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CONSISTENCY BETWEEN THE STRING BACKGROUND FIELD EQUATION
OF MOTION AND THE VANISHING OF THE CONFORMAL ANOMALY

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ABSTRACT

We prove to all orders in bosonic σ -model perturbation theory that at the fixed ultra-violet point the central charge is constant.

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1. - INTRODUCTION

The study of the ultra-violet properties of non-linear σ -models in two-dimensional curved space-time has recently received much attention for its connection with string theories¹⁾⁻⁷⁾. It is believed that the background configuration for the propagation of the string is given by a conformal invariant solution of the corresponding σ -model. This requirement implies the vanishing of the trace of the stress-energy tensor, or Weyl anomaly, for the two-dimensional quantum field theory. Roughly speaking, the Weyl anomaly is given in the renormalized theory by the sum of the β -functions associated with the low-lying excitations of the string.

In this work, we analyze the simplest bosonic string model describing the graviton, dilaton and tachion fields. The ultra-violet properties of the model are described by the β -functions β_{ij} , β_ϕ and β_H for these fields.

An interesting connection between β_{ij} and β_ϕ has been pointed out for this model^{2),3)}. At the first orders in perturbation theory, the vanishing of β_{ij} implies that β_ϕ is a c-number. This feature is necessary for the consistency of the approach, as at the fixed point $\beta_{ij} = 0$, β_ϕ is proportional to the vacuum expectation value of the trace of the stress tensor, i.e., to the central charge of the conformal invariant theory. We shall show that this remarkable property is true to all orders in perturbation theory.

To obtain this result, we first generalize to two-dimensional non-linear σ -models the programme developed for quantum field theories in curved space-time in the dimensional regularization scheme⁸⁾⁻¹¹⁾. This analysis shows how to compute renormalization group coefficients in a curved space-time from flat space calculations. We get a recursive relation expressing β_ϕ in terms of quantities computed at lower orders. Finally, as a nice by-product of this relation, we obtain the proof that β_ϕ is a c-number if $\beta_{ij} = 0$.

2. - RENORMALIZATION OF THE NON-LINEAR σ -MODEL

As is known, the generalized σ -model^{12),13)} is defined by the action in $d = 2 + \varepsilon$ dimensions

$$S = \frac{1}{2\pi\alpha'} \int d\sigma \left[\frac{1}{2} \gamma^{\mu\nu} g_{ij}^B \partial_\mu X^i \partial_\nu X^j - \frac{\alpha'}{4} \bar{R}^{(2)} \phi^B - H^B \right] \quad (1)$$

where X^i is a field with values in a target space M . The composite fields $g_{ij}(X)$, $\phi(X)$ and $H(X)$ are defined on M and they represent the graviton, dilaton and tachion fields in the string approach. Moreover, $\gamma_{\mu\nu}$ and $\bar{R}^{(2)} = R^{(2)}/(d-1)$ are the metric and the curvature on the world sheet^{*}.

The partition function of the system

$$Z = \int [dX] \exp(-S) \quad (2)$$

is invariant under reparametrization of M at the classical level. As shown in Ref. 12), the minimal dimensional renormalization scheme explicitly preserves this property (we consistently ignore measure terms). We choose a renormalization procedure that is only defined for invariant correlations so we can neglect reparametrizations of the X^i fields. Moreover, the quadratic term in the expansion of $H(X)$ in Eq. (1) is a generalized mass term for the X^i fields that provides an infra-red cut-off for the model¹²⁾.

