Superstrings at finite temperature

Enrique Alvarez*

CERN, CH-1211 Geneva 23, Switzerland

M. A. R. Osorio

Departamento de Física Teórica, Universidad Autónoma de Madrid, 28049 Madrid, Spain

(Received 20 February 1987)

The free energy of the heterotic string is computed in the one-loop approximation. Its infrared and ultraviolet behaviors are carefully analyzed, and their physical meaning is discussed. The properties of the zero-temperature limit are explored, as well as the possibility of a phase transition at a temperature of the order of the Planck mass.

I. INTRODUCTION

In the standard "big bang" cosmological model, the "initial state" of the Universe (i.e., the state at a time $t \sim 0$, but nevertheless well after the Planck era) can be described as a high-temperature statistical system. The initial singularity predicted by the Einstein classical theory of general relativity, as well as the fact that a consistent quantum theory of gravity has not yet been discovered, prevents any further extrapolation back in time of the physics in the initial state.

On the other hand, superstring theories aim at unifying all four fundamental interactions; they should then give quantitative answers to all quantum gravity problems and, in particular, they must predict the dynamics of the initial state of the Universe itself.

It is then natural, in this context, to study a system of superstrings in a mixed state (i.e., described by a density matrix). In the absence of an analogue of the "master equation" for superstrings,¹ we do not have any concrete information about the relaxation times involved in the approach to equilibrium. There is no guarantee that the initial state will be dominated by the canonical equilibrium density matrix. It could perhaps even be possible to argue that superstrings must be in pure states only, and that the observed entropy is due to an incomplete description because of the use of quantum fields.

The object of this paper is to study a system of strings in canonical equilibrium, putting aside, for the moment, all questions about the dynamical approach to this state. In Refs. 2 and 3, this problem has already been studied for the closed bosonic strings in the one-loop approximation.

In Ref. 2, the Polyakov-covariant first quantization was used. In this language, a thermal state of closed strings is equivalent to having one of the dimensions of the external space-time compactified down to a circle of radius β . This means that there is a winding number associated to the corresponding $\pi_1(S^1) = \mathbb{Z}$. As a consequence of these boundary conditions, the integrand of the amplitude is not modular invariant anymore. The integration over the Teichmüller parameter is then not restricted to the funda-

mental region of the modular group $\text{PSL}(2, \mathbb{Z})$. As we shall see in detail later on, this fact has deep implications on the zero-temperature limit of the theory.

In this paper we shall follow the approach of Ref. 3 and work in the light-cone gauge, which is exceedingly convenient when performing one-loop computations.

The one-string partition function at temperature β , $Z_1(\beta)$, can be obtained from the transverse partition function at temperature λ , $z_T(\lambda)$, by means of a Laplace transform:

$$Z_1(\beta) = \frac{L}{\pi} \int_0^\infty dp^+ e^{-\beta p^+} z_T(\beta/2p^+) , \quad (1)$$

where we have assumed that the spatial coordinates of the external Minkowskian space-time are compactified in a torus of circumference L .

In order to get the partition function of a system of bosonic strings, $Z(\beta)$, a further exponentiation has to be made:

$$\ln Z(\beta) = \sum_{r=1}^{\infty} \frac{1}{r} Z_1(\beta r) . \quad (2)$$

For a system of supersymmetric strings (i.e., involving an equal number of bosonic and fermionic states at each mass level) the corresponding formula is

$$\ln Z(\beta) = \frac{1}{2} \sum_{r=1}^{\infty} \frac{1}{r} [1 - (-1)^r] Z_1(\beta r) . \quad (3)$$

The strategy to follow is now straightforward. We first compute the transverse partition function $z_T(\lambda)$. We then compute the Laplace transform to get $Z_1(\beta)$ and finally use (2) or (3) to get the free energy of the system. In the following paragraphs, the different types of strings will be studied in turn, our final aim being the computation for the heterotic string in $d = 10$ dimensions.

II. OPEN BOSONIC STRINGS

The light-cone gauge action for the bosonic string (see, for instance, Ref. 4, whose notation we follow) is, in units in which $\alpha' = \frac{1}{2}$,

$$S = \frac{1}{2\pi} \int d^2\xi \partial_a x^i \partial^a x^i, \quad (4)$$

where ξ^a ($a=1,2$) are coordinates in the two-dimensional string world sheet and $x^i(\xi^a)$, $i=1, \dots, 24$, represent the embedding of the surface in the (transverse) space-time.

The transverse partition function

$$z_T(\lambda) \equiv \text{Tre}^{-\lambda H_T} = \text{Tre}^{-\lambda p^+ p^-} \quad (5)$$

can be expressed using standard manipulations³ as a Euclidean path integral, the coordinate ξ_2 being compactified into a circle S^1 of radius λ :

$$z_T(\lambda) = \int \mathcal{D}x^i e^{-S[x^i]}. \quad (6)$$

In the open-string case, the world sheet is then topologically a cylinder. In order to evaluate the functional integral,⁵ perhaps the simplest method is to use a Fourier expansion:

$$x^i = \hat{x}^i + \sum_{m=-\infty}^{\infty} \hat{x}_m^i e^{i2\pi m \xi_2 / \lambda} - \sum_{n_1=-\infty}^{\infty} \sum_{n_2=1}^{\infty} x_{n_1 n_2}^i e^{i2\pi n_1 \xi_2 / \lambda} \frac{\cos n_2 \xi_1}{n_2}, \quad (7)$$

where $\hat{x}^i$ are the zero modes and $\sum'$ means that the term $m=0$ is excluded from the sum. Reality implies $x_{n_1 n_2}^i = x_{-n_1 n_2}^i$, $\hat{x}_m^i = \hat{x}_{-m}^i$.

