atiyah-Singer index theorem which applies to families of elliptic operators. In order to make things more explicit, we consider in detail the case of a Weyl fermion in $2n$ dimensions in Euclidean space coupled to an arbitrary external gauge field. The starting

point of the analysis is always to consider the regularized fermionic effective action

$$\exp(-\Gamma[A]) = \int d\lambda d\bar{\lambda} \exp - \int \bar{\lambda} D(A) \lambda \quad (5)$$

where $D(A)$ is the Weyl operator, and it is understood that some regulator has been chosen for the functional integral. We take A to be an anti-Hermitian one-form valued in the representation of the Lie algebra of the gauge group carried by the fermions. An infinitesimal gauge transformation of A is given by

$$\delta A = dv + [A, v] = Dv \quad (6)$$

and v is a scalar valued in the same representation as A . Gauge invariance means that up to trivial local counterterms, the effective action does not change under gauge transformation

$$\begin{aligned} \Gamma[A+Dv] - \Gamma[A] &= - \int dx \operatorname{Tr} v D \delta \Gamma[A] / \delta A \\ &= \int dx \operatorname{Tr} v D_A \langle j(A) \rangle \end{aligned} \quad (7)$$

where $j(A)$ is the fermion current, and the angular brackets represent as usual the expectation value with respect to the measure defined by (5). As a consequence of Eq. (7), we learn that gauge invariance is equivalent to current conservation. From the topological point of view, $\exp(-\Gamma[A])$ defines a line bundle over the space of gauge configurations. In this language, the anomaly is simply expressed by saying that the line bundle is not flat; in other words the connection which determines the parallel transport of the effective action over the space of gauge fields is not flat and has a non-trivial curvature. In analogy with the monopole theory, we can test for the non-triviality of this line bundle by restricting it to topologically non-trivial two surfaces in the space of gauge connections modulo gauge transformations¹⁵. Here is where the index theorem for families of elliptic operators¹⁶ becomes useful, because it provides a local representative of the first Chern class of this line bundle (its local curvature). We can now briefly summarize the results obtained over the last three years concerning the form of this curvature (for further details and references, see Ref. 1). Let F be the gauge field strength two-form $F = dA + A^2$, and let R be the curvature two-form associated to the external gravitational field. If we work in orthonormal frames, R is a two-form valued on the Lie algebra of the frame rotation group, $SO(d)$: The index for the Dirac operator coupled to A and the spin connection ω , $R = d\omega + \omega^2$, is obtained by the following procedure. Since R is an antisymmetric matrix of two-forms, we can formally skew diagonalize R so that it looks like

$$\begin{bmatrix} 0 & x_1 & & \\ -x_1 & 0 & & \\ & & \ddots & \\ & & & 0 & x_n \\ & & & -x_n & 0 \end{bmatrix} \quad (8)$$

and construct the formal polynomial

$$A(M) = \prod_i (x_i/4n) / \sinh(x_i/4n) \quad (9)$$

known as the Dirac genus of the manifold M where the fermion lives. Notice that even though $A(M)$ is an infinite series in the curvature eigenvalues, the series actually terminates because the x 's are two-forms. Then the index for the Dirac operator is obtained by expanding the polynomial

$$\text{Ind } D(\omega, A) = A(M) \text{Tr} \exp F/2\pi i \quad (10)$$

until we find a term whose degree is equal to the dimension of M , and then integrating over M . This local polynomial gives the anomaly for the global axial current. Notice that due to the symmetries of $A(M)$, all the terms in its Taylor series expansion can be re-expressed in terms of R as combinations of monomials of the form $\text{Tr } R^n$. In four dimensions, (10) takes a familiar form

$$(2n)^2 \text{Ind } D(\omega, A) = -\text{Tr } F^2/2 + r \text{Tr } R^2/48 \quad (11)$$

where r is the number of fermions. We can easily recognize the ABJ anomaly in (11) when we set $R = 0$. (We are using a notation where the wedge product between forms is left implicit.) The anomaly for gauge currents can also be stated in very similar terms. A fundamental property of the differential forms entering the Taylor series expansion in (7) is that they are closed, therefore they can be expressed locally as total derivatives. For example, if we consider the $2N+2$ -form $\text{Tr } F^{N+1}$, one can write

$$\begin{aligned} \text{Tr } F^{N+1} &= (N+1) d \int dt \text{Tr } A (t dA + t^2 A^2)^N \\ &= d Q_{2N+1}^0(A, F) \end{aligned} \quad (12)$$

where Q is known as the Chern-Simons form¹⁷. If we make a gauge transformation of Q_{2N+1}^0 , it can be shown that¹⁸

$$\delta_v Q_{2N+1}^0 = d Q_{2N}^1(v, A) \quad (13)$$

and that the change in the effective action under a gauge transformation is given by Q_{2n}^1 if the dimension of space time is $2n$. The prescription to calculate the anomaly is then to first compute the index polynomial for the operator involved, depending on the spin fields we work with (spin $\frac{1}{2}$ or spin $3/2$ or self-dual tensor fields...) and the couplings involved. Next expand this polynomial to obtain the form of dimension $2n+2$. Using (12) and (13) but now applied to the term in the index polynomial of degree $2n+2$, the variation of the effective action is finally given by

$$\delta_v \Gamma[A] = \int Q_{2n}^1(v, A) \quad (14)$$

in four dimensions for example we have

$$Q_4^1(v, A) = \frac{1}{24n^2} v d (A dA + \frac{1}{2} A^3) \quad (15)$$

and similar expressions can be found for higher dimensions (see Ref. 1 for more details). In particular, if we concentrate in $d = 10$, and consider the contribution to the anomaly due to all the fields of the $N = 1$ supergravity and super Yang-Mills multiplets, the result can be succinctly summarized by the following 12-form