The bare quantities in Eq. (1) are expressed in terms of the renormalized ones as

$$g_{ij}^B = \mu^\varepsilon \left(g_{ij} + \sum_L (\alpha')^L \frac{1}{\varepsilon^L} T_{ij}^{(L,\nu)}(g) \right) \quad (3)$$

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^{*}) The choice of the $d-1$ factor will be clear in the following.

$$H^B = \mu^{2+\epsilon} Z \cdot H$$

$$Z \cdot H = H + \sum_L (\alpha')^L \frac{1}{\epsilon^L} Z^{(L,\nu)}(g) \cdot H \quad (4)$$

$$\phi^B = \mu^\epsilon (Z \cdot \phi + \Psi)$$

$$\Psi = \sum_\nu \frac{1}{\epsilon^\nu} \Psi^{(\nu)}(g) \quad (5)$$

where μ is the unit of mass in the dimensional scheme and L is the order in the loops expansion. The tensors $T_{ij}^{(L,\nu)}\{g\}$ are the counterterms induced by the metric which are needed to renormalize H -correlation functions. The differential operator Z defined in Eq. (4) represents a generalized renormalization constant for scalar fields. An alternative and equivalent prescription to compute these counterterms is to use background field methods as explained in Refs. 13) and 14).

The additive renormalization for the dilaton field, the term ϕ in Eq. (5), arises from the mixing of the metric operator with the world-sheet curvature according to general power-counting arguments¹⁰⁾. We note that g_{ij}^B and H^B have the same expression of the flat space theory as it follows from simple dimensional analysis. In principle, the determination of ϕ would require an involved computation with propagators defined in curved d -dimensional space-time. We will show that ϕ is more simply fixed by requiring consistency between the finiteness of the energy-stress tensor and the renormalization group properties of the model.

The first step of this programme is to define the renormalization-group coefficients according to the equation

$$\mathcal{D} Z = 0 \quad (6)$$

where

$$\begin{aligned} \mathcal{D} = & \mu \frac{\partial}{\partial \mu} + \left(-\varepsilon g_{ij} + \beta_{ij} \right) \frac{\partial}{\partial g_{ij}} \\ & + \left(-\varepsilon \phi + \tilde{\gamma}(\phi) \right) \frac{\partial}{\partial \phi} + \left(-(2+\varepsilon)H + \gamma(H) \right) \frac{\partial}{\partial H} \end{aligned} \quad (7)$$

The functional derivatives in Eq. (7) are to be understood as parametric ones. In the minimal subtraction scheme β_{ij} , $\gamma(H)$ and $\tilde{\gamma}(\phi)$ are given by the expressions

$$\beta_{ij} = - \sum_L (\alpha')^L L T_{ij}^{(L,1)} \quad (8)$$

$$\gamma(H) = - \sum_L (\alpha')^L L Z^{(L,1)} \cdot H \quad (9)$$

$$\tilde{\gamma}(\phi) = \gamma(\phi) + \theta(g)$$

$$\theta(g) = \left(g \frac{\partial}{\partial g} - 1 \right) \Psi^{(1)} \quad (10)$$

In addition, the coefficients of higher poles in $1/\varepsilon$ satisfy the consistency conditions^{12),13),15)}

$$\sum_L (\alpha')^L L T_{ij}^{(L,\nu+1)} + \beta \frac{\partial}{\partial g} \sum_L (\alpha')^L T_{ij}^{(L,\nu)} = 0 \quad (11)$$

$$\begin{aligned} & \sum_L (\alpha')^L \left(L Z^{(L,\nu+1)} \cdot H + Z^{(L,\nu)} \cdot \gamma(H) \right) \\ & + \beta \frac{\partial}{\partial g} \sum_L (\alpha')^L Z^{(L,\nu)} \cdot H = 0 \end{aligned} \quad (12)$$

$$\begin{aligned} & \left(1 - g \frac{\partial}{\partial g}\right) \Psi^{(v+1)} + \beta \frac{\partial}{\partial g} \Psi^{(v)} \\ & + \sum_L (\alpha')^L Z^{(L,v)} \cdot \left[\left(g \frac{\partial}{\partial g} - 1\right) \Psi^{(1)} \right] = 0 \end{aligned} \quad (13)$$