The functional measure is represented as an infinite product of definite integrals:

$$\mathcal{D}x^i = \prod_{i=1}^{24} d\hat{x}^i \prod_{m=-\infty}^{\infty} d\hat{x}_m^i \prod_{m_1=-\infty}^{\infty} \prod_{n_2=1}^{\infty} dx_{m_1 n_2}^i. \quad (8)$$

When the action in (4) is quadratic, all of the integrals are Gaussian, and we are left with

$$z_T = L^{24} \prod_{i=1}^{24} \prod_{m=1}^{\infty} \frac{\lambda}{2m^2} \prod_{\substack{n_2=1 \\ n_1 \in \mathbb{Z}}}^{\infty} \left[\frac{\lambda}{4\pi} + \frac{\pi}{\lambda} \frac{n_1^2}{n_2^2} \right]^{-1/2}. \quad (9)$$

This infinite product can easily be computed by means of ζ -function regularization (cf. Appendix A) and the answer turns out to be

$$z_T(\lambda) = L^{24} \eta \left[\frac{i\lambda}{2\pi} \right]^{-24} (2\pi\lambda)^{-12}, \quad (10)$$

where $\eta(z)$ is Dedekind's function. The corresponding free energy is

$$F(\beta) = -2^{-27} \pi^{-26} \int_0^\infty d\tau_2 \tau_2^{-14} \left[\theta_3 \left[0, \frac{i\beta^2}{4\pi^2 \tau_2} \right] - 1 \right] \eta(i\tau_2)^{-24}, \quad (11)$$

where θ_3 is one of the Jacobi θ functions.⁶

This result coincides with the infinite sum of the free energies of the quantum fields contained in the string, i.e., with masses given by the corresponding mass formula of the excitation of the string theory (cf. Appendix B). This observation was first made in Ref. 2 in the context of the theory of closed bosonic strings, but it remains true for all other types of strings considered in this paper.

Our formula (11) has both infrared and ultraviolet divergences. The parameter τ_2 roughly measures the length of the circumference of the cylinder. When $\tau_2 \rightarrow \infty$ the integral behaves as $\tau_2^{-14} \exp(2\pi\tau_2)$. This infrared divergence is due to the presence of a tachyon in the spectrum, which causes the very long cylinders to be weighted with a positive measure. On the other hand, when $\tau_2 \rightarrow 0$ (i.e., for very thin tubes) the behavior of the integrand is

$$\tau_2^{-2} \exp(-\beta^2/4\pi\tau_2) \exp(2\pi d_T/\tau_2),$$

in such a way that there is ultraviolet convergence only if $\beta < \beta_c = \pi\sqrt{8}$. This is the critical (Hagedorn) temperature which can either be a true maximum temperature of any physical system in the Universe or else an indication of the existence of a phase transition, depending on $\lim_{\beta \rightarrow \beta_c^+} F(\beta) < \infty$ (Ref. 5). If the free energy tends to a finite limit when the critical temperature is approached from below, there is indeed the possibility of a phase transition. If, on the contrary, this limit is divergent, then the meaning of β_c is that of a true maximum temperature. This issue has obviously important cosmological implica-

tions.^{7,8,3,9}

In our case, it is easy to check that because of the τ_2^{-2} factor, the free energy diverges in such a way that there is no possibility of phase transition in this system.

III. CLOSED BOSONIC STRINGS

The action is still the same as in (4), but the world sheet now has the topology of a two-torus T_2 . It turns out that in order to get the correct result, it is necessary to integrate over all possible nondiffeomorphic tori which, as is well known,¹⁰ are characterized by a single Teichmüller parameter τ . The imaginary part of τ , τ_2 , is the "length" of the torus [i.e., the length of one of the generators of $H_1(T_2)$; the length of the other generator is fixed to be 1]. The real part τ_1 is a "twist" to be done before identification. The periodicity properties of physical quantities (like the string coordinates themselves) are then

$$x^i(\xi_1 + 1, \xi_2) = x^i(\xi_1, \xi_2), \quad (12)$$

$$x^i(\xi_1, \xi_2 + \tau_2) = x^i(\xi_1 - \tau_1, \xi_2).$$

In our case, the length τ_2 is fixed to the value λ/π , where λ is the transverse temperature. It is possible to work directly with the action (4) and the twisted boundary conditions as in Eq. (12), but it is far more convenient to employ untwisted physical fields and a metric

$$ds^2 = d\xi d\bar{\xi}, \quad (13)$$

where $\xi \equiv \xi^1 + \tau \xi^2$, $\bar{\xi} \equiv \xi^1 + \bar{\tau} \xi^2$. The fact that both

descriptions of the torus are equivalent can easily be checked by an explicit change of coordinates.