$$\begin{aligned} I_{12} &= -\frac{1}{15} \text{Tr } F^6 + \frac{1}{24} \text{Tr } F^4 \text{Tr } R^2 - \\ &\quad - \frac{\text{Tr } F^2}{960} [5(\text{Tr } R^2)^2 + 4 \text{Tr } R^4] + \\ &\quad + \frac{N-496}{7560} \text{Tr } R^6 + \left[\frac{N-496}{5760} + \frac{1}{8} \right] \text{Tr } R^4 \text{Tr } R^2 \\ &\quad + \left[\frac{N-496}{13824} + \frac{1}{32} \right] (\text{Tr } R^2)^3 \end{aligned} \quad (16)$$

in the standard way if cancelling anomalies one would require that (16) vanishes identically when the contribution from all fields has been included. It is easy to see that (16) cannot possibly vanish for any gauge group. What Green and Schwarz noticed is that one can use the antisymmetric tensor field which appears in the massless sector of the string in order to construct counterterms whose variation will cancel the gauge variation of the effective action represented by (16). The new prescription for anomaly cancellation can be summarized by saying that if (16) factorizes in the form

$$\mathbb{I}_{12}(F, R) = (\text{Tr } R^2 - k \text{Tr } F^2) X_8 \quad (17)$$

where X_8 is an invariant form, then such counterterms always exists. In the case of (16), (17) is satisfied only if the gauge group is $E_8 \times E_8$ or $SO(32)$ [and also $U(1)^{496}$]. The factorization condition requires the gauge group to obey the equation

$$\text{Tr } F^6 = \frac{1}{48} \text{Tr } F^2 \text{Tr } F^4 - \frac{1}{1440} (\text{Tr } F^2)^3 \quad (18)$$

as well as $k = 1/30$.

It is worth pointing out that the prescription (12)-(13) is part of the descent equations of Stora and Zumino¹⁹, and can be obtained via purely algebraic methods using the BRS algebra, or by purely topological methods using the index theorem for families of operators.

Other interesting aspects of the recent developments in anomalies which we do not have time to review are the following.

a) The Hamiltonian interpretation of anomalies²⁰. If one starts from a purely Hamiltonian formulation of a theory of Weyl fermions coupled to an external gauge field, the anomaly is found as a Schwinger term in the commutator of the gauge currents. The Schwinger term represents a two-cocycle in the algebra of infinitesimal gauge transformations, and it measures the obstruction to the possibility of constructing a fermionic Fock vacuum satisfying the Gauss law.

b) Non-linear σ -model anomalies²¹. These are anomalies appearing in low energy effective Lagrangians where the symmetries are realized non-linearly. Apart from the many applications in constraining the chiral structure of low energy effective theories, the σ -model anomalies provide a novel way of understanding the 't Hooft anomaly matching conditions²².

c) The equivalence of local Lorentz and gravitational anomalies. When considering the coupling of fermions to gravity, we have to introduce local Lorentz frames. The reason is that the frame rotation group does admit spinor representations, whereas the general linear group does not. Thus a theory of fermions coupled to gravity possesses two types of invariances: general co-ordinate transformations and local frame rotations. If the fermions are chiral, one would naively expect two different kinds of anomalies. Happily this is not the case. In Ref. 22, Bardeen and Zumino showed that they are equivalent by explicitly exhibiting the local counterterm interpolating between them.

d) Anomalies in odd dimensions^{23,24}. These "parity" anomalies appear for certain representations of fermions in odd dimensions [for example, an odd number of $SU(2)$ doublets in three dimensions²⁴]. The fermionic effective action cannot be regulated in such a way as to preserve both gauge invariance and parity. The parity violating part of the effective action is given by the Chern-Simons form (12) if we work in flat space, and by the Atiyah, Patodi-Singer η -invariant (see part B) if we work in curved

space²⁴. These anomalies have a number of applications in problems related to fermion fractionalization (see Ref. 25 for details).

These remarks conclude our quick tour of local anomalies.

B) Global Anomalies

Even when we have cancelled the local anomalous variation of the effective action under infinitesimal gauge transformations or diffeomorphisms, we still have to make sure that the theory is also invariant under those transformations not connected to the identity. We can illustrate this problem by working again with a gauge theory with fermions in Euclidean space-time compactified to a sphere S^{2n} . The group of gauge transformations is the set of maps from S^{2n} into the gauge group G . Since we have compactified Euclidean space by adding the point at infinity, we will require that the gauge transformations be equal to the identity in the north pole. These maps fall into homotopy classes given by the elements of the homotopy group $\pi_0(S^{2n} \rightarrow G)$. The elements in the trivial class of π_0 are those relevant in the computation of local anomalies, and the cancellation of local anomalies is equivalent to the statement that $\Gamma[A]$ is invariant under those gauge transformations in the identity of π_0 . A more difficult, but an equally relevant question, is whether $\Gamma[A]$ will be invariant under the other elements of $\pi_0(S^{2n} \rightarrow G) = \pi_{2n}(G)$. To answer this question we cannot consider a one-parameter family of gauge transformations interpolating between the identity and a non-trivial element of $\pi_{2n}(G)$. However, we can take advantage of the fact that the space of gauge connection on the sphere for a fixed topological class is a contractible space. The deformation retract is given as follows. Given any two gauge connections A and A' in the same topological class we can construct the one-parameter family of gauge fields $A(t) = A + t(A' - A)$ which interpolates between A and A' as t changes from 0 to 1. Using this property we can interpolate continuously between A and A^g , where g is a gauge transformation in a non-trivial class of $\pi_{2n}(G)$. This is the procedure used by Witten² to evaluate global anomalies. Thus if we want to compare $\Gamma[A^g]$ with $\Gamma[A]$ we consider instead a one-parameter family of effective actions $\Gamma[A(t)]$. The conclusion² is that