For completeness we report the first orders^{12),13)}:

$$\begin{aligned} T_{ij}^{(1,1)} &= -R_{ij} & T_{ij}^{(2,1)} &= -\frac{1}{4} R_{iabc} R_j{}^{abc} \\ Z^{(1,1)} &= \frac{1}{2} \nabla^2 & Z^{(2,1)} &= 0 \end{aligned} \quad (14)$$

$\phi^{(1)}$ starts from the free theory value. By taking into account the contribution of the two-dimensional ghosts this value is given by¹⁶⁾

$$\Psi^{(1)} = -\frac{D-26}{3} \quad (15)$$

The equations (6)-(10) contain all the information we need about the renormalization of the action. In the following, we shall discuss the operatorial expression of the Weyl anomaly as a sum of renormalized composite operators. A preliminary step is to define in the dimensional scheme normal products¹⁷⁾⁻¹⁹⁾ of the form $N[F_{ij}(X) \partial_\mu X^i \partial^\mu X^j]$ and $N[F(X)]$.

The integrated normal products can be easily defined by using

$$\begin{aligned} & \int d\sigma \mu^\varepsilon N \left[\frac{1}{2} F_{ij} \partial_\mu X^i \partial^\mu X^j \right] = 2\pi\alpha' F \frac{\partial S}{\partial g} \\ & = \int d\sigma \mu^\varepsilon \left[\frac{1}{2} \left(F_{ij} + F \frac{\partial}{\partial g} \sum_{L,v} (\alpha')^L \frac{1}{\varepsilon^v} T_{ij}^{(L,v)} \right) \partial_\mu X^i \partial^\mu X^j \right. \\ & \quad \left. - \frac{\alpha'}{4} \bar{R}^{(2)} \left(F \frac{\partial Z}{\partial g} \cdot \Phi + F \frac{\partial \Psi}{\partial g} \right) - \mu^2 F \frac{\partial Z}{\partial g} \cdot H \right] \end{aligned} \quad (16)$$

and

$$\int d\nu \mu^{2+\epsilon} N[F(X)] = 2\pi\alpha' F \frac{\partial S}{\partial H} = \int d\nu \mu^{2+\epsilon} Z \cdot F \quad (17)$$

The finiteness of these operators is ensured by the action principle^{17),18)} as their insertion into a Green's function corresponds to a parametric derivative of the renormalized partition function. Moreover, Eq. (16) shows explicitly the mixing of a metric-like operator with the dilaton and tachion field. For the scalar operator, the renormalization is multiplicative as dictated by power counting.

The next step is to define non-integrated normal products. For the scalar operator,

$$N[F(X)] = Z \cdot F(X) \quad (18)$$

defines a finite operator, as other mixings are forbidden by power counting. For the metric-like operator, a mixing with a term proportional to a total derivative is possible. As pointed out in Ref. 7), this term can be written in the form $\partial_\mu (\partial^\mu X^i A_i(F))$ where $A_i(F) = A_i^{km}(g) F_{km}$. By using the equation of motion, we can write

$$\begin{aligned} N\left[\frac{1}{2} F_{ij} \partial_\mu X^i \partial^\mu X^j\right] &= \frac{1}{2} \left(F_{ij} + \sum_{L,\nu} (\alpha')^L \frac{1}{\epsilon^\nu} P_{ij}^{(L,\nu)km} F_{km} \right) \partial_\mu X^i \partial^\mu X^j \\ &- \frac{\alpha'}{4} \bar{R}^{(2)} \left(\sum_{L,\nu} (\alpha')^L \frac{1}{\epsilon^\nu} K^{(L,\nu)}(F) \cdot \Phi + \sum_{L,\nu} (\alpha')^{L-1} \frac{1}{\epsilon^\nu} \tilde{P}^{(L,\nu)km} F_{km} \right) \\ &- \mu^2 \sum_{L,\nu} (\alpha')^L \frac{1}{\epsilon^\nu} K^{(L,\nu)}(F) \cdot H \end{aligned} \quad (19)$$

