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EXACT S-MATRICES FOR ANOMALY-FREE NON-LINEAR σ MODELS
ON SYMMETRIC SPACES

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A B S T R A C T

We compute explicitly the two-body S-matrix for the five classical series of non-linear σ models on symmetric spaces that are known to admit an anomaly-free quantum non-local charge: SU(N) (A), SO(N) (BD), Sp(N) (C), SU(N)/SO(N) (AI) and SU(2N)/Sp(N) (AII). The results for the last two series are new.

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Two-dimensional non-linear σ models, or chiral models, are known to be classically integrable whenever the fields take values in a Riemannian symmetric space $M = G/H$ ¹⁾⁻³⁾. Moreover, it is known that integrability survives in the quantized theory, through existence of a conserved quantum non-local charge, if either H is simple (when appropriate "minimal" choices for the symmetry group G and the stability group H are made)⁴⁾ or - at least in certain families of models such as the $\mathbb{CP}^{N-1}$ models⁵⁾ or the Grassmannian models⁶⁾ - the bosonic fields are coupled to fermionic fields in a minimal or supersymmetric way. [For a general analysis of this second possibility, see Ref. 7).] If one of these conditions for the absence of an anomaly is met, the existence of a conserved quantum non-local charge leads to factorization of the S-matrix into two-body amplitudes, and in addition it imposes strong constraints on the latter. Depending on the concrete model, however, these constraints may or may not suffice to determine the S-matrix completely.

In the present article, we concentrate on pure (i.e., purely bosonic) chiral models defined on irreducible Riemannian symmetric spaces $M = G/H$ of the compact type. [Technically speaking, we assume, as in Ref. 4), that G is compact and connected and acts almost effectively on $M = G/H$.] These spaces fall into two types, according to whether G is simple (type I) or is the direct product $\tilde{G} \times \tilde{G}$ of two copies of the same simple group $\tilde{G}$ (type II), both types being completely classified [see Ref. 8), pp. 516 and 518]. In particular, the chiral model on M is anomaly-free whenever M is either of type II, i.e., any compact connected simple Lie group $\tilde{G}$, or one of the following spaces of type I:

- AI : $M = SU(N)/SO(N)$,
- AII : $M = SU(2N)/Sp(N)$,
- AIII : $M = SU(2)/U(1) = \mathbb{CP}^1$,
- BDI : $M = SO(N)/SO(N-1) = S^{N-1}$
- EI : $M = E_6/Sp(4)$
- EIV : $M = E_6/F_4$
- EV : $M = E_7/SU(8)$
- EVIII : $M = E_8/SO(16)$
- FII : $M = F_4/SO(9)$

We shall disregard the type I spaces $\mathbb{CP}^1 = S^2$ and S^{N-1} , whose S-matrices are well known^{9),10)}, and are therefore left with the following five series of classical ($\equiv$ non-exceptional) spaces:

- A (type II) : $M = SU(N)$
 BD (type II) : $M = SO(N)$
 C (type II) : $M = Sp(N)$
 AI (type I) : $M = SU(N)/SO(N)$
 AII (type I) : $M = SU(2N)/Sp(N)$

The S-matrices for the type II spaces (classical groups) have been determined in Refs. 11) and 12) and, by a different technique (Bethe ansatz), in Refs. 13) and 14). Our principal objective here is to extend the analysis of Refs. 11) and 12) to the remaining two series - where the technique of Refs. 13) and 14) does not appear to be applicable.

In order to unify and streamline the presentation as much as possible, we shall employ a manifestly gauge invariant formulation, which is based on the fact that in each of the five cases under consideration here, the target manifold in question can be represented as a (totally geodesic) submanifold of a special unitary group, defined by an appropriate constraint. More precisely^{*)}, consider the involutive automorphism σ_I of $SU(N)$ given by

$$\sigma_I(q) = q^* \quad \text{for } q \in SU(N), \quad (1.I)$$

and the involutive automorphism σ_{II} of $SU(2N)$ given by

$$\sigma_{II}(q) = -Jq^*J \quad \text{for } q \in SU(2N), \quad (1.II)$$

where J is the matrix

$$J = \begin{pmatrix} 0 & +1_N \\ -1_N & 0 \end{pmatrix} \in SU(2N). \quad (2.II)$$

Both of these can be written in the common form

$$\sigma(q) = Iq^*I^{-1} \quad \text{for } q \in SU(N) \text{ resp. } SU(2N), \quad (1)$$

where I is the matrix

^{*)} We follow the convention that $*$, τ and $+$ stand for complex conjugation, transposition and Hermitian conjugation of matrices, respectively - even when the matrix entries are operators in a Hilbert space, for which "conjugate" means "adjoint".

$$\begin{aligned} I &= 1_N \in SU(N) \text{ for the cases BD and AI, } I^{-1} = I^T = +I = +I^* \\ I &= J \in SU(2N) \text{ for the cases C and AII, } I^{-1} = I^T = -I = -I^* \end{aligned} \quad (2)$$

Then, obviously,

$$\sigma(q) = q \iff q \in SO(N) \text{ resp. } Sp(N), \quad (3)$$

or more specifically,

$$\sigma_I(q) = q \iff q \in SO(N) \quad \text{for } q \in SU(N). \quad (3.I)$$

$$\sigma_{II}(q) = q \iff q \in Sp(N) \quad \text{for } q \in SU(2N). \quad (3.II)$$

Similarly,

$$\sigma(q) = q^{-1} \iff q \in SU(N)/SO(N) \text{ resp. } SU(2N)/Sp(N), \quad (4)$$

or more specifically,

$$\sigma_I(q) = q^{-1} \iff q \in SU(N)/SO(N) \quad \text{for } q \in SU(N). \quad (4.I)$$

$$\sigma_{II}(q) = q^{-1} \iff q \in SU(2N)/Sp(N) \quad \text{for } q \in SU(2N). \quad (4.II)$$

Here, we have identified the coset spaces $SU(N)/SO(N)$ and $SU(2N)/Sp(N)$ with their images under the Cartan immersion; see Ref. 8) p. 347 and Ref. 2) for details. Finally, we extend our convention (2) to the remaining case by demanding

$$I = 1_N \in SU(N) \text{ for the case A, } I^{-1} = I^T = +I = +I^*. \quad (5)$$

The merit of introducing the matrix I is that this will allow us to give common formulae for all five cases, and when this is not possible, at least to deal simultaneously with the "subgroup cases" BD and C on the one hand and with the "quotient space cases" or "coset space cases" AI and AII on the other hand: we shall in such situations indicate the type of model to which a given formula refers by adding a letter A (for the case A) or S (for the subgroup cases) or Q (for the quotient space cases). Moreover, certain formulae will contain $\pm$ or $\mp$ signs: then the general convention is that the upper sign will refer to the cases

A and/or BD and AI (where I is symmetric) while the lower sign will refer to the cases C and AII (where I is antisymmetric).