$$\frac{e^{-\Gamma[A^g]}}{e^{-\Gamma[A]}} = e^{i\pi \eta(0)} \quad (19)$$

where η is computed as follows. Since we have constructed a one-parameter family of gauge fields, we can construct the mapping cylinder $(S^{2n} \times [0,1])_g$, which means that on the cylinder $S^{2n} \times [0,1]$, the top and the bottom are identified after we twist by the gauge transformation g . The mapping cylinder is a compact odd dimensional manifold, and we can consider the $2n+1$ dimensional Dirac operator coupled to the gauge field $A(t)$. In odd dimensions, the index of an elliptic operator vanishes (there is no chirality in odd dimensions), but there is another object which plays a central role in the index theorem for manifolds with boundaries²⁶: the η -invariant of Atiyah Patodi and Singer. The η -invariant for a self-adjoint operator H on a compact manifold is defined by

$$\eta(s) = \sum_{\lambda} \text{sgn}(\lambda) |\lambda|^{-s} \quad (20)$$

for $\text{Re}(s)$ large enough, the series (20) converges (the sum is over the eigenvalues of H), and it can be analytically continued to the rest of the complex s -plane. For a large class of operators (including the Dirac operator), $\eta(0)$ is well defined, and roughly speaking, it measures the

number of positive minus the number of negative eigenvalues of the operator involved. It is often called the spectral asymmetry of H. Witten's result implies that global gauge anomalies are determined by the spectral asymmetry of the Dirac operator on the mapping cylinder. When the local anomalies cancel between several representations, the exponent on the right-hand side of (19) becomes

$$i\pi \left(\sum_L \eta_L(b) - \sum_R \eta_R(b) \right) \quad (21)$$

where the subscript L(R) stands for the left- (right-) handed representations of fermions. Using the index theorem for manifolds with boundary, it is easy to show that (21) is a topological invariant depending only on the homotopy class of g and not on the particular gauge field we choose or even what metric we have on the sphere. To see this explicitly, we write down the formula for the index of the Dirac operator in a manifold M with boundary $\partial M = B^{26}$:

$$\text{ind } D = \int_M \text{Ind } D(A, \omega) - (\eta(b) + h)/2 \quad (22)$$

where $\text{Ind } D(A, \omega)$ is the index polynomial constructed in (10), η is the η -invariant calculated for the restriction of the Dirac operator to the boundary B , and h is the dimension of the kernel of the boundary Dirac operator. To show that (21) is a topologically invariant object when the local anomalies cancel, consider a deformation of the configuration that we chose for the mapping cylinder (for simplicity, we assume that h is equal to zero, although a more careful argument can be carried through where this is not the case). If we call I the interval where the parameter that defines the deformation varies, we can construct a $2n+2$ dimensional manifold $D = (S^{2n} \times [0, 1]) \times I$ whose boundary is given by the original mapping cylinder and its deformation. Using now (22), we find

$$\begin{aligned} \sum_L \Delta \eta_L(b) - \sum_R \Delta \eta_R(b) = & -2 \sum_L \text{ind } D_L + 2 \sum_R \text{ind } D_R \\ & + \int_D \left(\sum_L \text{Ind } D(A, \omega) - \sum_R \text{Ind } D(A, \omega) \right) (\text{mod } \mathbb{Z}) \end{aligned} \quad (23)$$

where $\Delta \eta$ represents the change in η in the interpolation between the initial and final mapping cylinders. Since the anomaly is given by the index theorem in two higher dimensions, the last two terms in (23) cancel if the combination of fermions chosen is free from local anomalies. Since the index is an integer, we conclude that (22) is a topological invariant which only depends on the topological class of the gauge transformation g . When the field considered is not simply a Weyl fermion, Eq. (16) needs some changes, but the basic idea is the same. When this analysis is applied to the chiral $N = 2$ ten-dimensional supergravity theory or to the $E_8 \times E_8$ or $SO(32)$ super-Yang-Mills theory coupled to $N = 1$ supergravity in ten dimensions, the result is that these theories are free of global gauge and gravitational anomalies².

In string theory, invariance under global diffeomorphisms on the string world sheet is very important in the determination of the string spectrum. In the NSR (Neveu-Schwarz-Ramond) formulation of the superstring²⁷, and of the heterotic string¹², the supersymmetric spectrum is obtained after one performs the GSO²⁸ projection. This projection can be interpreted geometrically as a sum over all the spin structures of the left- and right-handed fermions involved in the first quantized formulation of the theory on the Riemann surface traced out by the string. The relevance of the global anomalies in this case is that spin structures are