Taking into account the contribution of the total derivative, the coefficients of the simple poles in Eq. (19) are given by the expressions

$$\begin{aligned}
 P_{ij}^{(L,1)km} F_{km} &= F \frac{\partial}{\partial g} T_{ij}^{(L,1)} + \nabla_i A_j^{(L)}(F) + \nabla_j A_i^{(L)}(F) \\
 K^{(L,1)}(F) \cdot H &= F \frac{\partial Z^{(L,1)}}{\partial g} \cdot H + \nabla^i H A_i^{(L)}(F) \\
 K^{(L,1)}(F) \cdot \Phi &= F \frac{\partial Z^{(L,1)}}{\partial g} \cdot \Phi + \nabla^i \Phi A_i^{(L)}(F) \\
 \sum_L (\alpha')^{L-1} \tilde{P}^{(L,1)km} F_{km} &= F \frac{\partial \Psi^{(1)}}{\partial g}
 \end{aligned} \tag{20}$$

The scaling properties of this operator are given by the renormalization group equation

$$\begin{aligned}
 &\mathcal{D} N[\tfrac{1}{2} F_{ij} \partial_\mu X^i \partial^\mu X^j] + N[\tfrac{1}{2} \gamma_{ij}^F \partial_\mu X^i \partial^\mu X^j] \\
 &- \frac{\alpha'}{4} \bar{R}^{(2)} N[\gamma^{FH}] - \mu^2 N[\gamma^{FH}] = 0
 \end{aligned} \tag{21}$$

The anomalous dimension operators can be expressed in terms of the simple poles of Eq. (20) as

$$\begin{aligned}
 \gamma_{ij}^F &= - \sum_L (\alpha')^L L P_{ij}^{(L,1)km} F_{km} = F \frac{\partial}{\partial g} \beta_{ij} - \sum_L (\alpha')^L L (\nabla_i A_j^{(L)}(F) + \nabla_j A_i^{(L)}(F)) \\
 \gamma^{FH} &= - \sum_L (\alpha')^L L K^{(L,1)}(F) \cdot H = F \frac{\partial}{\partial g} \gamma(H) - \sum_L (\alpha')^L L A_i^{(L)}(F) \nabla^i H \\
 \gamma^{F\Phi} &= - \sum_L (\alpha')^L L K^{(L,1)}(F) \cdot \Phi - \sum_L (\alpha')^{L-1} L \tilde{P}^{(L,1)km} F_{km} \\
 &= F \frac{\partial}{\partial g} \gamma(\Phi) - \sum_L (\alpha')^L L A_i^{(L)}(F) \nabla^i \Phi + F \frac{\partial}{\partial g} \theta(g)
 \end{aligned} \tag{22}$$

The coefficients of the higher poles in Eq. (19) are determined by a generalization of Eqs. (11) and (12). For the scalar function we easily get

$$\mathcal{D} N[F(X)] + N[\gamma(F)] = 0 \quad (23)$$

where γ is the differential operator defined in Eq. (9). We report the values of $A_i^{(L)}(F)$ and $P_{ij}^{(L,1)km} F_{km}$ up to two loops⁷⁾

$$A_i^{(1)}(F) = \frac{1}{2} \left(\nabla^k F_{ki} - \frac{1}{2} \nabla_i F^k_k \right)$$

$$A_i^{(2)}(F) = \frac{1}{4} R_i^{abc} \nabla_c F_{ab} \quad (24)$$

$$P_{ij}^{(1,1)km} F_{km} = \frac{1}{2} \Delta_L F_{ij} \quad (25)$$

where Δ_L is the Lichnerowicz Laplacian

$$\Delta_L F_{ij} = \nabla^2 F_{ij} - R_{is} F_j^s - R_{js} F_i^s - 2 R_{iskj} F^{sk}$$

and

$$P_{ij}^{(2,1)km} F_{km} = -\frac{1}{4} \left[R_i^{abc} \left(\nabla_a \nabla_b F_{cj} - \frac{1}{2} R_j^d{}_{bc} F_{da} \right) \right. \\ \left. - \nabla_i R_j^{abc} \nabla_c F_{ab} + i \leftrightarrow j \right] \quad (26)$$

The above construction shows that besides the flat-space renormalizations, the only extra information we need is the value of $\phi(g)$ in order to renormalize the action and the metric-like operators in the curved world sheet.