With these conventions, the basic field of the model is a matrix field g which, classically, satisfies the unitarity and determinant constraints

$$g^{-1} = g^+ , \quad (6)$$

$$\det g = 1 , \quad (7)$$

plus additional constraints depending on the model at hand:

$$g = \sigma(g) \equiv I g^* I^+ \quad \text{or} \quad g^T = I g^+ I^+ . \quad (8.S)$$

$$g^+ = \sigma(g) \equiv I g^* I^+ \quad \text{or} \quad g^T = I g I^+ . \quad (8.Q)$$

In particular, (8.S) means that for the subgroup cases, the field g is invariant under charge conjugation $g \rightarrow g^C$, which is defined by

$$g^C = I g^* I^+ . \quad (9)$$

The action is the standard one:

$$S = \frac{1}{2} \int d^2x \quad \text{trace} (g^+ \partial^\mu g g^+ \partial_\mu g) . \quad (10)$$

It admits various global internal symmetries, namely continuous ones, described by the connected Lie group

$$G = SU(N)_L \times SU(N)_R , \quad (11.A)$$

$$G = SO(N)_L \times SO(N)_R , \quad (11.BD)$$

$$G = Sp(N)_L \times Sp(N)_R , \quad (11.C)$$

$$G = SU(N) , \quad (11.AI)$$

$$G = SU(2N) , \quad (11.AII)$$

which acts according to

$$g \rightarrow g_L g g_R^+ , \quad (12.A)$$

$$g \rightarrow g_L g g_R^+ , \quad (12.S)$$

$$g \rightarrow \sigma(g_0) g g_0^+ = I g_0^* I^+ g g_0^+ , \quad (12.Q)$$

as well as discrete ones, namely charge conjugation $g \rightarrow g^c = I g^* I^+$ and the transformation $g \rightarrow g^+$, or equivalently, the transformation $g \rightarrow I g^T I^+$. As usual, the continuous global symmetries lead to a Noether current J_μ taking values in the Lie algebra $\mathfrak{g}$ of G :

$$J_\mu = (J_\mu^L, J_\mu^R) , \quad J_\mu^L = -\partial_\mu g g^+ , \quad J_\mu^R = +g^+ \partial_\mu g . \quad (13.A)$$

$$J_\mu = (J_\mu^L, J_\mu^R) , \quad J_\mu^L = -\partial_\mu g g^+ , \quad J_\mu^R = +g^+ \partial_\mu g . \quad (13.S)$$

$$J_\mu = g^+ \partial_\mu g - \sigma(\partial_\mu g g^+) . \quad (13.Q)$$

In the group manifold cases, the currents J_μ^L and J_μ^R are exchanged under the discrete symmetry $g \rightarrow g^+$, while in the coset space cases, the expression for J_μ can be simplified due to the fact that the constraints (6) and (8.Q) imply $\sigma(\partial_\mu g g^+) = -g^+ \partial_\mu g$. Therefore, it is sufficient to work with the current

$$j_\mu = g^+ \partial_\mu g \quad (14)$$

whose conservation

$$\partial^\mu j_\mu = 0 \quad (15)$$

is completely equivalent to the equations of motion derived from (10), and which, classically, also satisfies the flatness condition

$$\partial_\mu j_\nu - \partial_\nu j_\mu + [j_\mu, j_\nu] = 0 , \quad (16)$$

thus giving rise not only to the standard conserved classical charge

$$Q^{(0)} = \int dy j_0(t, y), \quad (17)$$

but also to a conserved classical non-local charge, namely

$$Q^{(1)} = \int dy_1 dy_2 \epsilon(y_1 - y_2) j_0(t, y_1) j_0(t, y_2) - \int dy j_1(t, y). \quad (18)$$

For the quantum theory, we note that all non-linear relations appearing in the classical context must be handled with care, while linear relations do not cause any problems: they can simply be required to hold as operator identities. Thus the non-linear conditions (6) and (7) must be reformulated as constraints

$$\mathcal{N}[q^+ q] - \text{const.} 1 = \mathcal{N}[q q^+] \quad (19)$$

$$\mathcal{N}[\det q] = \text{const.} \quad (20)$$

on normal products, with renormalization scheme dependent constants, while the additional linear constraints (8) remain as they stand. Similarly, interpreting (14) in the sense of normal products,

$$j_\mu = \mathcal{N}[q^+ \partial_\mu q], \quad (21)$$

the conservation law (15) remains as it stands, while the flatness condition (16) is replaced by a short-distance expansion for the (matrix) commutator of two currents at nearby (spacelike separated) points

$$j_\mu(x+\epsilon) j_\nu(x-\epsilon) - j_\nu(x-\epsilon) j_\mu(x+\epsilon) \sim C_{\mu\nu}^g(\epsilon) j_g(x) + D_{\mu\nu}^{\sigma g}(\epsilon) (\partial_\sigma j_g)(x), \quad (22)$$

where $\sim$ means equality up to terms that go to zero as $\epsilon \rightarrow 0$ ⁴⁾, and the coefficients can be determined completely^{7), 15), 16)} in terms of a single function of $-\epsilon^2$. In the usual way, this gives rise not only to the standard conserved quantum charge

$$Q^{(0)} = \int dy j_0(t, y), \quad (23)$$

but also to a conserved quantum non-local charge, namely

$$Q^{(1)} = \lim_{\delta \rightarrow 0} Q_{\delta}^{(1)}(t), \quad (24)$$

with

$$Q_{\delta}^{(1)}(t) = \int_{|y_1 - y_2| \geq \delta} dy_1 dy_2 \varepsilon(y_1 - y_2) j_0(t, y_1) j_0(t, y_2) - Z(\delta) \int dy j_1(t, y), \quad (25)$$

where $Z(\delta)$ is a function that diverges logarithmically as $\delta \rightarrow 0$ (4), (7), (15), (16). Moreover, if T is the generator of Lorentz transformations, we have the commutation relation

$$[T, Q^{(1)}] = -\frac{N'}{2\pi} Q^{(0)}, \quad (26)$$

where

$$N' = N. \quad (27.A)$$

$$N' = N - 2. \quad (27.BD)$$

$$N' = 2N + 2. \quad (27.C)$$

$$N' = N. \quad (27.AI)$$

$$N' = 2N. \quad (27.AII)$$

[The proof of an analogous relation, with an appropriate constant, for the spherical models can be found in Ref. 16); we shall not give any details here.]

The next step is to introduce asymptotic free fields g_{ex} , where "ex" stands for "in" or "out": these are matrix fields with Fourier representations

$$g_{ex}(t, y) = \int d\mu(q) \{ b_{ex}(q) \exp(-iE(q)t + iqy) + d_{ex}^*(q) \exp(+iE(q)t - iqy) \}, \quad (28)$$

where the integration measure is

$$d\mu(q) = \frac{dq}{2\pi 2E(q)} = \frac{d\theta}{4\pi}, \quad E(q) = \sqrt{q^2 + m^2} = m \cosh \theta, \quad q = m \sinh \theta, \quad (29)$$

with $m > 0$ the lowest (dynamically generated) mass in this sector of the model, and the creation and annihilation operators appearing in (28) are supposed to satisfy the following canonical commutation relations:

$$\begin{aligned} [t_{ex}^{ij}(q_1), t_{ex}^{*kl}(q_2)] &= [d_{ex}^{ij}(q_1), d_{ex}^{*kl}(q_2)] = 2\pi 2E \tau^{ij,kl} \delta(q_1 - q_2), \\ [t_{ex}^{ij}(q_1), d_{ex}^{*kl}(q_2)] &= [d_{ex}^{ij}(q_1), t_{ex}^{*kl}(q_2)] = 2\pi 2E s^{ij,kl} \delta(q_1 - q_2), \end{aligned} \quad (30)$$

with

$$\tau^{ij,kl} = \delta^{ik} \delta^{jl}, \quad s^{ij,kl} = 0 \quad (31.A)$$

$$\tau^{ij,kl} = \delta^{ik} \delta^{jl}, \quad s^{ij,kl} = I^{ik} I^{jl} \quad (31.S)$$

$$\tau^{ij,kl} = \delta^{ik} \delta^{jl} + I^{il} I^{jk}, \quad s^{ij,kl} = 0 \quad (31.Q)$$