not invariant under arbitrary non-trivial diffeomorphisms on the Riemann surface. Thus the absence of global anomalies in the string world sheet imposes severe constraints on the allowed spectrum of consistent string theories. For the known string theories, Witten²⁹ used his general analysis of global anomalies to show that once the local conformal anomalies are cancelled, there are no global anomalies with respect to diffeomorphisms which do not change the spin structure of the fermions. As a consequence, one can always choose signs in the sum over the spin structures (the GSO projection) consistent with invariance under global diffeomorphisms on the Riemann surface (modular invariance). The proof³ of invariance under global diffeomorphisms in two dimensions is much simpler than in the ten-dimensional case. It follows from some general theorems concerning three-manifolds. For illustrative purposes, let us look at the computation of (19) in the case of a spin- $\frac{1}{2}$ Weyl fermion on a compact Riemann surface Σ of genus g (g is the number of handles). If $\text{Diff}^+\Sigma$ stands for all diffeomorphisms of Σ preserving orientation, and $\text{Diff}_0^+\Sigma$ is the normal subgroup of $\text{Diff}^+\Sigma$ of those diffeomorphisms connected to the identity, the group of non-trivial diffeomorphisms of Σ is defined as $\Omega(\Sigma) = \text{Diff}^+\Sigma / \text{Diff}_0^+\Sigma$ [in the mathematic literature, $\Omega(\Sigma)$ is known as the mapping class group. Following the physics literature we will call $\Omega(\Sigma)$ the modular group of Σ]. There is a good deal of information about $\Omega(\Sigma)$ ³⁰. For instance, it is known that one can always find a representative of each class in $\Omega(\Sigma)$ in terms of Dehn twists: given a closed curve γ on Σ , we can cut Σ along γ . Keeping one of the lips of the cut fixed, we twist the other one by 2π and glue them back together. Thus the mapping cylinder $(\Sigma \times I)_\phi$ is now constructed by gluing the top and bottom of $\Sigma \times I$ up to a Dehn twist. If the diffeomorphism preserves the spin structure chosen on Σ , then we get a spin 3-manifold bounds $(\Sigma \times I)_\phi$, and from (19), we have to compute the η -invariant to find any possible anomalies. The first useful mathematical result in this direction is the fact that any spin three-manifold bounds a spin 4-manifold³¹. Using the Atiyah-Patodi-Singer index theorem²⁶, Eq. (22), in four dimensions we can write

$$\frac{\eta(0)}{2} = \frac{\sigma}{8} - \int_B \hat{A}(R) \quad \partial B = (\Sigma \times I)_\phi \quad (24)$$

where σ is the signature of the 4-manifold B . The second mathematical result needed is a theorem of Rohlin that the signature of any spin 4-manifold with boundary is always equal to 0 (mod 8). The computations in $d = 10$ are much more subtle², but the main idea is the same: one always looks for a $d+2$ -dimensional manifold whose topological invariants are known, and such that its boundary is the mapping cylinder. For more details, see Ref. 2.

Once modular invariance is proved, the understanding of the various ways of realizing modular invariance in the string world sheet is the reason behind the construction of the tachyon-free non-supersymmetric heterotic string $O(16) \times O(16)$ ³². In all known examples of string theories (even those with tachyons) with gauge groups other than $E_8 \times E_8$ or $SO(32)$, one always finds that once the theory is free of global anomalies in the string world sheet, it is also free of both local and global anomalies in ten dimensions. Modular invariance is one of the central elements in the formulation of string theories. Unfortunately we do not yet have a good understanding of this intimate interplay between the discrete group of global diffeomorphisms in two dimensions and the continuous group of gauge and co-ordinate transformations in $d = 10$. A first step in this direction is provided by the work of Ref. 33 where, for the first time, one can find a direct link between modular invariance and absence of local anomalies in the critical dimension.

2. - TOPOLOGICAL METHODS IN STRING PERTURBATION THEORY

In the first quantized formulation of string perturbation theory²⁷, one considers a string propagating in a fixed external background space-time. If we restrict our attention to closed strings, the loop expansion for on-shell amplitudes is an expansion in the number of handles of the two-dimensional surface describing the history of the string. After taking proper account of gauge fixing, one finds that in a given order of perturbation theory the Polyakov measure³⁴ requires that we sum over all inequivalent complex structures on a given surface of fixed genus. The space of all such complex structures is known as the moduli space of the surface, and it has $6g-6$ dimensions. Even though the topology of this space is rather complicated, it has a fundamental property, namely it has the structure of a complex manifold (more precisely, a complex algebraic variety) whose boundary is made out of Riemann surfaces with nodes. A very particular property of moduli space is that even though it is a non-compact complex space, any globally defined holomorphic function is a constant³⁵. When one considers string perturbation theory in the critical dimension, it can be shown³⁶ that the integrand of the closed string partition function factorizes into a holomorphic function times an anti-holomorphic function (or sums of factors with this general form if one wants to consider non-supersymmetric strings). Thus one can rephrase many questions of string perturbation theory (and perhaps even non-perturbative properties of the theory if one follows the infinite genus approach of Friedan and Shenker (see Friedan lectures in this volume) in terms of complex algebraic geometry. In the string path integral one integrates the partition and correlation functions of two dimensional field theories over the moduli space of conformally inequivalent Riemann surfaces. On a Riemann surface any field can be described in terms of q -differentials: $t = t(z, z^*) dz^q$, $q = 0, \pm\frac{1}{2}, \pm 1, \dots$; when q is half integral, one has to be careful in choosing consistent branches for the square root (spin structures, see for instance 37). The kinetic term giving the dynamics of the field t always involves in the cases of interest in string theory, the Cauchy-Riemann operator $\bar{\partial}$ for the given complex structure. Thus the evaluation of string partition and correlation functions is reduced to the computation of functional determinants for $\bar{\partial}$ -operators acting on q -differentials as well as their Green's functions. A very powerful procedure to carry out these calculations is to use bosonization as was shown by Friedan, Martinec and Shenker³⁸ when the fields live on a simply connected surface (string-tree level). The generalization of the bosonization prescription to surfaces of higher genus presents several difficulties, in particular the lack of a canonical quantization procedure for fields on compact surfaces of genus $g > 1$. The approach taken in 39 is to carry out the proof of bosonization in terms of Euclidean path integrals using algebraic geometry. The main ingredients in the mathematical proof of the results presented below for bosonization in surfaces of high genus are: (i) a theorem due to Quillen⁴⁰ which describes the determinant of a holomorphic family of wave operators, and (ii) the application of Faltings' work on Arakelov geometry⁴¹, in particular, Faltings' inductive construction of determinants for the $\bar{\partial}$ -operator acting on sections of any holomorphic line bundle over the Riemann surface in terms of spin- $\frac{1}{2}$ determinants. It is very intriguing that methods which seem to be very powerful in arithmetic geometry appear to be also intimately connected with the mathematical structure of string theories. Whether these mathematical structures reflect some fundamental property of string theory and not just of its perturbation expansion, is something that remains to be clarified.