3. - THE WEYL ANOMALY

The standard definition of the trace of the stress-energy tensor, the Weyl anomaly, is

$$T^\mu_\mu = \frac{2}{\sqrt{\gamma}} \gamma^{\mu\nu} \frac{\delta S}{\delta \gamma^{\mu\nu}} = \frac{1}{2\pi\alpha'} \left(-\epsilon \frac{1}{2} g_{ij}^\beta \partial_\mu X^i \partial^\mu X^j + \epsilon \frac{\alpha'}{4} \bar{R}^{(2)} \Phi_B - \frac{\alpha'}{2} \square \Phi_B \right) \quad (27)$$

Our choice for $\bar{R}^{(2)}$ avoids a tedious factor $1+\epsilon$ in the Laplacian term¹¹⁾. The action principle guarantees that T^μ_μ is a finite operator. From Eq. (27) it is evident that the ϕ function plays a double role. It is needed to renormalize the coupling with the two-dimensional curvature and at the same time it represents the needed term to define an improved tensor in flat space²⁰⁾. From this observation it is clear that only flat-space calculations are needed to determine ϕ .

We first consider the integrated form of the trace connected to the scale anomaly of the model

$$\begin{aligned} \int dV \, 2\pi\alpha' T^\mu_\mu = & \int dV \left[-\epsilon N \left[\frac{1}{2} g_{ij}^\beta \partial_\mu X^i \partial^\mu X^j \right] + N \left[\frac{1}{2} \beta_{ij} \partial_\mu X^i \partial^\mu X^j \right] \right. \\ & + \epsilon \frac{\alpha'}{4} \bar{R}^{(2)} N[\Phi] - \frac{\alpha'}{4} \bar{R}^{(2)} N[\tilde{\gamma}(\Phi)] \\ & \left. - \mu^2 \epsilon N[H] - \mu^2 N[2H + \gamma(H)] \right] \end{aligned} \quad (28)$$

It is easy to verify Eq. (28) by expressing the normal products in terms of bare quantities and using the consistency relations¹¹⁾⁻¹³⁾. To define the trace of the stress-energy tensor we follow the same argument we used for the renormalization of the metric-like operator. We can write the general expression suggested by power counting

$$2\pi\alpha'T_\mu^\mu = \mu^\varepsilon \left\{ -\varepsilon N\left[\frac{1}{2}\partial_{ij}\partial_\mu X^i\partial^\mu X^j\right] + \right. \\ \left. + \varepsilon \frac{\alpha'}{4} \bar{R}^{(2)} N[\Phi] - \mu^2 \varepsilon N[H] \right. \\ \left. + N\left[\frac{1}{2}\beta_{ij}^* \partial_\mu X^i\partial^\mu X^j\right] + \frac{\alpha'}{8} \bar{R}^{(2)} N[\beta_\phi^*] + \frac{\mu^2}{2} N[\beta_H^*] \right\} \quad (29)$$

where

$$\beta_{ij}^* = \beta_{ij} - \alpha' \nabla_i \nabla_j \Phi + \nabla_i \mathcal{S}_j + \nabla_j \mathcal{S}_i \quad (30)$$

$$\beta_\phi^* = -2\gamma(\Phi) - 2\theta(g) + \alpha' \nabla_i \Phi \nabla^i \Phi - 2\nabla^i \Phi \mathcal{S}_i \quad (31)$$

$$\beta_H^* = -2\gamma(H) - 4H + \alpha' \nabla_i \Phi \nabla^i H - 2\nabla^i H \mathcal{S}_i \quad (32)$$