Taking into account that the metric is now given by Eq. (13) and performing a normal-mode expansion

$$x^i = \sum_{n_1, n_2 \in \mathbb{Z}} x_{n_1 n_2}^i e^{2\pi i n_1 \xi_1} e^{2\pi i n_2 \xi_2},$$

the transverse partition function can be written in terms of a particular Epstein function (cf. Appendix A of Ref. 3):

$$z_T(\lambda) = 2^{24} L^{24} \lambda^{12} e^{12Z_2'(0)}, \quad (14)$$

where the coefficient of the quadratic form in the Epstein function is given by

$$a_{ij} = \tau_2^2 (g^{-1})_{ij}. \quad (15)$$

This can eventually be expressed in terms of an integral over the twist of the Dedekind functions:

$$z_T(\lambda) = \frac{L^{24} 2^{12}}{(2\pi)^{24}} \left[\frac{\pi}{\lambda} \right]^{12} \int_{-1/2}^{1/2} d\tau_1 |\eta(\tau)|^{-48}. \quad (16)$$

Using standard manipulations, the free energy of the system of closed bosonic strings is found to be

$$F(\beta) = -\pi^{-26} 2^{-14} \int_0^\infty \frac{d\tau_2}{\tau_2^{14}} \int_{-1/2}^{1/2} d\tau_1 \left[\theta_3 \left[0, \frac{i\beta^2}{2\pi^2 \tau_2} \right] - 1 \right] |\eta(\tau)|^{-48}. \quad (17)$$

This formula was already obtained in Refs. 2 and 3. It still presents a tachyonic infrared divergence when $\tau_2 \rightarrow \infty$ because then the integral behaves as $\tau^{-27/2} \exp(4\pi\tau_2)$. In the ultraviolet regime ($\tau_2 \rightarrow 0$) the integral behaves as

$$\tau_2^{-14} |\tau|^{24} \exp(4\pi\tau_2 / |\tau|^2) \exp(-\beta^2 / 2\pi\tau_2),$$

so that the critical temperature is now $\beta_c = \pi\sqrt{8}$. The new feature with respect to the open-string case is that now (forgetting for a moment the infrared divergence) the free energy remains finite when the critical temperature is approached from below; this fact opens up the possibility of a phase transition in this class of theories.

IV. OPEN SUPERSTRINGS (WITHOUT GAUGE GROUP)

The action is now

$$S = \int d^2\xi \left[-\frac{1}{2\pi} \partial_a x^i \partial^a x^i + \frac{i}{4\pi} \bar{S} \gamma^{-\xi a} \partial_a S \right] \quad (18)$$

(we follow here the notation of Refs. 4 and 11). The transverse partition function factorizes now into a bosonic part (which we computed already) and a fermionic part.

In terms of $O(8)$ real spinors (due to the structure of the constrained fermionic space), the fermionic part of the action reads

$$S_F = -\frac{i\sqrt{2}}{4\pi} \int d^2\xi [S^2(\partial_2 - \partial_1)S^2 + S^1(\partial_2 + \partial_1)S^1]. \quad (19)$$

The general Fourier expansion involves only modes antiperiodic in ξ_2 ; this is because we are representing a fermionic trace:

$$\begin{aligned} S^1 &= \sum_{n,m \in \mathbb{Z}} S_{nm} e^{in\xi_1} e^{(2m+1)\pi i \xi_2 / \lambda}, \\ S^2 &= \sum_{n,m \in \mathbb{Z}} S_{nm} e^{-in\xi_1} e^{(2m+1)\pi i \xi_2 / \lambda}. \end{aligned} \quad (20)$$

The Euclidean fermionic action reads

$$S_E = -\frac{\lambda}{\sqrt{2}} \sum_{n,m \in \mathbb{Z}} \left[in + \frac{(2m+1)\pi}{\lambda} \right] S_{nm}^+ S_{nm} \quad (21)$$

and the measure is

$$\mathcal{D}S = \prod_{n,m \in \mathbb{Z}} dS_{nm}.$$

Performing the corresponding Gaussian integrations over anticommuting c numbers, the fermionic contribution to z_T is found to be

$$\left[\prod_{n,m \in \mathbb{Z}} \frac{\lambda}{\sqrt{2}} \left[n - \frac{(2m+1)i\pi}{\lambda} \right] \right]^4$$

which can easily be written using formulas of Appendix A as

$$2^{-4} \prod_{n=1}^{\infty} \cosh^{-8} \frac{\lambda n}{2}$$

is such a way that the total transverse partition function is

$$z_T(\lambda) = L^8 \prod_{n=1}^{\infty} \left[\frac{e^{\lambda n/2} + e^{-\lambda n/2}}{e^{\lambda n/2} - e^{-\lambda n/2}} \right]^8. \quad (22)$$

The corresponding free energy is

$$\begin{aligned} F(\beta) &= -\frac{1}{8} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \left[\theta_3 \left[0, \frac{i\beta^2}{2\tau_2} \right] - \theta_4 \left[0, \frac{i\beta^2}{2\tau_2} \right] \right] \\ &\quad \times \tau_2^{-4} \theta_4 \left[0, \frac{i\tau_2}{\pi^2} \right]^{-8}. \end{aligned} \quad (23)$$

The infrared behavior is now regular because when $\tau_2 \rightarrow \infty$ the integrand behaves as $\tau_2^{-11/2}$. This is a consequence of the fact that there are now no tachyons in the spectrum because of $d=10$ supersymmetry.

The ultraviolet behavior is

$$\tau_2^{-2} \exp(-\beta^2 \pi / 2\tau_2) \exp(2\pi^3 / \tau_2),$$

so that the critical temperature is $\beta_c = 2\pi$. The free energy diverges when $\beta \rightarrow \beta_c^+$, implying that T_c is really a maximum temperature in the Universe.