In the following we will describe what bosonization says about the partition function and correlation functions of generalized ghost systems.

Following the treatment of Ref. 38, we consider a first-order ghost system whose dynamics is described by the action

$$S = \int_{\Sigma} b \bar{\partial} c + \text{c.c.} \quad (25)$$

where b is a λ -differential and c is a $1-\lambda$ -differential. Note that $b\bar{\partial}c$ is a $(1,1)$ -form and therefore can be integrated over Σ without any additional factor. Geometrically the field c is a section of $L^{2-2\lambda}$ where L is a twisted spin bundle. In general the $\bar{\partial}$ operator may have a kernel and/or a co-kernel. In order to obtain a non-vanishing partition function we have to insert the appropriate number of b and c fields to soak up the zero modes. This also follows from the fact that the ghost number current bc has an anomaly leading to a net violation of ghost charge $\Delta Q = k = (2\lambda-1)(g-1)$, as required by the Riemann-Roch theorem (for more details, see for example 38). In particular, for $\lambda = \frac{1}{2}$, no insertions are required; for $\lambda = 1$, we need one insertion of c and g insertions of b ; and k insertions of b for $\lambda > 1$. Thus, if ϕ_i , $i = 1, \dots, k$ denotes a basis of holomorphic λ -differentials ($\lambda > 1$), and ω_i , $i = 1, \dots, g$ is a basis of holomorphic one-forms (Abelian differentials), we can write the partition functions to be computed as

$$\begin{aligned} Z(\lambda = \frac{1}{2}) &= \det \bar{\partial}_L^+ \bar{\partial}_L \\ Z(\lambda = 1) &= \int db d\bar{b} dc d\bar{c} \prod_{i=1}^g b\bar{b}(P_i) c\bar{c}(Q) e^{-S} = \\ &= \frac{\det \omega_i(P_j) \wedge \det \bar{\omega}_i(P_j)}{\det \langle \omega_i | \omega_j \rangle} \frac{\det' -\nabla^2}{\int \sqrt{g}} \\ Z(\lambda > 1) &= \int db d\bar{b} dc d\bar{c} \prod_{i=1}^k b\bar{b}(P_i) e^{-S} = \\ &= \frac{\det \psi_i(P_j) \wedge \det \bar{\psi}_i(P_j)}{\det \langle \psi_i | \psi_j \rangle} \det' \bar{\partial}_L^+ \bar{\partial}_L^{2-2\lambda} \end{aligned} \quad (26)$$

Since we have inserted b fields, $Z(\lambda)$ should be considered as a differential form on Σ^k . In (26), the factor of $\sqrt{g}$ represents the volume of the surface for the metric chosen. The prime means that zero modes have been removed. The computation of the determinants (26) is greatly simplified from a mathematical point of view if we choose the Arakelov metric on Σ . Since the integrand in the string path integral is conformally invariant, it does not matter which metric we choose. There are two metrics which have proved useful in string theory: (i) the Poincaré metric, obtained from the general uniformization theorem⁴² stating that any Riemann surface of genus $g > 1$ can be represented as the upper half plane divided by a hyperbolic fuchsian group; a discrete subgroup of $SL(2, \mathbb{R})$, the isometry group of the upper half plane with the Poincaré metric $ds^2 = dzd\bar{z}/(\text{Im}z)^2$. With this representation of the Riemann surface, one can use the Selberg trace formula to obtain useful expressions for the determinants in (26)⁴³. (ii) The Arakelov metric: on any Riemann surface of genus $g > 0$, there are g holomorphic one-forms, $\omega_i = \omega_i(z)dz$, $i = 1, \dots, g$. A canonical basis for the homology of the surface is given by a collection of one-cycles a_i , b_i , $i = 1, \dots, g$ whose intersection pairing is given by

$$\begin{aligned} (a_i, a_j) &= (b_i, b_j) = 0 \\ (a_i, b_j) &= -(b_i, a_j) = \delta_{ij} \end{aligned} \quad (27)$$

a convenient normalization of the basis of Abelian differentials is to require that

$$\int_{a_i} \omega_j = \delta_{ij} \quad (28)$$

then the periods over the b-cycles are determined

$$\int_{b_i} \omega_j = \Omega_{ij} \quad (29)$$

Ω is known as the period matrix of the Riemann surface for the given marking (choice of a canonical homology basis), and it is a symmetric matrix with positive definite imaginary part. Using the Abelian differentials we can construct the (1,1)-form

$$\mu = \frac{i}{2g} \sum_{i,j} \omega_i (\text{Im } \Omega)^{-1}_{ij} \omega_j \quad (30)$$

satisfying

$$\int_{\Sigma} \mu = 1 \quad (31)$$

Given a holomorphic line bundle ξ over Σ , a metric on it is admissible if its curvature is proportional to μ . Thus if s is a section of ξ , and $\|s\|^2$ its norm, the curvature R_{ξ} is given by

$$R_{\xi} = \bar{\partial} \partial \log \|s\|^2 \propto \mu \quad (32)$$

Using the Arakelov metric on Σ , we can construct the Arakelov Green's function $G(P, Q)$ satisfying

$$\begin{aligned} \bar{\partial} \partial \log G(P, Q) &= \text{in } \mu(P) - \text{in } \delta(P, Q) \\ \int_{\Sigma} \log G(P, Q) \mu(P) &= 0 \quad G(P, Q) = G(Q, P) \end{aligned} \quad (33)$$

the explicit form of the determinants (26) is given in terms of $G(P, Q)$.