(All other commutators are supposed to vanish.) Moreover, asymptotic states are defined by

$$| \theta k l \rangle_{ex} = (I t_{ex}^*(q(\theta)))^{kl} | 0 \rangle, \quad | \bar{\theta} \bar{k} \bar{l} \rangle_{ex} = (d_{ex}^*(q(\theta)) I)^{kl} | 0 \rangle. \quad (32)$$

(These are the asymptotic one-body states; asymptotic two-body states are defined correspondingly^{*)}.) This fixes the structure of the asymptotic free theory, since the commutation relations (30) and (31) contain, among other things, the rules for defining composite operators by means of Wick ordering $\therefore$. Examples of such operators are provided by the asymptotic free currents $j_{\mu,ex}$: these are traceless anti-Hermitian matrix fields given by

$$\tilde{j}_{\mu,ex} = \frac{1}{2} (: q_{ex}^+ \partial_{\mu} q_{ex} : - : \partial_{\mu} q_{ex}^+ q_{ex} :), \quad (33)$$

and

$$j_{\mu,ex} = \tilde{j}_{\mu,ex} - \frac{1}{N} \text{trace } \tilde{j}_{\mu,ex}, \quad (34)$$

^{*)} The term "body" is synonymous for "particle or antiparticle".

where N' is given by (27). [For the subgroup cases, the value of N' in (34) is irrelevant since the constraint (35.S) below implies that $\tilde{j}_{\mu,ex}$ is already traceless.] Notice also that the g_{ex} do not satisfy the free field analogue of the non-linear constraints (19) and (20): this was the reason why we had to make the $j_{\mu,ex}$ traceless and anti-Hermitian by hand. (Other consequences will emerge later on.) The g_{ex} can and will, however, be made to satisfy the free field analogue of the additional linear constraints (8): this is achieved by imposing the following simple identities between the various creation and annihilation operators:

$$d_{ex}^{(*)}(q) I = I t_{ex}^{(*)}(q) . \quad (35.S)$$

$$t_{ex}^{(*)T}(q) I = I t_{ex}^{(*)}(q) , \quad d_{ex}^{(*)T}(q) I = I d_{ex}^{(*)}(q) . \quad (35.Q)$$

For asymptotic states, this means that

$$|\bar{\theta} \bar{k} \bar{l}\rangle_{ex} = |\theta k l\rangle_{ex} . \quad (36.S)$$

$$|\theta k l\rangle_{ex} = \pm |\theta \bar{k} \bar{l}\rangle_{ex} , \quad |\bar{\theta} \bar{k} \bar{l}\rangle_{ex} = \pm |\bar{\theta} \bar{l} \bar{k}\rangle_{ex} \quad (36.Q)$$

Another condition is that the g_{ex} should have the same transformation laws under global symmetries as the interacting field g : this is achieved by imposing appropriate transformation laws on the various creation and annihilation operators. More precisely, (11) and (12), with g replaced by g_{ex} , lead to

$$t_{ex}^{(*)}(q) \rightarrow g_L^{(*)} t_{ex}^{(*)}(q) g_R^{(*)+} , \quad d_{ex}^{(*)}(q) \rightarrow g_L^{(*)*} d_{ex}^{(*)}(q) g_R^{(*)T} . \quad (37.A)$$

$$t_{ex}^{(*)}(q) \rightarrow g_L^{(*)} t_{ex}^{(*)}(q) g_R^{(*)+} , \quad d_{ex}^{(*)}(q) \rightarrow g_L^{(*)*} d_{ex}^{(*)}(q) g_R^{(*)T} . \quad (37.S)$$

$$t_{ex}^{(*)}(q) \rightarrow I g_0^{(*)*} I^+ t_{ex}^{(*)}(q) g_0^{(*)+} , \quad d_{ex}^{(*)}(q) \rightarrow I g_0^{(*)} I^+ d_{ex}^{(*)}(q) g_0^{(*)T} . \quad (37.Q)$$

For asymptotic states, this means that

$$|\theta k l\rangle_{ex} \rightarrow g_L^{*ku} g_R^{lu} |\theta u u\rangle_{ex} , \quad |\bar{\theta} \bar{k} \bar{l}\rangle_{ex} \rightarrow g_L^{ku} g_R^{*lu} |\bar{\theta} \bar{u} \bar{u}\rangle_{ex} . \quad (38.A)$$

$$|\theta kl\rangle_{ex} \rightarrow g_L^{km} g_R^{ln} |\theta mn\rangle_{ex} \quad (38.S)$$

$$|\theta kl\rangle_{ex} \rightarrow g_0^{km} g_0^{ln} |\theta mn\rangle_{ex} , \quad |\bar{\theta} \bar{k} \bar{l}\rangle_{ex} \rightarrow g_0^{c km} g_0^{c ln} |\bar{\theta} \bar{m} \bar{n}\rangle_{ex} \quad (38.Q)$$

Similarly, under charge conjugation $g_{ex} \rightarrow g_{ex}^c = I g_{ex}^* I^+$,

$$t_{ex}^{(*)}(q) \rightarrow I d_{ex}^{(*)}(q) I^+ , \quad d_{ex}^{(*)}(q) \rightarrow I t_{ex}^{(*)}(q) I^+ , \quad (39)$$

so

$$|\theta kl\rangle_{ex} \rightarrow |\bar{\theta} \bar{k} \bar{l}\rangle_{ex} , \quad |\bar{\theta} \bar{k} \bar{l}\rangle_{ex} \rightarrow |\theta kl\rangle_{ex} \quad (40)$$

Finally, under the discrete symmetry $g_{ex} \rightarrow I g_{ex}^T I^+$,

$$t_{ex}^{(*)}(q) \rightarrow I t_{ex}^{(*)T}(q) I^+ , \quad d_{ex}^{(*)}(q) \rightarrow I d_{ex}^{(*)T}(q) I^+ , \quad (41)$$

so

$$|\theta kl\rangle_{ex} \rightarrow \pm |\theta lk\rangle_{ex} , \quad |\bar{\theta} \bar{k} \bar{l}\rangle_{ex} \rightarrow \pm |\bar{\theta} \bar{l} \bar{k}\rangle_{ex} \quad (42)$$

For consistency, we have to require, of course, that the additional linear constraints (35) and the various symmetries (37), (39) and (41) are compatible with the commutation relations (30): this imposes severe restrictions on the group-theoretical structure of the coefficients r and s . It is straightforward to verify, however, that the choice made in (31) does satisfy all these restrictive conditions.