V. CLOSED SUPERSTRINGS

The world sheet now has the topology of a torus, and we must introduce the metric $ds^2 = d\xi d\bar{\xi}$ on it. The Euclidean fermionic action is given by

$$S_E = \frac{-i\sqrt{2}}{4\pi} \int d^2\xi \left\{ S^2 \left[-\tau \partial_1 + \left[\frac{\tau^2 - \bar{\tau}^2}{2|\tau|^2} - 1 \right] \partial_2 \right] S^2 \right. \\ \left. - S^1 \left[\bar{\tau} \partial_1 + \frac{\tau}{\bar{\tau}} \partial_2 \right] S^2 \right\}. \quad (24)$$

The only spin structure to be considered is the one corresponding to periodic boundaries in the direction ξ_1 and antiperiodic ones in the direction ξ_2 . This is again because of the fact that we are functionally representing a fermionic trace.

The corresponding normal mode expansion is

$$z_T = \frac{L^8 2^{12}}{(2\pi)^8} \left[\frac{\lambda}{\pi} \right]^{-4} \int_{-1/2}^{1/2} d\tau_1 \prod_{n=1}^{\infty} \left[\frac{1+e^{2\pi i n \tau}}{1-e^{2\pi i n \tau}} \frac{1+e^{-2\pi i n \bar{\tau}}}{1-e^{-2\pi i n \bar{\tau}}} \right]^8$$

and the total free energy of the system can be written as

$$F(\beta) = -\frac{2}{\pi^{10}} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1 \tau_2^{-4} \left[\theta_3 \left[0, \frac{i\beta^2}{2\pi^2 \tau_2} \right] - \theta_4 \left[0, \frac{i\beta^2}{2\pi^2 \tau_2} \right] \right] |\theta_4(0, 2\tau)|^{-16}. \quad (25)$$

The infrared properties of the above expression are identical to the ones for open superstrings. In the ultraviolet regime, the behavior of the integrand is

$$\tau_2^2 \exp(-\beta^2/2\tau_2\pi) \exp(2\pi/\tau_2),$$

leading to a limiting temperature of $\beta_c = 2\pi$.

The main difference though, from the physical point of view, between open and closed superstrings stems from the fact that the free energy now tends to a finite limit when $\beta \rightarrow \beta_c^+$ because the integral behaves as $\int_0 \tau_2^2$. This means that there is the possibility of a phase transition at $T = T_c$.

VI. THE HETEROTIC STRING IN $d = 10$

This particular string¹² seems the most promising one, from the phenomenological point of view, as a unified theory of the fundamental interactions. Its construction involves a hybrid of bosonic and supersymmetric strings. The action is

$$S = -\frac{1}{2\pi} \int d^2\xi [\partial_a x^i \partial^a x^i + \partial_a x^I \partial^a x^I \\ + i\bar{S}\gamma^-(\partial_\tau + \partial_\sigma)S]. \quad (26)$$

The right-handed sector of the theory is analogous to the one of an ordinary closed superstring, whereas the left-handed one is divided into two sets: one part x^I , $i=1, \dots, 8$ corresponding to the ordinary transverse coordinates of the external space-time, and the other part x^I , $I=1, \dots, 16$ standing for coordinates in an internal

$$S^{1,2} = \sum_{n_1, n_2 \in \mathbb{Z}} S_{n_1 n_2}^{1,2} e^{2\pi i n_1 \xi_1} e^{(2n_2+1)\pi i \xi_2}.$$

The fermionic part of the transverse partition function reads

$$z_T = \prod_{n_1, n_2 \in \mathbb{Z}} \left[\tau n_1 - \frac{\tau}{\bar{\tau}} (2n_2 + 1) \right]^4 \left[\bar{\tau} n_1 - \frac{\bar{\tau}}{\tau} (2n_2 + 1) \right]^4.$$

This determinant can be easily worked out with the help of the formulas of Appendix A, leading to

$$2^8 \prod_{n=1}^{\infty} (1 + e^{2\pi i n \tau})^8 (1 + e^{-2\pi i n \bar{\tau}})^8 e^{(2\pi i/3)(\tau - \bar{\tau})}$$

in such a way that the complete transverse partition function is

manifold, a torus defined by quotient with one of the lattices Γ_{16} or $\Gamma_8 \oplus \Gamma_8$.

The transverse partition function consists then of the product of the holomorphic part of the superstrings times the antiholomorphic contribution of a bosonic string in eight transverse dimensions times the contribution from the internal torus. To be specific,

$$z_T = \frac{L^8}{\lambda^4 \pi^4} \int_{-1/2}^{1/2} d\tau_1 \prod_{n=1}^{\infty} \left[\frac{1+e^{2\pi i n \tau}}{1-e^{2\pi i n \tau}} \right]^8 \eta(-\bar{\tau})^{-8} z_{\text{int}}. \quad (27)$$

The partition function corresponding to the internal torus can be computed from the functional integral

$$z_{\text{int}} = \int \mathcal{D}x^I \exp \left[\frac{1}{2\pi} \int d^2\xi \partial_a x^I \partial^a x^I + \lambda^I [(\partial_\tau - \partial_\sigma)x^I]^2 \right] \quad (28)$$

with twisted boundary conditions as in (12). The simplest way to compute (28) is to use the standard operator formalism, i.e., compute the trace

$$z_T(\lambda) = \int_{-1/2}^{1/2} d\tau_1 \text{tr} e^{-\lambda H_T} U(\pi\tau_1), \quad (29)$$

where $U(\Delta)$ is the "shift operator" introduced in Ref. 13:

$$U(\Delta) \equiv \exp \left[2i\Delta \left[N - \tilde{N} + 1 - \frac{1}{2} \sum_I (p^I)^2 \right] \right] \quad (30)$$

which satisfies

$$U(\Delta)F(\tau, \sigma)U^\dagger(\Delta) = F(\tau, \sigma + \Delta), \quad (31)$$

where F can be either x^i , x^I , or S^a . Using the explicit form for the mass operator, the transverse Hamiltonian can be written as

$$H_T = p^+ p^- = \frac{P_T^2 + m^2}{2} = \frac{P_T^2}{2} + 2 \left[N + \tilde{N} - 1 + \frac{1}{2} \sum_I (p^I)^2 \right]. \quad (32)$$

It is perhaps worth remarking that the standard constraint

$$N = \tilde{N} - 1 + \frac{1}{2} \sum_I (p^I)^2 \quad (33)$$

stemming from invariance under (31) should not be implemented, since it is valid on shell only.