Bosonization is the statement that the first-order field theories (25) and (26) can be replaced by equivalent bosonic field theories. In Ref. 30, it was shown that the local properties of the weight λ -system can be reproduced by the scalar action

$$S[\varphi] = 4\pi i \int \partial \varphi \bar{\partial} \varphi + (2\lambda - 1) \int R_{K^1} \varphi \quad (34)$$

the first term is the standard kinetic term, and the second term accounts for the anomaly in the ghost number current. R_{K^1} is the curvature of the holomorphic tangent bundle. K is the canonical line bundle; the bundle of holomorphic one-forms. The curvature in (34) is normalized so that $\int R = 4\pi i(g-1)$. The bosonization prescription for ghost bilinears is then

$$\begin{aligned} b \bar{b} &= : \exp 4\pi i \varphi : \\ c \bar{c} &= : \exp -4\pi i \varphi : \end{aligned} \quad (34)$$

and the normal ordering prescription cancels the co-ordinate dependence so that b and c are well-defined forms. Normal ordering requires one to define the Green's functions at coincident points. In the Arakelov metric, the prescription is³⁹:

$$: \log G(P, Q) : = \lim_{P \rightarrow Q} \left\{ G(P, Q) - 2\lambda \log |z(P) - z(Q)| - (1-2\lambda) \log d(P, Q) \right\} \quad (35)$$

z is a local co-ordinate in a neighbourhood of P and $d(P, Q)$ is computed with the Arakelov metric. On a multiply connected surface, the field ϕ is

not single-valued and it can have non-trivial shifts around the homology cycles. Since mass terms and currents should be well defined on the surface, (34) forces the shifts of ϕ to be either integral or half-integral. In the evaluation of the functional integral for the scalar field (34), we can split the field ϕ into a single-valued piece ϕ and an instanton $\phi(n,m)$. The instanton is constructed such that $d\phi(n,m)$ is a globally defined harmonic one-form with n_i periods over the a-cycles and m_i periods over the b-cycles, n_i, m_i integers or half-integers, $i = 1, \dots, g$. To evaluate the second term in (34) in an instanton section we must define the multiple-valued field ϕ by choosing a system of curves intersecting in a single point A and being homologous to the homology basis chosen. Such a choice lets us cut open the surface to obtain a $4g$ -sided polygon Σ_c . Choosing now a base point P_0 , the field

$$\varphi(n,m) = \int_{P_0}^P d\varphi(n,m) \quad (36)$$

is well defined on Σ_c , and we may evaluate the action (34). In doing so, one finds that the bosonic action and path integral depend explicitly on the marking. This is unacceptable because for non-chiral ghost systems [as in (25)], the partition function does not depend on the choice of homology basis (it is modular invariant): Therefore we must change the action (34) in order to achieve marking independence. In fact the new terms to be added to (34) can be uniquely determined by very plausible requirements. Namely³⁹:

- (i) independence of A and P_0 ,
- (ii) independence of the representative cycles on the homology classes of our homology basis,
- (iii) the conformal anomaly and local conformal properties should not be changed,
- (iv) modular invariance of the action for multivalued fields (independence of the homology basis),
- (v) factorization of the instanton partition function.

Since the instanton quantum numbers are only related to the homology basis, one would expect that when the surface Σ is split into two pieces Σ_1, Σ_2 of genus g_1, g_2 (resp.), $g_2, g_2 > 0$ by shrinking to zero a homologically trivial cycle, the instanton partition function would also factorize. Before writing down the modified action obtained after use has been made of (i)-(v), we need to introduce a bit more notation. After a marking has been chosen there is a preferred spin structure L_Δ associated to the vector of Riemann constants⁴⁴, and it is natural to use this spin bundle to parametrize the line bundles whose sections are represented by b and c. Since two holomorphic line bundles with the same first Chern number only differ by a flat line bundle⁴⁵, we can write $(L \otimes L_\Delta^{-1})^{2-2\lambda} = F(\theta_1, \theta_2)$, where F is a holomorphic line bundle with holonomy $2\pi i \theta_1, -2\pi i \theta_2$ around the a and b-cycles, respectively. After all these preliminaries we can finally write down the complete scalar action

$$\begin{aligned} S[\varphi] = & 4\pi i \int \partial\varphi \bar{\partial}\varphi + (2\lambda-1) \int R_{\kappa^{-1}} \varphi \\ & + \sum_{\kappa=1}^g \left\{ \int_{a_\kappa} d\varphi f(b_\kappa) - \int_{b_\kappa} d\varphi f(a_\kappa) \right\} \\ & + \end{aligned}$$

$$+ 4\pi i \sum_{k=1}^g \int_{a_k} d\phi \int_{b_k} d\phi \quad (37)$$

where $f(\gamma)$ is a function defined for any closed curve and it is given by twice the holonomy of the admissible connection for the line bundle $L^{2-2\lambda} \otimes L_{\Delta}^{-1}$

$$f(\gamma) = 2h(\gamma; L^{2-2\lambda} \otimes L_{\Delta}^{-1}) \quad (38)$$

By using (37) and (38) and the prescription (34) to evaluate (26) (assuming that bosonization works) we obtain the following formulae for the determinants appearing in (26)³⁹, computed with the Arakelov metric

$$\det \bar{\partial}_L^+ \bar{\partial}_L^- = \left(\frac{\det' - \nabla^2}{\int g \det \operatorname{Im} \Omega} \right)^{-1/2} \| \vartheta(\theta_1 + \Omega \theta_2 | \Omega) \|^2 \quad (39a)$$

$$\left(\frac{\det' - \nabla^2}{\int g \det \operatorname{Im} \Omega} \right)^{3/2} = \frac{\prod_{i,j} G^2(p_i, p_j) \| \vartheta(z_{\lambda+1} | \Omega) \|^2}{\| \det w_i(p_j) \|^2 \prod G(p_i, Q)^2} \quad (39b)$$