Similar consistency considerations can be used to write down the general ansatz for the elastic two-body S-matrix. Namely, charge conjugation invariance means that

$$\begin{aligned} {}_{out} \langle \bar{\theta}'_1 \bar{k}'_1 \bar{l}'_1, \bar{\theta}'_2 \bar{k}'_2 \bar{l}'_2 | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{in} &= {}_{out} \langle \bar{\theta}'_1 \bar{k}'_1 \bar{l}'_1, \bar{\theta}'_2 \bar{k}'_2 \bar{l}'_2 | \bar{\theta}_1 \bar{k}_1 \bar{l}_1, \bar{\theta}_2 \bar{k}_2 \bar{l}_2 \rangle_{in} \\ {}_{out} \langle \bar{\theta}'_1 \bar{k}'_1 \bar{l}'_1, \bar{\theta}'_2 \bar{k}'_2 \bar{l}'_2 | \bar{\theta}_1 \bar{k}_1 \bar{l}_1, \bar{\theta}_2 \bar{k}_2 \bar{l}_2 \rangle_{in} &= {}_{out} \langle \bar{\theta}'_1 \bar{k}'_1 \bar{l}'_1, \bar{\theta}'_2 \bar{k}'_2 \bar{l}'_2 | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{in} \end{aligned} \quad (43)$$

In addition, the particle-particle = antiparticle-antiparticle scattering amplitudes are symmetric under the exchanges $1 \leftrightarrow 2$ and $1' \leftrightarrow 2'$. The main

restriction, however, comes from G-invariance, which implies that among all possible contractions of indices, only certain combinations are allowed. More concretely, this works as follows.

For the case A, indices $k_1, k_2, \bar{k}_1, \bar{k}_2, k'_1, k'_2, \bar{k}'_1, \bar{k}'_2$ transform under $SU(N)_L$ while indices $\ell_1, \ell_2, \bar{\ell}_1, \bar{\ell}_2, \ell'_1, \ell'_2, \bar{\ell}'_1, \bar{\ell}'_2$ transform under $SU(N)_R$, so they cannot be contracted with each other. Moreover, the representations N and N^* of $SU(N)$ are inequivalent. This leads to

$$\begin{aligned} \text{out} \langle \bar{\theta}'_1 \bar{k}'_1 \bar{\ell}'_1, \bar{\theta}'_2 \bar{k}'_2 \bar{\ell}'_2 \mid \theta_1 k_1 \ell_1, \theta_2 k_2 \ell_2 \rangle_{in} \\ = +16\pi^2 \delta(\theta'_1 - \theta_1) \delta(\theta'_2 - \theta_2) \{ +u_1(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} + u_2(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} \\ + u_3(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} + u_4(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} \} \\ + 16\pi^2 \delta(\theta'_1 - \theta_2) \delta(\theta'_2 - \theta_1) \{ +u_4(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} + u_3(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} \\ + u_2(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} + u_1(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} \} \end{aligned} \quad (44.A)$$

$$\begin{aligned} \text{out} \langle \bar{\theta}'_1 \bar{k}'_1 \bar{\ell}'_1, \bar{\theta}'_2 \bar{k}'_2 \bar{\ell}'_2 \mid \bar{\theta}_1 \bar{k}_1 \bar{\ell}_1, \bar{\theta}_2 \bar{k}_2 \bar{\ell}_2 \rangle_{in} \\ = +16\pi^2 \delta(\theta'_1 - \theta_1) \delta(\theta'_2 - \theta_2) \{ +t_1(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} + t_2(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} \\ + t_3(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} + t_4(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} \} \\ + 16\pi^2 \delta(\theta'_1 - \theta_2) \delta(\theta'_2 - \theta_1) \{ +\tau_4(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} + \tau_3(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} \\ + \tau_2(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} + \tau_1(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} \} \end{aligned} \quad (45.A)$$

with $\theta = |\theta_1 - \theta_2|$; invariance under the discrete symmetry (41) and (42) is then guaranteed by the additional conditions

$$u_2 = u_3, \quad t_2 = t_3, \quad \tau_2 = \tau_3. \quad (46.A)$$

For the subgroup cases BD resp. C, indices k_1, k_2, k'_1, k'_2 transform under $SO(N)_L$ resp. $Sp(N)_L$ while indices $\lambda_1, \lambda_2, \lambda'_1, \lambda'_2$ transform under $SO(N)_R$ resp. $Sp(N)_R$, so they cannot be contracted with each other. However, the representations N and N^* of $SO(N)$ resp. $2N$ and $2N^*$ of $Sp(N)$ are equivalent, with the matrix I as the intertwining operator. This leads to

$$\begin{aligned}
 & \text{out} \langle \theta'_1 k'_1 \ell'_1, \theta'_2 k'_2 \ell'_2 \mid \theta_1 k_1 \ell_1, \theta_2 k_2 \ell_2 \rangle_{in} \\
 & = +16\pi^2 \delta(\theta'_1 - \theta_1) \delta(\theta'_2 - \theta_2) \{ +u_1(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} + u_2(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} \\
 & \quad + u_3(\theta) I^{k'_1 k'_2} I^{k_1 k_2} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} + u_4(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} \\
 & \quad + u_5(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} + u_6(\theta) I^{k'_1 k'_2} I^{k_1 k_2} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} \\
 & \quad + u_7(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} I^{\ell'_1 \ell'_2} I^{\ell_1 \ell_2} + u_8(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} I^{\ell'_1 \ell'_2} I^{\ell_1 \ell_2} \\
 & \quad + u_9(\theta) I^{k'_1 k'_2} I^{k_1 k_2} I^{\ell'_1 \ell'_2} I^{\ell_1 \ell_2} \} \\
 & + 16\pi^2 \delta(\theta'_1 - \theta_2) \delta(\theta'_2 - \theta_1) \{ +u_5(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} + u_4(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} \\
 & \quad \pm u_6(\theta) I^{k'_1 k'_2} I^{k_1 k_2} \delta^{\ell'_1 \ell_1} \delta^{\ell'_2 \ell_2} + u_2(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} \\
 & \quad + u_1(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} \pm u_3(\theta) I^{k'_1 k'_2} I^{k_1 k_2} \delta^{\ell'_1 \ell_2} \delta^{\ell'_2 \ell_1} \\
 & \quad \pm u_8(\theta) \delta^{k'_1 k_1} \delta^{k'_2 k_2} I^{\ell'_1 \ell'_2} I^{\ell_1 \ell_2} \pm u_7(\theta) \delta^{k'_1 k_2} \delta^{k'_2 k_1} I^{\ell'_1 \ell'_2} I^{\ell_1 \ell_2} \\
 & \quad + u_9(\theta) I^{k'_1 k'_2} I^{k_1 k_2} I^{\ell'_1 \ell'_2} I^{\ell_1 \ell_2} \} \\
 & \hspace{20em} (44.S)
 \end{aligned}$$

with $\theta = |\theta_1 - \theta_2|$; invariance under the discrete symmetry (41) and (42) is then guaranteed by the additional conditions

$$u_2 = u_4, \quad u_3 = u_7, \quad u_6 = u_8. \quad (46.S)$$

For the quotient space cases AI resp. AII, we must take into account the symmetry resp. antisymmetry under exchanges $k_1 \leftrightarrow \lambda_1, k_2 \leftrightarrow \lambda_2, \bar{k}_1 \leftrightarrow \bar{\lambda}_1, \bar{k}_2 \leftrightarrow \bar{\lambda}_2, k'_1 \leftrightarrow \lambda'_1, k'_2 \leftrightarrow \lambda'_2, \bar{k}'_1 \leftrightarrow \bar{\lambda}'_1, \bar{k}'_2 \leftrightarrow \bar{\lambda}'_2$. Moreover, the representations N' and N'^* of $SU(N')$, with $N' = N$ resp. $2N$, are inequivalent. This leads to

$$_{out} \langle \theta'_1 k'_1 l'_1, \theta'_2 k'_2 l'_2 | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{in}$$