Collecting now the different terms, the final formula for the internal partition function is

$$z_{\text{int}} = \eta(-\bar{\tau})^{-16} \theta_{\Gamma}(-\bar{\tau}), \quad (34)$$

where θ_{Γ} is the θ function of the lattice.¹⁴

The free energy of the heterotic string is then found to be

$$F(\beta) = -\frac{1}{8\pi^{10}} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1 \left[\theta_3 \left[0, \frac{i\beta^2}{2\pi^2\tau_2} \right] - \theta_4 \left[0, \frac{i\beta^2}{2\pi^2\tau_2} \right] \right] \tau_2^{-4} \eta(-\bar{\tau})^{-24} \theta_4(0, 2\tau)^{-8} \theta_{\Gamma}(-\bar{\tau}). \quad (35)$$

The ultraviolet behavior of the integrand of $F(\beta)$ is

$$\exp(-\beta^2/2\tau_2\pi) \exp(\pi/\tau_2) \exp(2\pi/\tau_2),$$

implying thus an upper bound on the critical temperature of $\beta_c \leq \beta_0 = \pi\sqrt{6}$ (which is very close, although not identical, to the one quoted in Ref. 12). This is an intermediate value between the ones corresponding to bosonic and supersymmetric strings.

The limiting value of the free energy, when $\beta \rightarrow \beta_c^+$, converges to a finite value, because the integrand then goes as τ_2^2 . There is then the possibility of a phase transition in the heterotic string—as in all other theories involving closed strings only.

The infrared behavior is more subtle: when $\tau_2 \rightarrow \infty$ the integrand of $F(\beta)$ behaves as $e^{2\pi i\tau_1} e^{2\pi/\tau_2}$. The tachyonic divergence disappears because of the integration over τ_1 , and in fact this implies that the leading term in this limit is zero. This is a novel mechanism for infrared convergence, which does not exist for purely bosonic strings, because the closed one depends on τ_1 via $|\eta(\tau)|$ only (which in turn does not depend on τ_1 at all in the infrared limit), and because the open one is dependent only via $\eta(i\tau_2)$ (there is no twist in this case).

VII. CONCLUSION

The one-loop free energy of a gas of strings in canonical equilibrium has been computed for several types of strings, and its main characteristics studied. The actual result is equivalent to the sum of the free energies of the quantum fields present in the normal modes of the string.

From the physical point of view, there are two main types of theories: those involving only closed strings and those involving only open strings (which is sensible in this approximation, although in the next order in the topological expansion of the functional integral it is necessary to also consider closed-string corrections when studying open strings).

The class of theories involving open strings admits only equilibrium descriptions at temperatures smaller than a critical one, the Hagedorn temperature, which would then determine an absolute upper value in nature. The other

class of theories, involving only closed strings (such as the standard heterotic string in $d = 10$) presents indications of a phase transition at a critical temperature.

The only energy scale available is the Planck mass; this means that this phase transition takes place at a quantum gravity regime. The knowledge of the characteristics of the phase at $T > T_c$ would be most interesting in studying the quantum properties of gravity dictated by superstrings. As a matter of fact, this is also the phase which will probably dominate the dynamics of the “initial state” of the Universe.

A related point is the value of the cosmological constant. It has already been remarked in Ref. 2 that for the closed bosonic string, the (finite) cosmological constant is not the one resulting from the quantum field theory content of the strings (which is divergent). Both admit an identical representation as integrals over the complex parameter of the torus of the same integrand; the region of integration is different, however. For the quantum fields, it is the rectangle $-\frac{1}{2} < \tau_1 < \frac{1}{2}$, $\tau_2 > 0$, whereas for the string, modular invariance implies the further restriction $|\tau| > 1$, in order to integrate over the fundamental region of the modular group $\text{PSL}(2, \mathbb{Z})$.

This whole question is related to the nature of the zero-temperature limit of string physics. The integration region on τ changes in this limit, because manifest full modular invariance is recovered only at $T = 0$.

We know, of course, that due to its duality properties, string physics cannot be reduced to the corresponding field-theory content. This result seems nevertheless a bit paradoxical, and should be understood better because of its obvious cosmological implications (after all, bulk matter in the Universe is always at a finite, whatever small, temperature). Although for supersymmetric strings the cosmological constant is zero, this whole problem reappears as soon as supersymmetry is broken.¹⁵

The main conclusion of this paper is the following. The divergence of the partition function, combined with the finiteness of the free energy in the proper limit, can be interpreted in two different ways: either as an indication of the existence of new physics for $T > T_c$ (described by an as yet unknown fundamental theory) or as evidence that the canonical ensemble fails for $T \sim T_c$ and

should be substituted by the microcanonical one.⁸ To the extent that the analogy with the situation in the old dual models is correct, new physics should be expected at $T > T_c$, which, in this example, was the deconfined phase of quantum chromodynamics.⁶

It is of course evident that for temperatures close to the Planck mass the use of a flat external space-time background is not justified anymore. A study should be made of the effect of arbitrary consistent curved backgrounds. Another point which is worth clarifying is the thermodynamical behavior of superstrings compactified down to four dimensions.¹⁶ Work is currently in progress in these matters.