$$\frac{\det \bar{\partial}_{L^{2-2\lambda}}^+ \bar{\partial}_{L^{2-2\lambda}}^-}{\det \langle \psi_i | \psi_j \rangle} = \left[\frac{\det' - \nabla^2}{\int g \det \operatorname{Im} \Omega} \right]^{-1/2} \frac{\prod_{i,j} G(p_i, p_j)^2 \| \vartheta(z_1 | \Omega) \|^2}{\| \det \psi_i(p_j) \|^2} \quad (39c)$$

where θ is the Riemann θ -function for the Riemann surface given by⁴⁴:

$$\vartheta(z | \Omega) = \sum_{(n_i) \in \mathbb{Z}^g} \exp(i\pi n_i \Omega_{ij} n_j + 2\pi i n_i z_i)$$

and

$$\| \vartheta(z | \Omega) \|^2 = e^{-2\pi(\operatorname{Im} z)_i (\operatorname{Im} \Omega)^{-1}_{ij} \operatorname{Im} z_j} | \vartheta(z | \Omega) |^2$$

$$z_{\lambda+1} = I \left(\sum_{i=1}^g p_i - Q \right) - \Delta$$

$$z_{\lambda} = I \left(\sum_{i=1}^g p_i - (2\lambda-1)\Delta \right) + \theta_1 + \Omega \theta_2$$

$$I \left(\sum_{i=1}^g p_i \right) = \sum_{i=1}^g \int_{\mathcal{P}_0}^{\mathcal{P}_i} \omega$$

and in the very last formula we have assembled the Abelian differentials to form a vector. Similar formulae can be obtained for correlation functions of an arbitrary number of insertions of b and c fields³⁸. As pointed out at the beginning of this section, the rigorous proof of these bosonization formulae³⁹ makes use of Quillen's theorem⁴⁰, some of Faltings' constructions⁴¹, and the very particular property of moduli space that any globally defined holomorphic function is a constant³⁵. The results expressed in Eq. (39) can be used to write fairly explicit formulae for the string partition and correlation functions in high orders of perturbation theory, and thus analyze issues concerning finiteness as

well as general properties of string perturbation theory with the help of rather powerful methods in algebraic geometry. Perhaps the most outstanding recent result of the interplay between algebraic geometry and string theory is Manin's⁴⁶ evaluation of the string integrand for the bosonic string in a surface of arbitrary genus. It should also be said, however, that if one is interested in obtaining definite numbers for a given string amplitude one is still left with the rather difficult problem of performing the integration of expressions similar to those in (36) over moduli space. Nevertheless, many qualitative properties of the theory can be formulated in terms of the partition and correlation functions at generic points in moduli space. Even more interesting is the possibility suggested by Friedan and Shenker⁴⁷ in their approach to quantum string theory that these beautiful structures which appear in perturbation theory, do capture some of the fundamental properties of string theory. These aspects of the theory and their connection with string field theory are being investigated vigorously, and hopefully we will have a more clear picture not too far in the future.

ACKNOWLEDGEMENTS

I would like to thank Prof. A Zichichi for his invitation to participate in this School and for his kind hospitality.

REFERENCES

1. For a review and references to the original literature, see:
L. Alvarez-Gaumé - "An Introduction to Anomalies" HUTP 85/A052, to appear in the Proceedings of the Erice School in Mathematical Physics; G. Velo and A. Wightman, eds. (Plenum Press, 1986).
2. E. Witten - Phys.Lett. 117B (1982) 324; Comm.Math.Phys. 100 (1985) 197.
3. E. Witten - in Proceedings of the Argonne Symposium on Anomalies Geometry and Topology. W.A. Bardeen and A. White, eds. (World Scientific, 1985).
4. S.L: Adler - Phys.Rev. 177 (1969) 2426, and in "Lectures in Elementary Particles and Quantum Field Theory", S. Deser et al., eds. (MIT Press, 1970);
J.S. Bell and R. Jackiw - Nuovo Cimento 60A (1969) 47.
See also: R. Jackiw - in "Lectures on Current Algebra and its Application", (Princeton U. Press, 1972).
5. G. 't Hooft - Phys.Rev.Lett. 37 (1976) 8;
C. Callan, R. Dashen and D. Gross - Phys.Lett. 63B (1976) 334;
R. Jackiw and C. Rebbi - Phys.Rev.Lett. 37 (1976) 172.
6. R. Delbourgo and A. Salam - Phys.Lett. 40B (1972) 381;
T. Eguchi and P. Freund - Phys.Rev.Lett. 37 (1976) 1251.
7. D.J. Gross and R. Jackiw - Phys.Rev. D6 (1972) 477;
C. Bouchiat, J. Iliopoulos and Ph. Meyer - Phys.Lett. 38B (1972) 519;
H. Georgi and S.L: Glashow - Phys.Rev. D6 (1972) 429.
8. P. Langacker - Physics Reports 72 (1981) 185.