$$\begin{aligned}
 &= +16\pi^2 \delta(\theta'_1 - \theta_1) \delta(\theta'_2 - \theta_2) \\
 &\quad \cdot \{ +u_1(\theta) (\delta^{k_1 k_1} \delta^{k_2 k_2} \delta^{l_1 l_1} \delta^{l_2 l_2} \pm k'_1 \leftrightarrow l'_1, k'_2 \leftrightarrow l'_2 \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2) \\
 &\quad + u_2(\theta) (\delta^{k_1 k_2} \delta^{k_2 k_1} \delta^{l_1 l_1} \delta^{l_2 l_2} \pm k'_1 \leftrightarrow l'_1, k'_2 \leftrightarrow l'_2 \text{ and } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2) \\
 &\quad + u_3(\theta) (\delta^{k_1 k_2} \delta^{k_2 k_1} \delta^{l_1 l_2} \delta^{l_2 l_1} \pm k'_1 \leftrightarrow l'_1, k'_2 \leftrightarrow l'_2 \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2) \} \\
 &+ 16\pi^2 \delta(\theta'_1 - \theta_2) \delta(\theta'_2 - \theta_1) \\
 &\quad \cdot \{ +u_3(\theta) (\delta^{k_1 k_1} \delta^{k_2 k_2} \delta^{l_1 l_1} \delta^{l_2 l_2} \pm k'_1 \leftrightarrow l'_1, k'_2 \leftrightarrow l'_2 \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2) \\
 &\quad + u_2(\theta) (\delta^{k_1 k_2} \delta^{k_2 k_1} \delta^{l_1 l_1} \delta^{l_2 l_2} \pm k'_1 \leftrightarrow l'_1, k'_2 \leftrightarrow l'_2 \text{ and } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2) \\
 &\quad + u_1(\theta) (\delta^{k_1 k_2} \delta^{k_2 k_1} \delta^{l_1 l_2} \delta^{l_2 l_1} \pm k'_1 \leftrightarrow l'_1, k'_2 \leftrightarrow l'_2 \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2) \}
 \end{aligned}
 \tag{44.Q}$$

$$_{out} \langle \bar{\theta}'_1 \bar{k}'_1 \bar{l}'_1, \bar{\theta}'_2 \bar{k}'_2 \bar{l}'_2 | \bar{\theta}_1 \bar{k}_1 \bar{l}_1, \bar{\theta}_2 \bar{k}_2 \bar{l}_2 \rangle_{in}$$

$$\begin{aligned}
 &= +16\pi^2 \delta(\theta'_1 - \theta_1) \delta(\theta'_2 - \theta_2) \\
 &\quad \cdot \{ +t_1(\theta) (\delta^{k_1 k_1} \delta^{k_2 k_2} \delta^{l_1 l_1} \delta^{l_2 l_2} \pm k'_1 \leftrightarrow l'_1, k'_2 \leftrightarrow l'_2 \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2) \\
 &\quad + t_2(\theta) (I^{k_1 k_2} I^{k_2 k_1} \delta^{l_1 l_1} \delta^{l_2 l_2} \pm k'_1 \leftrightarrow l'_1, k'_2 \leftrightarrow l'_2 \text{ and } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2) \\
 &\quad + t_3(\theta) (I^{k_1 k_2} I^{k_2 k_1} I^{l_1 l_2} I^{l_2 l_1} \pm k'_1 \leftrightarrow l'_1, k_1 \leftrightarrow l_1 \text{ or } k'_2 \leftrightarrow l'_2, k_2 \leftrightarrow l_2) \} \\
 &+ 16\pi^2 \delta(\theta'_1 - \theta_2) \delta(\theta'_2 - \theta_1) \\
 &\quad \cdot \{ +r_3(\theta) (\delta^{k_1 k_1} \delta^{k_2 k_2} \delta^{l_1 l_1} \delta^{l_2 l_2} \pm k'_1 \leftrightarrow l'_1, k'_2 \leftrightarrow l'_2 \text{ or } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2) \\
 &\quad + r_2(\theta) (I^{k_1 k_2} I^{k_2 k_1} \delta^{l_1 l_1} \delta^{l_2 l_2} \pm k'_1 \leftrightarrow l'_1, k'_2 \leftrightarrow l'_2 \text{ and } k_1 \leftrightarrow l_1, k_2 \leftrightarrow l_2) \\
 &\quad + r_1(\theta) (I^{k_1 k_2} I^{k_2 k_1} I^{l_1 l_2} I^{l_2 l_1} \pm k'_1 \leftrightarrow l'_1, k_1 \leftrightarrow l_1 \text{ or } k'_2 \leftrightarrow l'_2, k_2 \leftrightarrow l_2) \}
 \end{aligned}
 \tag{45.Q}$$

with $\theta = |\theta_1 - \theta_2|$; invariance under the discrete symmetry (41) and (42) is then automatic.

Further constraints on the as yet undetermined functions u_i , t_i and r_i appearing in the scattering amplitudes (44) and (45) follow from the existence of the conserved quantum non-local charge. To derive these constraints, we must

first compute the action of this charge on asymptotic one-body and two-body states, or equivalently, its expression in terms of asymptotic creation and annihilation operators. This was first carried out for the spherical models in Ref. 15), but the strategy used there can be equally well applied in all the cases under consideration here: it essentially amounts to repeating the relevant definitions with interacting fields replaced by asymptotic free fields. (This implies, of course, that the normal ordering prescription and, hence, all renormalization constants for the interacting theory must also be replaced by their counterparts for the free theory.) More specifically, apart from the standard conserved asymptotic quantum charges

$$Q_{\text{ex}}^{(0)} = \int dy \, j_{0,\text{ex}}(t, y) , \quad (47)$$

we define asymptotic quantum non-local charges

$$Q_{\text{ex}}^{(1)}(t) = \lim_{\delta \rightarrow 0} Q_{\text{ex},\delta}^{(1)}(t) , \quad (48)$$

with

$$Q_{\text{ex},\delta}^{(1)}(t) = \int_{|y_1 - y_2| \geq \delta} dy_1 dy_2 \, \varepsilon(y_1 - y_2) j_{0,\text{ex}}(t, y_1) j_{0,\text{ex}}(t, y_2) - Z_{\text{ex}}(\delta) \int dy \, j_{1,\text{ex}}(t, y) , \quad (49)$$

where

$$Z_{\text{ex}}(\delta) = -\frac{N'}{4\pi} \left(\ln(m\delta/2) + \gamma - 1 \right) , \quad (50)$$