ACKNOWLEDGMENTS

One of us (E.A.) is indebted to L. Alvarez-Gaumé, J. Ellis, F. Englert, M. B. Gavela, and C. Gómez for many discussions on the contents of the present paper. This work has been partially supported by Comisión Asesora de Investigación Científica y Técnica, Spain.

APPENDIX A: ζ -FUNCTION REGULARIZATION OF SOME INFINITE PRODUCTS

When bosonic strings are considered, the general form of the determinants of the different operators of interest is (expressed as products of eigenvalues)

$$P_+ \equiv \prod_{n=1}^{\infty} \left[\frac{n}{u} + v \right]. \quad (\text{A1})$$

As an example

$$\prod_{n=1}^{\infty} \left[A + \frac{n^2}{B} \right] = \prod_{n=1}^{\infty} \left[\frac{n}{\sqrt{B}} - i\sqrt{A} \right] \left[\frac{n}{\sqrt{B}} + i\sqrt{A} \right].$$

We shall explain in detail only the computation of (A1); all other results contained in the appendix can be easily obtained using similar techniques.

The ζ function associated to P_+ is

$$\begin{aligned} \zeta_{P_+}(s) &= \frac{1}{\Gamma(s)} \int_0^{\infty} \tau^{s-1} d\tau \left[\frac{e^{-v\tau}}{1 - e^{-\tau/u}} - e^{-v\tau} \right] \\ &= u^s \zeta(s, uv) - v^{-s}, \end{aligned}$$

where the Riemann function $\zeta(s, z)$ is defined as⁶

$$\zeta(z, q) \equiv \sum_{n=0}^{\infty} \left[\frac{1}{q+n} \right]^z \quad (\text{Re } z > 1).$$

Working out the derivative, and using

$$\begin{aligned} \zeta(0, q) &= \frac{1}{2} - q, \\ \zeta(0, q) &= \ln \Gamma(q) - \frac{1}{2} \ln 2\pi, \end{aligned}$$

the result obtained is

$$P_+ \equiv e^{-\zeta'_P(0)} = \frac{\sqrt{2\pi}}{v \Gamma(uv)} u^{uv-1/2}. \quad (\text{A2})$$

This implies, using

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin \pi z},$$

so that

$$\prod_{n=1}^{\infty} (A + n^2/B) = \frac{2}{\sqrt{A}} \sinh \pi \sqrt{AB}. \quad (\text{A3})$$

It is amusing to realize that this gives a way to derive the modular properties of the Dedekind function, performing the computation in two different orders:

$$\begin{aligned} \prod_{n,m=1}^{\infty} \left[\frac{m^2}{\alpha^2} + \frac{n^2}{\beta^2} \right] &= \prod_{m=1}^{\infty} \frac{2\alpha}{m} \sinh \pi \frac{m\beta}{\alpha} \\ &= \prod_{m=1}^{\infty} \frac{2\beta}{m} \sinh \frac{\pi \alpha m}{\beta}, \end{aligned} \quad (\text{A4})$$

which implies the well-known property $[\eta(i\mu)] = (1/\sqrt{2}) \prod_{n=1}^{\infty} \sinh \pi n \mu$

$$\sqrt{\beta} \eta \left[\frac{i\beta}{\alpha} \right] = \sqrt{\alpha} \eta \left[i \frac{\alpha}{\beta} \right]. \quad (\text{A5})$$

When fermionic strings are studied, the basic infinite product to regulate is

$$P_- \equiv \prod_{n=0}^{\infty} \left[\frac{2n+1}{u} + v \right]. \quad (\text{A6})$$

This can easily be shown to be equal to

$$P_- = \frac{\sqrt{2\pi}}{\Gamma \left[\frac{1+uv}{2} \right]} (u/2)^{uv/2}, \quad (\text{A7})$$

implying, for example, that

$$\begin{aligned} \prod_{m=0}^{\infty} \left[\frac{(2m+1)^2}{B} + A \right] &= \prod_{m=0}^{\infty} \left[\frac{2m+1}{\sqrt{B}} - i\sqrt{A} \right] \\ &\quad \times \left[\frac{2m+1}{\sqrt{B}} + i\sqrt{A} \right] \\ &= 2 \cosh \frac{\pi \sqrt{AB}}{2}. \end{aligned} \quad (\text{A8})$$

Many other useful formulas can easily be obtained by straightforward manipulation of the ones we have included here.