9. P.H. Frampton and T.W. Kephart - Phys.Rev.Lett. 50 (1983) 1343, 1347;
P.K. Townsend and G. Sierra - Nucl.Phys. B222 (1983) 493;
B. Zumino, W.Y. Shi and A. Zee - Nucl.Phys. B239 (1984) 477.
10. L. Alvarez-Gaumé and E. Witten - Nucl.Phys. B234 (1983) 269.
11. M. Green and J. Schwarz - Phys.Lett. 149B (1984) 117;
M. Green, J. Schwarz and P. West - Nucl.Phys. B254 (1984) 377.
12. D.J. Gross, J. Harvey, E. Martinec and R. Rohm - Phys.Rev.Lett. 54 (1985) 502; Nucl.Phys. B256 (1985) 253; B267 (1986).
13. M.F. Atiyah and I.M. Singer - Ann. of Math. 87 (1968) 485, 546; 93 (1971) 1, 119, 139;
M.F. Atiyah and G.B. Segal - Ann. of Math. 87 (1968) 531.
14. R. Jackiw, C. Nohl and C. Rebbi - in "Particles and Fields", D. Boch and A. Kamal, eds. (Plenum Press, NY, 1978);
N.K. Nielsen, H. Roemer and B. Schroer - Nucl.Phys. B136 (1978) 478.
15. L. Alvarez-Gaumé and P. Ginsparg - Nucl.Phys. B243 (1984) 449.
16. M.F. Atiyah and I.M. Singer - Proc.Nat.Acad.Sci. U.S.A. 81 (1984) 2597.
17. S.S. Chern - "Complex Manifolds without Potential Theory" (Van Nostrand, 1967).
18. See Ref. 1 for a thorough discussion.
19. For very clear discussions, see:
B. Zumino - in "Relativity, Groups and Topology - II", Proceedings of the Les Houches Summer School, B.S. DeWitt and R. Stora, eds. (North Holland, 1984);
R. Stora - in "Progress in Gauge Theories", Proceedings of the 1984 Cargèse School, G. 't Hooft et al., eds. (Plenum Press, 1984).
20. L.D. Faddeev - Phys.Lett. 145B (1984) 81;
L.D. Faddeev and S. Shatashvili - Teor. and Math.Phys. 60 (1984) 206;
B. Zumino - Nucl.Phys. B253 (1985) 477;
R. Jackiw - MIT Preprint CTP 1298 (198), to appear in Comm.Nucl. Part.Phys.;
P. Nelson and L. Alvarez-Gaumé - Comm.Math.Phys. 99 (1985) 103.
21. G. 't Hooft - in "Recent Progress in gauge Theories", G. 't Hooft et al., eds. (Plenum Press, N.Y., 1980).
22. W.A. Bardeen and B. Zumino - Nucl.Phys. b244 (1984) 241.
23. A.N. Redlich - Phys.Rev.Lett. 52 (1984) 1.
See also Ref. 10.
24. L. Alvarez-Gaumé, S. Della Pietra and G. Moore - Ann.Phys. 163 (1985) 288.
25. A. Niemi and G. Semenoff - Physics Reports 136 (1986) 100.

26. M.F. Atiyah, V.I. Patodi and J.M. Singer - Proc.Camb.Phil.Soc. 77 (1975) 43; 78 (1975) 405; 79 (1976) 71.
27. J. Schwarz - Physics Reports 89 (1982) 223; and Proceedings of the Santa Barbara Workshop on Superstrings, M. Green and D. Gross, eds. (World Scientific, 1986).
28. F. Gliozzi, J. Scherk and D. Olive - Nucl.Phys. B122 (1977) 253.
29. See Ref. 3.
30. J. Birman - "Braids, Links and Mapping Class Groups", Lecture Notes in Mathematics (Princeton University Press, 1974).
31. R. Stong - "Lectures on Cobordism Theory", Lecture Notes in Mathematics (Princeton University Press, 1968).
32. L. Alvarez-Gaumé, P. Ginsparg, G. Moore and C. Vafa - HUTP 86/A013; L. Dixon and J. Harvey - Princeton Preprint
33. B.A. Schellekens and N. Warner - CERN Preprints TH. 4464 and 4465 (1986).
34. A.M. Polyakov - Phys.Lett. 103B (1981) 207, 211;
D. Friedan - in "Recent Advances in Field Theory and Statistical Mechanics", J.B. Zuber and R. Stora, eds. (Elsevier, 1984);
O. Alvarez - Nucl.Phys. B216 (1983) 125;
J. Polchinski - Comm.Math.Phys. 104 (1986) 37;
G. Moore and P. Nelson - Nucl.Phys. B266 (1986) 58;
E. D'Hoker and D. Phong - Nucl.Phys. B269 (1986) 206.
35. H. Harris - in "Proceedings of the International Congress of Mathematicians", Warszawa, Poland, C. Olech and Z. Ciesielski, eds. (Elsevier, Amsterdam, 1984).
36. A. Belavin and V.I. Knizhnik - "Algebraic Geometry and the Geometry of Quantum Strings", Landau Institute Preprint (to appear in Nucl.Phys.);
R. Catenacci, M. Cornalba, M. Martellini and C. Reina - "Algebraic Geometry and Path Integrals for Closed Strings", Pavia Preprint;
J.B. Bost and T. Jolicoeur - Saclay PhT/86-28 (1986).
37. L. Alvarez-Gaumé, G. Moore and C. Vafa - Comm.Math.Phys. 106 (1986) 40;
See also: J.B. Bost and P. Nelson - Phys.Rev.Lett. 57 (1986) 795.
38. D. Friedan, E. Martinec and S. Shenker - Nucl.Phys. B271 (1986) 93.
39. L. Alvarez-Gaumé, J.B. Bost, G. Moore, P. Nelson and C. Vafa - HUTP 86/A036 (1986) (to appear in Phys.Lett. B) and in preparation.
40. D. Quillen - Funct.Anal.Appl. 19 (1986) 31.
41. G. Faltings - Ann. of Math. 119 (1984) 387.
42. H. Farkas and J. Kra - "Riemann Surfaces", Springer (1980).
43. E. D'Hoker and D. Phong - Comm.Math.Phys. 104 (1986) 537.

- 44. D. Mumford - "Tata Lectures on Theta", Birkhauser (1983).
- 45. G. Gunning - "Introduction to Riemann Surfaces", Princeton Lecture Notes in Mathematics (Princeton University Press, 1966).
- 46. Yu. Manin - JETP Lett. 43 (1986) 204.
- 47. D. Friedan and S. Shenker - Chicago Preprints EFI-86-18A,B (1986).