The terms proportional to $\mathcal{S}_i$ are due to a total derivative term possibly needed to define the improved stress-energy tensor in flat space-time. The crucial point for the determination of $\theta(g)$ in Eq. (31) is to note that T_μ^μ is an "observable", as is clear by its expression in terms of bare quantities. This property implies that

$$\mathcal{D}T_\mu^\mu = 0 \quad (33)$$

and this equation gives very useful consistency relations between the quantities defined in Eqs. (30)-(32). These relations are obtained by applying the $\mathcal{D}$ operator, Eq. (7), to both sides of Eq. (29). Using the definitions of the anomalous dimensions in Eqs. (21) and (23), and by comparing the various orders in the ε -expansion, we get generalized consistency relations. We only quote those useful for the subsequent analysis:

$$\gamma_{ij}^{km} g_{km} - g \frac{\partial}{\partial g} \beta_{ij} + \left(1 - g \frac{\partial}{\partial g}\right) (\nabla_i S_j + \nabla_j S_i) = 0 \quad (34)$$

$$\gamma_{ij}^{km} \beta_{km}^* - \beta \frac{\partial}{\partial g} \beta_{ij}^* + \alpha' \nabla_i \nabla_j \tilde{\gamma}(\Phi) = 0 \quad (35)$$

$$\begin{aligned} \gamma^{\beta H} - \beta \frac{\partial}{\partial g} \gamma^*(H) - \gamma \left(\frac{\alpha'}{2} \nabla^i H \nabla_i \Phi - \nabla^i H S_i \right) \\ + \frac{\alpha'}{2} \nabla^i H \nabla_i \tilde{\gamma}(\Phi) + \nabla^i \gamma(H) \cdot \left(\frac{\alpha'}{2} \nabla_i \Phi - S_i \right) = 0 \end{aligned} \quad (36)$$

Equation (34) relates S_i to the renormalization of the operator $N[g_{ij} \partial_\mu X^i \partial^\mu X^j]$ as it implies that

$$S_i = \sum_L (\alpha')^L A_i^{(L)}(g) \quad (37)$$

and from the expressions (24) we conclude that

$$S_i = O((\alpha')^3) \quad (38)$$

Let us now show how to extract $\theta(g)$ from Eqs. (35) and (36) using a recursive procedure. From Eq. (35), it follows that

$$\begin{aligned} \alpha' \nabla_i \nabla_j \theta(g) = \sum_L (\alpha')^L \left(\nabla_i A_j^{(L)km} \left(\beta_{km} + \nabla_k S_m + \nabla_m S_k \right) + i \leftrightarrow j \right) \\ + \nabla_i \left(\beta \frac{\partial S_j}{\partial g} - \beta_{jp} S^p \right) + i \leftrightarrow j \end{aligned} \quad (39)$$

and from Eq. (36) we obtain

$$\begin{aligned} \frac{\alpha'}{2} \nabla_i \theta(g) = \sum_L (\alpha')^L L A_i^{(L)km} (\beta_{km} + \nabla_k s'_m + \nabla_m s'_k) \\ + \beta \frac{\partial s'_i}{\partial g} - \beta_{ip} s'^p \end{aligned} \quad (40)$$

In this derivation we have used the definition of the Lie derivatives of tensors. A necessary consistency check is that Eq. (40) is a solution of Eq. (39). Using the known two-loop calculations in flat space-time, i.e., Eqs. (14), (24) and (38), we obtain the value of $\theta(g)$ up to the three-loop level

$$\begin{aligned} \nabla_i \theta(g) = \alpha' 2 A_i^{(1)km} R_{km} \\ + (\alpha')^2 \left(A_i^{(1)km} R_{kab c} R_m^{abc} + 4 A_i^{(2)km} R_{km} \right) \end{aligned} \quad (41)$$