APPENDIX B: THE FREE ENERGY OF THE QUANTUM FIELDS ASSOCIATED TO THE STRING

In this appendix we shall compute the free energy of the quantum fields with masses given by the mass formula corresponding to each particular type of string. In order to perform the different sums, it is useful to introduce the “proper time” representation of the free energy of a quantum field in d dimensions:

$$F_{B,F} = \pm \frac{1}{\beta} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \ln(1 \mp e^{-\beta \omega_k}), \quad (\text{B1})$$

where $\omega_k = (\mathbf{k}^2 + m^2)^{1/2}$ is the energy of a free particle of mass m . By expanding the logarithm and performing an elementary integration, we get, in the bosonic case,

$$F_B = - \sum_{r=1}^{\infty} (\beta r)^{-d/2} \pi^{-d/2} 2^{-d/2+1} m^{d/2} K_{d/2}(\beta r m), \quad (\text{B2})$$

where $K_{d/2}$ are the modified Bessel functions. In the fermionic case the corresponding formula is

$$F_F = \sum_{r=1}^{\infty} (-1)^r (\beta r)^{-d/2} \pi^{-d/2} 2^{-d/2+1} m^{d/2} K_{d/2}(\beta r m). \quad (\text{B3})$$

Using now the integral representation of the Bessel function

$$K_{d/2}(z) = \frac{1}{2} (z/2)^{d/2} \int_0^{\infty} ds \frac{e^{-s-z^2/4s}}{s^{1+d/2}}$$

one obtains the proper-time representation:

$$F_B = - \int_0^{\infty} ds \pi^{-d/2} 2^{-d/2-1} s^{-1-d/2} e^{-m^2 s/2} \times \left[\theta_3 \left[0, \frac{i\beta^2}{2\pi s} \right] - 1 \right], \quad (\text{B4})$$

$$F_F = \int_0^{\infty} ds \pi^{-d/2} 2^{-1-d/2} s^{-1-d/2} e^{-m^2 s/2} \times \left[\theta_4 \left[0, \frac{i\beta^2}{2\pi s} \right] - 1 \right]. \quad (\text{B5})$$

When expressed in terms of Jacobi θ functions,⁶ the cosmological constant corresponds (up to a sign) to the β -independent terms in the expressions for the free energy. Actually, in the infinite-volume limit we can use the regularization

$$\begin{aligned} \lambda_{B,F} &\equiv \pm \frac{L^{-(d-1)}}{2} \sum_{\mathbf{p}} \omega_{\mathbf{p}} = \pm \frac{1}{2} (2\pi)^{-(d-1)} \left[- \frac{\partial}{\partial \xi} \right], \\ \int_0^{\infty} d^{d-1} p e^{-\xi \omega_p} \Big|_{\xi=0} &= \mp \frac{1}{2} (2\pi)^{-d/2} \int_0^{\infty} ds s^{-1-d/2} e^{-m^2 s/2} \end{aligned}$$

which contains an ultraviolet divergence for very small “proper times”, $s \rightarrow 0$ (this is the usual zero-point energy of the quantum fields). Let us now examine in detail the field content of the different string theories.

1. Open bosonic strings

The mass formula is⁴

$$M^2 = 2 \left[\sum_{i=1}^{24} \sum_{n=1}^{\infty} n N_n^i - 1 \right],$$

where N_n^i are the occupation numbers of the transverse oscillators.

The total free energy of the fields contained in the strings (which is an extensive quantity) will be

$$F = - \int_0^{\infty} \frac{ds}{s} (2\pi s)^{-d/2} \sum_{N_u^i} \sum_{r=1}^{\infty} \exp \left[- \frac{r^2 \beta^2}{2s} - s \left[\sum_{i=1}^{24} \sum_{n=1}^{\infty} n N_n^i - 1 \right] \right].$$

Performing the sum, and changing variables to $\tau_2 = s/2\pi$ we finally obtain

$$F = - 2^{-27} \pi^{-26} \int_0^{\infty} \frac{d\tau_2}{\tau_2^{14}} \left[\theta_3 \left[0, \frac{i\beta^2}{2\tau_2} \right] - 1 \right] \eta(i\tau_2)^{-24}. \quad (\text{B6})$$

2. Closed bosonic strings

There are now two independent types of oscillators for left and right movers, and the mass operator is⁴

$$M^2 = 4 \left[\sum_{i=1}^{24} \sum_{n=1}^{\infty} n (N_n^i + \tilde{N}_n^i) - 2 \right].$$

There is, moreover, a constraint (expressing the invariances of the theory under shifts of the origin of the σ coordinate)

$$\sum_{i=1}^{24} \sum_{n=1}^{\infty} n (N_n^i - \tilde{N}_n^i) = 0.$$

In deriving the total free energy of the field-theory content, this constraint has to be implemented (for example, by means of Kronecker delta functions):

$$\begin{aligned} F = - \pi^{-13} \int_0^{\infty} \frac{d\tau_2}{\tau_2} (2\pi\tau_2)^{-13} \sum_{r=1}^{\infty} \exp \left[- \frac{r^2 \beta^2}{2\pi\tau_2} + 4\pi\tau_2 \right] \sum_{N_n^i, \tilde{N}_n^i} \exp \left[- 2\pi\tau_2 \sum_{i,n} n (N_n^i + \tilde{N}_n^i) \right] \\ \times \int_{-\pi}^{\pi} \frac{d\tau_1}{2\pi} \exp \left[i\tau_1 \sum_{i=1}^{24} \sum_{n=1}^{\infty} n (N_n^i - \tilde{N}_n^i) \right]. \end{aligned}$$

Working out the sums, and using $\eta(-\tau) = \overline{\eta(\tau)}$ we get

$$F(\beta) = -2^{-14} \pi^{-26} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1 \tau_2^{-12} \left[\theta_3 \left[0, \frac{i\beta^2}{2\pi^2 \tau_2} \right] - 1 \right] |\eta(\tau)|^{-48}. \quad (\text{B7})$$