γ is Euler's constant, and N' is given by (27). Notice that since the g_{ex} do not satisfy the free field analogue of the non-linear constraints (19) and (20), and the $j_{\mu,\text{ex}}$ therefore do not satisfy the free field analogue of the short-distance expansion (22), the limits in (48) are still explicitly time-dependent. We may, however, take the limits

$$Q_{\text{in}}^{(1)} = \lim_{t \rightarrow -\infty} Q_{\text{in}}^{(1)}(t) , \quad Q_{\text{out}}^{(1)} = \lim_{t \rightarrow +\infty} Q_{\text{out}}^{(1)}(t) , \quad (51)$$

and after a rather tedious calculation along the lines of Ref. 15), we arrive at the following expressions in terms of asymptotic creation and annihilation operators:

$$Q_{ex}^{(0)} = -\frac{i}{2} \int d\mu(q) : (t_{ex}^+(q) t_{ex}(q) - d_{ex}^T(q) d_{ex}^*(q)) - \frac{1}{N'} \text{trace} (t_{ex}^+(q) t_{ex}(q) - d_{ex}^T(q) d_{ex}^*(q)) : , \quad (52)$$

$$Q_{in}^{(1)} = \frac{N'}{4} (A_{in} + B_{in}) , \quad Q_{out}^{(1)} = \frac{N'}{4} (-A_{out} + B_{out}) , \quad (53)$$

$$A_{ex} = \frac{1}{N'} \int d\mu(q_1) d\mu(q_2) \varepsilon(q_1 - q_2) : (t_{ex}^+(q_1) t_{ex}(q_1) - d_{ex}^T(q_1) d_{ex}^*(q_1)) (t_{ex}^+(q_2) t_{ex}(q_2) - d_{ex}^T(q_2) d_{ex}^*(q_2)) : , \quad (54)$$

$$B_{ex} = \frac{i}{\pi} \int d\mu(q) \ln \frac{E(q)+q}{u} : (t_{ex}^+(q) t_{ex}(q) - d_{ex}^T(q) d_{ex}^*(q)) - \frac{1}{N'} \text{trace} (t_{ex}^+(q) t_{ex}(q) - d_{ex}^T(q) d_{ex}^*(q)) : , \quad (55)$$

where N' is given by (27). [For the subgroup cases, the value of N' in (52) and (55) is irrelevant since the constraint (35.8) forces the trace terms to vanish. It does appear, however, in the relative normalizations of the terms A_{ex} and B_{ex} .] This in turn leads to the following explicit formulae in terms of asymptotic states^{*})

$$A_{ex}^{ij} | \theta k l \rangle_{ex} = 0 , \quad (56)$$

$$A_{ex}^{ij} | \bar{\theta} \bar{k} \bar{l} \rangle_{ex} = 0$$

$$\begin{aligned} B_{ex}^{ij} | \theta k l \rangle_{ex} &= + \frac{i\theta}{\pi} (\delta^{il} | \theta k i \rangle_{ex} - \frac{1}{N} \delta^{ij} | \theta k l \rangle_{ex}) , \\ B_{ex}^{ij} | \bar{\theta} \bar{k} \bar{l} \rangle_{ex} &= - \frac{i\theta}{\pi} (\delta^{il} | \bar{\theta} \bar{k} \bar{i} \rangle_{ex} - \frac{1}{N} \delta^{ij} | \bar{\theta} \bar{k} \bar{l} \rangle_{ex}) . \end{aligned} \quad (57.A)$$

^{*}) The formulae for $Q_{ex}^{(0)}$ can be obtained from those for B_{ex} by the simple substitution $i\theta/\pi \rightarrow -i/2$.

$$B_{ex}^{ij} |\theta k l\rangle_{ex} = + \frac{i\theta}{\pi} (\delta^{il} |\theta k i\rangle_{ex} - I^{il} I^{jn} |\theta k n\rangle_{ex}) . \quad (57.S)$$

$$\begin{aligned} B_{ex}^{ij} |\theta k l\rangle_{ex} &= + \frac{i\theta}{\pi} (\delta^{il} |\theta k i\rangle_{ex} \pm \delta^{ik} |\theta l i\rangle_{ex} \\ &\quad - \frac{2}{N'} \delta^{ij} |\theta k l\rangle_{ex}) . \\ B_{ex}^{ij} |\bar{\theta} \bar{k} \bar{l}\rangle_{ex} &= - \frac{i\theta}{\pi} (I^{il} I^{jn} |\bar{\theta} \bar{k} \bar{n}\rangle_{ex} \pm I^{ik} I^{jn} |\bar{\theta} \bar{l} \bar{n}\rangle_{ex} \\ &\quad - \frac{2}{N'} \delta^{ij} |\bar{\theta} \bar{k} \bar{l}\rangle_{ex}) . \end{aligned} \quad (57.Q)$$

$$\begin{aligned} R_{ex}^{ij} |\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{ex} \\ &= \frac{1}{N} \varepsilon(\theta_1 - \theta_2) (+ \delta^{in_1} \delta^{jl_2} \delta^{l_1 n_2} - \delta^{in_2} \delta^{jl_1} \delta^{l_2 n_1}) |\theta_1 k_1 n_1, \theta_2 k_2 n_2\rangle_{ex} . \\ R_{ex}^{ij} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 k_2 l_2\rangle_{ex} \\ &= \frac{1}{N} \varepsilon(\theta_1 - \theta_2) (+ \delta^{in_2} \delta^{jn_1} \delta^{l_2 l_1} - \delta^{il_1} \delta^{jl_2} \delta^{n_2 n_1}) |\bar{\theta}_1 \bar{k}_1 n_1, \theta_2 k_2 n_2\rangle_{ex} . \end{aligned} \quad (58.A)$$

$$\begin{aligned} R_{ex}^{ij} |\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{ex} \\ &= \frac{1}{N'} \varepsilon(\theta_1 - \theta_2) (+ \delta^{in_1} \delta^{jl_2} \delta^{l_1 n_2} - \delta^{in_2} \delta^{jl_1} \delta^{l_2 n_1} \\ &\quad - \delta^{in_1} I^{jn_2} I^{l_1 l_2} + \delta^{in_2} I^{jn_1} I^{l_2 l_1} \\ &\quad - I^{il_1} \delta^{jl_2} I^{n_2 n_1} + I^{il_2} \delta^{jl_1} I^{n_1 n_2} \\ &\quad + I^{il_1} I^{jn_2} \delta^{l_2 n_1} - I^{il_2} I^{jn_1} \delta^{l_1 n_2}) |\theta_1 k_1 n_1, \theta_2 k_2 n_2\rangle_{ex} . \end{aligned} \quad (58.S)$$

$$\begin{aligned} R_{ex}^{ij} |\theta_1 k_1 l_1, \theta_2 k_2 l_2\rangle_{ex} \\ &= \frac{1}{N'} \varepsilon(\theta_1 - \theta_2) \{ (+ (\delta^{in_1} \delta^{jl_2} \delta^{l_1 n_2} - \delta^{in_2} \delta^{jl_1} \delta^{l_2 n_1}) |\theta_1 k_1 n_1, \theta_2 k_2 n_2\rangle_{ex} \\ &\quad \pm (\delta^{in_1} \delta^{jl_2} \delta^{k_1 n_2} - \delta^{in_2} \delta^{jk_1} \delta^{l_2 n_1}) |\theta_1 l_1 n_1, \theta_2 k_2 n_2\rangle_{ex} \\ &\quad \pm (\delta^{in_1} \delta^{jk_2} \delta^{l_1 n_2} - \delta^{in_2} \delta^{jl_1} \delta^{k_2 n_1}) |\theta_1 k_1 n_1, \theta_2 l_2 n_2\rangle_{ex} \\ &\quad + (\delta^{in_1} \delta^{jk_2} \delta^{k_1 n_2} - \delta^{in_2} \delta^{jk_1} \delta^{k_2 n_1}) |\theta_1 l_1 n_1, \theta_2 l_2 n_2\rangle_{ex} \} . \end{aligned}$$