Now $A_i^{(1)km} R_{km} = 0$, so $\theta(g)$ starts at the three-loop level and is given by

$$\nabla_i \theta(g) = (\alpha')^2 \nabla_i \left(-\frac{1}{8} R^{abcd} R_{abcd} \right) \quad (42)$$

The final expression for β_ϕ^* is

$$\beta_\phi^* = \frac{2}{3} (D-26) + \alpha' \left(\nabla^2 \Phi + (\nabla \Phi)^2 \right) + \frac{\alpha'^2}{4} R_{abcd} R^{abcd} + O(\alpha'^3) \quad (43)$$

From this expression we get the dilaton equation of motion (for $D = 26$)

$$\nabla^2 \Phi + (\nabla \Phi)^2 = -\frac{\alpha'}{4} R_{abcd} R^{abcd} \quad (44)$$

that is in agreement with the conjecture of Ref. 3). The value of $\theta(g)$ is in agreement with a calculation in curved space-time quoted in Ref. 7). Moreover, we have verified with an explicit three-loop calculation that the same result is obtained by requiring the finiteness of the improved tensor in flat space-time.

We conclude this section with a comment on the definitions of the β functions in the literature. Our results are written in terms of local operators, while in Refs. 2) and 3) the expectation value of $\langle T_\mu^\mu \rangle$ is quoted as β_ϕ . In our opinion this is quite unnatural as $\langle T_\mu^\mu \rangle$ is necessarily non-local, and it cannot provide a consistent definition of the renormalization group coefficients.

4. - CONFORMAL INVARIANCE AND CONSISTENCY CONDITIONS

As we have mentioned in the Introduction, the vanishing of the β functions Eqs. (30)-(32) defines the conformal invariant theory and provides the equations of motion for the background fields of the string. It is crucial to show that these equations are compatible. The linear equation for the tachion field does not raise any problem, so we devote our attention to the graviton and dilaton equations of motion. With the definition

$$V_i = \frac{\alpha'}{2} \nabla_i \Phi - S_i \quad (45)$$

we rewrite (30) and (31) in the form

$$\beta_{ij}^* = \beta_{ij} - \nabla_i V_j - \nabla_j V_i \quad (46)$$

$$\beta_\phi^* = -2 \left(\tilde{\gamma}(\phi) - V^i \nabla_i \phi \right) \quad (47)$$

To show the compatibility of (46) and (47), we use Eqs. (35), (36), (39) and (40) that relate β_ϕ^* to β_{ij}^* . It is simple to show that from Eqs. (35) and (39) we obtain

$$\begin{aligned} \alpha' \nabla_i \nabla_j \beta_\phi^* = -2 \left\{ \sum_L (\alpha')^L L \left(\nabla_i A_j^{(L)}(\beta^*) + i \leftrightarrow j \right) \right. \\ \left. + \nabla_i \left(\beta_j^{*S} V_S - \beta^* \frac{\partial V_j}{\partial g} \right) + i \leftrightarrow j \right\} \end{aligned} \quad (48)$$

and from Eqs. (36) and (40)

$$\alpha' \nabla_i \beta_\phi^* = -4 \left\{ \sum_L (\alpha')^L L A_i^{(L)}(\beta^*) + \beta_{is}^* V^s - \beta^* \frac{\partial V_i}{\partial g} \right\} \quad (49)$$

We again obtain a consistency check, as (49) is a solution of (48). We note that the expression for $\nabla_i \beta_\phi^*$ is linear in β_{ij}^* and we finally obtain the consistency relation

$$\nabla_i \beta_\phi^* = 0 \quad \text{if} \quad \beta_{ij}^* = 0 \quad (50)$$

The consequence of Eq. (50) is very simple as it means that at the fixed ultra-violet point $\beta_{ij}^* = 0$ the Weyl anomaly is a c-number. When this c-number vanishes, the theory is conformally invariant.

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