3. Open superstrings (without gauge group)

There are now two types of oscillators, bosonic and fermionic, and the spectrum is given by⁴

$$M^2 = 2 \sum_{n=1}^\infty \sum_{i=1}^\infty (n N_{n_i}^B + n N_{n_i}^F).$$

The final formula for the free energy is

$$F = -\frac{1}{8} \int_0^\infty d\tau_2 \tau_2^{-6} \left[\theta_3 \left[0, \frac{i\beta^2}{2\tau_2} \right] - \theta_4 \left[0, \frac{i\beta^2}{2\tau_2} \right] \right] \times \theta_4 \left[0, \frac{i\tau_2}{\pi^2} \right]^{-8}. \quad (\text{B8})$$

4. Closed superstrings

Four types of oscillators are present in this case, left and right, bosonic and fermionic. The mass formula is⁴

$$M^2 = 4 \sum_{i=1}^8 \sum_{n=1}^\infty n (N_{n_i}^B + N_{n_i}^F + \tilde{N}_{n_i}^B + \tilde{N}_{n_i}^F)$$

and the constraint is

$$\sum_{i=1}^8 \sum_{n=1}^\infty n (N_{n_i}^B + N_{n_i}^F - \tilde{N}_{n_i}^B - \tilde{N}_{n_i}^F) = 0.$$

The free energy can be expressed as

$$F = -\frac{1}{8\pi^{10}} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1 \tau_2^{-4} \left[\theta_3 \left[0, \frac{i\beta^2}{2\pi^2 \tau_2} \right] - \theta_4 \left[0, \frac{i\beta^2}{2\pi^2 \tau_2} \right] \right] \theta_4(0, 2\tau)^{-8} \eta(-\bar{\tau})^{-24} \theta_\Gamma(-\bar{\tau}), \quad (\text{B10})$$

where $\theta_\Gamma(z)$ is the θ function corresponding to the even, self-dual lattice Γ (Ref. 14). To be explicit,

$$\theta_\Gamma(z) = \sum_{x \in \Gamma} e^{i\pi z(x, x)}$$

is a modular form of weight $n/2$ ($n=16$ in our case):

$$\theta_\Gamma(-1/z) = (iz)^{n/2} \theta_\Gamma(z).$$

$$F = -\frac{2}{\pi^{10}} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1 \tau_2^{-4} \left[\theta_3 \left[0, \frac{i\beta^2}{2\pi^2 \tau_2} \right] - \theta_4 \left[0, \frac{i\beta^2}{2\pi^2 \tau_2} \right] \right] \times |\theta_4(0, 2\tau)|^{-16}. \quad (\text{B9})$$

5. The heterotic string in $d=10$

In this case there are¹² two types of oscillators in the right sector, N^B and N^F , and another two in the left sector, $\tilde{N}^B$ and $\tilde{N}^I$; the latter being the one corresponding to the maximal internal 16-dimensional torus. There are, moreover, solitonic states characterized by the internal momentum P_I .

The mass formula is

$$M^2 = 4 \left[\sum_{i=1}^8 \sum_{n=1}^\infty n (N_{n_i}^B + N_{n_i}^F + \tilde{N}_{n_i}^B) + \sum_{I=1}^{16} \sum_{n=1}^\infty n \tilde{N}_n^I + \sum_{I=1}^{16} P_I^2 / 2 \right].$$

The closed-string constraint is now

$$\sum_{i=1}^8 \sum_{n=1}^\infty n (N_{n_i}^B + N_{n_i}^F) = \sum_{n=1}^\infty n \left[\sum_{i=1}^\infty \tilde{N}_{n_i}^B + \sum_{I=1}^{16} \tilde{N}_n^I \right] - 1 + \sum_{I=1}^{16} P_I^2 / 2.$$

The free energy of the heterotic string reads

*Permanent address: Departamento de Física Teórica, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain.

¹L. Reichl, *A Modern Course in Statistical Physics* (University of

Texas, Texas, 1980).

²J. Polchinski, *Commun. Math. Phys.* **104**, 37 (1986).

³E. Alvarez, *Nucl. Phys.* **B269**, 596 (1986).

⁴J. Schwarz, *Phys. Rep.* **89**, 223 (1982).

- ⁵N. Cabibbo and G. Parisi, Phys. Lett. **59B**, 67 (1975).
- ⁶A. Erdelyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953).
- ⁷K. Huang, and S. Weinberg, Phys. Rev. Lett. **25**, 895 (1970).
- ⁸M. J. Bowick and L. C. R. Wijewardhana, Phys. Rev. Lett. **54**, 2485 (1985).
- ⁹E. Alvarez, CERN Report No. TH-4478, 1986 (unpublished).
- ¹⁰L. Alvarez-Gaumé, CERN Report No. TH-4480, 1986 (unpublished).
- ¹¹M. Peskin, in *High Energy Physics 1985*, proceedings of the Theoretical Advanced Study Institute on Elementary Particle Physics, New Haven, Connecticut, 1985, edited by M. J. Bowick and F. Gursey (World Scientific, Singapore, 1985), p. 217.
- ¹²D. Gross, J. Harvey, E. Martinec, and R. Rohm, Nucl. Phys. **B256**, 253 (1985); **B267**, 75 (1986).
- ¹³J. Igusa, *Theta Functions* (Springer, New York, 1982).
- ¹⁴J. P. Serre, *A Course in Arithmetic* (Springer, New York, 1973).
- ¹⁵R. Rohm, Nucl. Phys. **B237**, 553 (1984).
- ¹⁶L. Dixon, J. A. Harvey, C. Vafa, and E. Witten, Nucl. Phys. **B261**, 678 (1985).