$$\begin{aligned}
 & R_{ex}^{ij} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 k_2 l_2 \rangle_{ex} \\
 &= \frac{1}{N'} \epsilon(\theta_1 - \theta_2) \{ (\delta^{iu_2} I^{ju_1} I^{l_2 l_1} - I^{il_1} \delta^{jl_2} I^{u_2 u_1}) |\bar{\theta}_1 \bar{k}_1 \bar{u}_1, \theta_2 k_2 u_2 \rangle_{ex} \\
 &\quad \pm (\delta^{iu_2} I^{ju_1} I^{l_2 k_1} - I^{ik_1} \delta^{jl_2} I^{u_2 u_1}) |\bar{\theta}_1 \bar{l}_1 \bar{u}_1, \theta_2 k_2 u_2 \rangle_{ex} \\
 &\quad \pm (\delta^{iu_2} I^{ju_1} I^{k_2 l_1} - I^{il_1} \delta^{jk_2} I^{u_2 u_1}) |\bar{\theta}_1 \bar{k}_1 \bar{u}_1, \theta_2 l_2 u_2 \rangle_{ex} \\
 &\quad + (\delta^{iu_2} I^{ju_1} I^{k_2 k_1} - I^{ik_1} \delta^{jk_2} I^{u_2 u_1}) |\bar{\theta}_1 \bar{l}_1 \bar{u}_1, \theta_2 l_2 u_2 \rangle_{ex} \}. \quad (58.Q)
 \end{aligned}$$

$$\begin{aligned}
 B_{ex}^{ij} |\theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{ex} &= + \frac{i\theta_1}{\pi} (\delta^{il_1} |\theta_1 k_1 i, \theta_2 k_2 l_2 \rangle_{ex} - \frac{1}{N} \delta^{ij} |\theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{ex}) \\
 &\quad + \frac{i\theta_2}{\pi} (\delta^{il_2} |\theta_1 k_1 l_1, \theta_2 k_2 i \rangle_{ex} - \frac{1}{N} \delta^{ij} |\theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{ex}) \\
 B_{ex}^{ij} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 k_2 l_2 \rangle_{ex} &= - \frac{i\theta_1}{\pi} (\delta^{il_1} |\bar{\theta}_1 \bar{k}_1 i, \theta_2 k_2 l_2 \rangle_{ex} - \frac{1}{N} \delta^{ij} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 k_2 l_2 \rangle_{ex}) \\
 &\quad + \frac{i\theta_2}{\pi} (\delta^{il_2} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 k_2 i \rangle_{ex} - \frac{1}{N} \delta^{ij} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 k_2 l_2 \rangle_{ex}). \quad (59.A)
 \end{aligned}$$

$$\begin{aligned}
 B_{ex}^{ij} |\theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{ex} &= + \frac{i\theta_1}{\pi} (\delta^{il_1} |\theta_1 k_1 i, \theta_2 k_2 l_2 \rangle_{ex} - I^{il_1} I^{ju_1} |\theta_1 k_1 u_1, \theta_2 k_2 l_2 \rangle_{ex}) \\
 &\quad + \frac{i\theta_2}{\pi} (\delta^{il_2} |\theta_1 k_1 l_1, \theta_2 k_2 i \rangle_{ex} - I^{il_2} I^{ju_2} |\theta_1 k_1 l_1, \theta_2 k_2 u_2 \rangle_{ex}). \quad (59.S)
 \end{aligned}$$

$$\begin{aligned}
 B_{ex}^{ij} |\theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{ex} &= + \frac{i\theta_1}{\pi} (\delta^{il_1} |\theta_1 k_1 i, \theta_2 k_2 l_2 \rangle_{ex} \pm \delta^{jk_1} |\theta_1 l_1 i, \theta_2 k_2 l_2 \rangle_{ex} \\
 &\quad - \frac{2}{N'} \delta^{ij} |\theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{ex}) \\
 &\quad + \frac{i\theta_2}{\pi} (\delta^{il_2} |\theta_1 k_1 l_1, \theta_2 k_2 i \rangle_{ex} \pm \delta^{jk_2} |\theta_1 k_1 l_1, \theta_2 l_2 i \rangle_{ex} \\
 &\quad - \frac{2}{N'} \delta^{ij} |\theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{ex}).
 \end{aligned}$$

$$\begin{aligned}
 B_{ex}^{ij} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 k_2 l_2 \rangle_{ex} &= - \frac{i\theta_1}{\pi} (I^{il_1} I^{ju_1} |\bar{\theta}_1 \bar{k}_1 \bar{u}_1, \theta_2 k_2 l_2 \rangle_{ex} \pm I^{ik_1} I^{ju_1} |\bar{\theta}_1 \bar{l}_1 \bar{u}_1, \theta_2 k_2 l_2 \rangle_{ex} \\
 &\quad - \frac{2}{N'} \delta^{ij} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 k_2 l_2 \rangle_{ex}) \\
 &\quad + \frac{i\theta_2}{\pi} (\delta^{il_2} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 k_2 i \rangle_{ex} \pm \delta^{jk_2} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 l_2 i \rangle_{ex} \\
 &\quad - \frac{2}{N'} \delta^{ij} |\bar{\theta}_1 \bar{k}_1 \bar{l}_1, \theta_2 k_2 l_2 \rangle_{ex}). \quad (59.Q)
 \end{aligned}$$

On the other hand, the standard quantum charge $Q^{(0)}$ and the quantum non-local charge $Q^{(1)}$ of the interacting theory are conserved, and since the interacting field g converges at large times to the asymptotic free fields g_{ex} , we conclude, as in Ref. 15), that

$$Q_{in}^{(0)} = Q^{(0)} = Q_{out}^{(0)} . \quad (60)$$

$$Q_{in}^{(1)} = Q^{(1)} = Q_{out}^{(1)} . \quad (61)$$

For the standard charge, which generates isospin symmetry, this is a legitimate procedure, but for the non-local charge, Eq. (61) is a non-trivial relation which does not follow from any of the arguments presented so far. This was first pointed out in Ref. 17), where it was also argued that the condition for (61) to hold is that the action of $Q^{(1)}$ on asymptotic incoming and outgoing one-body states coincides with that of $Q_{in}^{(1)}$ and $Q_{out}^{(1)}$, respectively, as given by (56) and (57). But on such states, the action of $Q^{(1)}$ can be computed from the action of $Q^{(0)}$, which is known due to (60), by making use of the commutation relation (26), and the result is (61) - just as for the spherical models where this was shown in Ref. 16).

The desired additional constraints on the scattering amplitudes now follow by combining (44), (45), (46) and (53), (58), (59) with the relations

$$\begin{aligned} \langle \theta'_1 k'_1 l'_1, \theta'_2 k'_2 l'_2 | Q_{in} | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{in} &= \langle \theta'_1 k'_1 l'_1, \theta'_2 k'_2 l'_2 | Q_{out} | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{in} \\ \langle \theta'_1 k'_1 l'_1, \theta'_2 k'_2 l'_2 | Q_{in} | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{in} &= \langle \theta'_1 k'_1 l'_1, \theta'_2 k'_2 l'_2 | Q_{out} | \theta_1 k_1 l_1, \theta_2 k_2 l_2 \rangle_{in} \end{aligned} \quad (62)$$

plus the fact that $Q_{ex}^+ = -Q_{ex}$. After tedious calculation, this gives

$$u_2(\theta) = u_3(\theta) = -\frac{2\pi i}{N\theta} u_1(\theta) , \quad u_4(\theta) = -\frac{4\pi^2}{N^2\theta^2} u_1(\theta) ,$$

$$t_2(\theta) = t_3(\theta) = -\frac{2\pi i}{N(i\pi - \theta)} t_1(\theta) , \quad t_4(\theta) = -\frac{4\pi^2}{N^2(i\pi - \theta)^2} t_1(\theta) ,$$

$$\tau_1(\theta) = \tau_2(\theta) = \tau_3(\theta) = \tau_4(\theta) = 0 . \quad (63.A)$$

$$\begin{aligned}
 u_2(\theta) &= u_4(\theta) = -\frac{2\pi i}{N'\theta} u_1(\theta) , \\
 u_3(\theta) &= u_7(\theta) = -\frac{2\pi i}{N'(i\pi-\theta)} u_1(\theta) , \\
 u_5(\theta) &= -\frac{4\pi^2}{N'^2\theta^2} u_1(\theta) \\
 u_6(\theta) &= u_8(\theta) = -\frac{4\pi^2}{N'^2\theta(i\pi-\theta)} u_1(\theta) , \\
 u_9(\theta) &= -\frac{4\pi^2}{N'^2(i\pi-\theta)^2} u_1(\theta) . \tag{63.S}
 \end{aligned}$$

$$\begin{aligned}
 u_2(\theta) &= \mp \left(1 \pm \frac{N'\theta}{2\pi i}\right)^{-1} u_1(\theta) , \quad u_3(\theta) = -\frac{4\pi i}{N'\theta} u_2(\theta) , \\
 t_2(\theta) &= \mp \left(1 \pm \frac{N'(i\pi-\theta)}{2\pi i}\right)^{-1} t_1(\theta) , \quad t_3(\theta) = -\frac{4\pi i}{N'(i\pi-\theta)} t_2(\theta) , \\
 \tau_1(\theta) &= \tau_2(\theta) = \tau_3(\theta) = 0 . \tag{63.Q}
 \end{aligned}$$

Unitarity then leads to

$$|u_1(\theta)|^2 = \left(\frac{\theta^2}{\theta^2 + (2\pi/N)^2}\right)^2 , \quad |t_1(\theta)|^2 = 1 , \tag{64.A}$$

$$|u_1(\theta)|^2 = \left(\frac{\theta^2}{\theta^2 + (2\pi/N')^2}\right)^2 , \tag{64.S}$$

$$|u_1(\theta)|^2 = \frac{\theta^2}{\theta^2 + (4\pi/N')^2} , \quad |t_1(\theta)|^2 = 1 , \tag{64.Q}$$

all other unitarity conditions being satisfied automatically as a consequence of (63).

Equations (63) and (64) are a priori valid only in the physical region, i.e., for θ real and positive. Analyticity tells us, however, that the amplitudes u_i , t_i can all be continued to meromorphic functions in the complex θ -plane in such a way that

$$u_i(\theta)^* = u_i(-\theta^*) , \quad t_i(\theta)^* = t_i(-\theta^*) . \tag{65}$$

[Namely, the Mandelstam variable s and the rapidity variable θ are related by $s = 2m^2(1+\cosh\theta)$ or $s-4m^2 = 4m^2 \sinh^2(\theta/2)$, so the square root branch cut in s starting at $s = 4m^2$ disappears when we go over from s to θ . Other square root branch cuts in s starting at $s = 9m^2, 16m^2, \dots$, should not appear at all since, as a consequence of the factorization equations which in turn follow from the existence of the conserved quantum non-local charge, the models under consideration here do not tolerate particle production. However, we have not analyzed this issue in any detail.] Notice that the conditions (65) are consistent with the equations (63), which now hold for θ complex, and that they lead to a form of the equations (64) which holds for θ complex as well: it is obtained upon replacing $|u_1(\theta)|^2$ by $u_1(\theta) u_1(-\theta)$ and $|t_1(\theta)|^2$ by $t_1(\theta)t_1(-\theta)$. Finally, crossing symmetry gives rise to one additional condition, namely

$$t_1(\theta) = u_1(i\pi - \theta) , \quad (66.A)$$

$$u_1(\theta) = u_1(i\pi - \theta) , \quad (66.S)$$

$$t_1(\theta) = u_1(i\pi - \theta) , \quad (66.Q)$$

all other crossing relations then being satisfied automatically as a consequence of (63).

The minimal solutions $u_1^{\min}(\theta)$ are given by

$$u_1^{\min}(\theta) = \left(\frac{\Gamma(1-\theta/2\pi i) \Gamma(1/N + \theta/2\pi i)}{\Gamma(\theta/2\pi i) \Gamma(1+1/N - \theta/2\pi i)} \right)^2 . \quad (67.A)$$

$$u_1^{\min}(\theta) = \left(\frac{\Gamma(1/2 + \theta/2\pi i) \Gamma(1/N' + \theta/2\pi i) \Gamma(1-\theta/2\pi i) \Gamma(1/2 + 1/N' - \theta/2\pi i)}{\Gamma(\theta/2\pi i) \Gamma(1/2 + 1/N' + \theta/2\pi i) \Gamma(1/2 - \theta/2\pi i) \Gamma(1 + 1/N' - \theta/2\pi i)} \right)^2 \quad (67.S)$$

$$u_1^{\min}(\theta) = \frac{\Gamma(1-\theta/2\pi i) \Gamma(2/N' + \theta/2\pi i)}{\Gamma(\theta/2\pi i) \Gamma(1+2/N' - \theta/2\pi i)} . \quad (67.Q)$$

Note that the solution (67.A)¹¹⁾ is the square of the class II solution given by Berg, Karowski, Kurak, Weisz¹⁸⁾ and the solution (67.S)¹²⁾, for the orthogonal case, is the square of the original solution given by Zamolodchikov and Zamolodchikov⁹⁾, while the solution (67.Q) seems to be new. They are minimal in

the sense that their zeros and poles are located on the imaginary axis in such a way that in the physical strip $0 < \text{Im}\theta < \pi$, there is a single zero at $\theta = 0$ and no pole. More precisely, in the strip $-\pi < \text{Im}\theta < +\pi$, there is a single zero at $\theta = 0$ and a single pole at $\theta = -2\pi i/\Delta$, where

$$\Delta = 1/N . \quad (68.A)$$

$$\Delta = 1/N' . \quad (68.S)$$

$$\Delta = 2/N' . \quad (68.Q)$$

The general solutions $u_1(\theta)$ are obtained, as usual, after multiplication by CDD factors¹⁹⁾. In particular, consider the solutions

$$u_1(\theta) = P(\theta) u_1^{\text{min}}(\theta) \quad (69)$$

with the pole factors

$$P(\theta) = \frac{\sinh^2(\theta + 2\pi i/N)}{\sinh^2(\theta - 2\pi i/N)} . \quad (70.A)$$

$$P(\theta) = \left(\frac{\sinh \theta + i \sin(2\pi/N')}{\sinh \theta - i \sin(2\pi/N')} \right)^2 . \quad (70.S)$$

$$P(\theta) = \frac{\sinh(\theta + 4\pi i/N')}{\sinh(\theta - 4\pi i/N')} . \quad (70.Q)$$

Obviously, their zeros and poles are located on the imaginary axis in such a way that in the physical strip $0 < \text{Im}\theta < \pi$, there is a single zero at $\theta = 0$ and a single pole at $\theta = +2\pi i/\Delta$. The latter represents the bound state that we expect to appear in view of the determinant condition (7), whose quantum analogue, in terms of asymptotic states, is interpreted as the statement that antiparticles become bound states of particles and vice versa.

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