DIMENSIONAL REDUCTION AND THE HIGGS POTENTIAL

K. Farakos and G. Koutsoumbas

Physics Department, National Technical University
Athens, Greece

M. Surridge

Physics Department, The University of Southampton
SO9 5NH Southampton, England

and

G. Zoupanos^{*)}

CERN - Geneva

A B S T R A C T

Dimensional reduction of pure gauge theories over a compact coset space S/R leads to four-dimensional gauge theories, where Higgs fields and the corresponding potential appear naturally. We derive and examine the Higgs potential in certain classes of dimensionally reduced models. In some of these models with Higgs potential of geometrical origin, the spontaneous symmetry breaking takes us a step closer towards the observed low energy gauge theory.

^{*)} On leave from Physics Department, National Technical University, Athens, Greece.

1. Introduction

The idea of achieving unification of all interactions in a space-time possessing more than four dimensions has recently gained considerable popularity. The most ambitious dimensional reduction program is based on extensions of the old Kaluza-Klein idea⁽¹⁾. Here one starts with a pure gravitational theory in $(4+d)$ dimensions, and takes the extra dimensions to correspond to a compact space with isometry group S . One then finds in 4-dimensions an effective Lagrangian that contains a gravitational theory plus a gauge theory with gauge symmetry S . However, this program has met with serious problems. For example, the space-time manifold is assumed to take the form $M^4 \times B$, where M^4 is 4-dimensional and non-compact whilst B is compact and corresponds to the d extra dimensions. However, such a configuration cannot be a solution of the original higher dimensional theory. Perhaps the most serious problem of such a scheme is that it cannot reproduce the observed quantum numbers of the fermionic sector and in particular its left-right asymmetry in 4-dimensions.

It is possible to avoid these problems by introducing Yang-Mills fields in the original higher dimensional theory.⁽²⁾ This destroys the beautiful simplicity of the scheme whereby all interactions were geometrical in origin (in Kaluza-Klein theories). On the other hand, it is possible to find a natural role for the new fields in string theories⁽³⁾ for example. An important step in the investigation of such a composite Einstein-Yang-Mills dimensional reduction scheme is to determine the effective theory emerging from the Yang-Mills sector after dimensional reduction.

In fact, there exists an attractive approach which permits the detailed examination of the various features of the effective four-dimensional theory, namely Coset-Space Dimensional Reduction (C.S.D.R). This is based on the imposition of symmetry conditions on the $(4+d)$ dimensional theory, whereupon one can obtain a Yang-Mills-Higgs theory in lower dimensions.⁽⁴⁾ This approach, known to mathematicians, was examined and given physical application by Forgacs and Manton⁽⁵⁾ and others⁽⁶⁻¹⁷⁾, resulting in a complete dimensional reduction prescription for Yang-Mills theories now known as C.S.D.R.

Various attempts have been made to extract the observed low-energy gauge interactions and fermion multiplets from a simple gauge theory via C.S.D.R. Eventually, when one has discovered a suitable arrangement for the Yang-Mills plus fermions sectors, one has to generate stable, compactifying solutions of the full Einstein-Yang-Mills equations corresponding to the assumed manifold $M^4 \times (S/R)$ of the C.S.D.R. scheme. Some aspects of this part of the problem have

already been examined⁽¹⁵⁾.

In the present paper, we shall concentrate on the Higgs potential of the four dimensional effective gauge theory which is obtained from a pure Yang-Mills theory through C.S.D.R. This allows a further contribution (through the usual Higgs mechanism) to the geometrical symmetry breaking, beyond that arising from the dimensional reduction itself. It is known that if a condition is imposed on the isometry group S of the compact coset space, then the symmetry breaking from the Higgs fields can be deduced by group theoretical arguments alone, without there being any need to consider the potential directly. Almost all C.S.D.R. models which have so far been considered depend on this condition for the full analysis of the symmetry breaking. Unfortunately, it was found recently that the same condition which permits an easy determination of the final gauge group also forces the fermion quantum numbers to lose their left-right asymmetry⁽¹³⁾, whilst the fermions develop large masses (comparable with the compactification scale).⁽¹⁴⁾

Furthermore, it appears that relaxing the aforementioned condition may be an important step in realizing compactifying solutions of the full E-Y-M theory.^(2,9) It turns out that even to write down explicitly the Higgs potential in such a case is far from trivial. The purpose of this paper is to indicate how such a potential can be written down and how we can approach the problem of the minimization in order to recover the vacuum expectation values of the scalar fields. This is done by presentation of a number of examples, based on gauge groups SU_5 and E_8 in higher dimensions. Although no particularly appealing model emerges from the analysis of the examples described here, the methods we have used and the extensive appendices may prove of considerable use in other investigations.

This paper is organized as follows. In section 2 we recall the general features of the C.S.D.R. scheme and note some new features which will be used in the two subsequent sections. In section 3 we present examples of four-dimensional theories which are obtained from dimensional reduction over coset spaces of the gauge group SU_5 in higher dimensions. We write down the scalar potential explicitly in each case and discuss the spontaneous symmetry breaking of the corresponding gauge groups. In section 4 we study examples based on the gauge group E_8 in ten dimensions and we discuss the breaking patterns of the resulting four dimensional theory. In section 5 we summarize our results, whilst in the extensive appendices can be found details of the commutation relations of group generators which are used in sections 3 and 4.

2. The Coset Space Dimensional Reduction Scheme (C.S.D.R).

Let us briefly recall the dimensional reduction procedure using symmetric gauge fields⁽⁵⁾. One starts with a pure gauge field theory with gauge group G defined on an extended space-time manifold M which is assumed to be a direct product:

$$M=M_4 \times S/R \quad (2.1)$$

where M_4 is a four dimensional Minkowski space-time and S/R is a compact coset space. In general S is assumed to be a compact semi-simple Lie group and R is a Lie subgroup of S . The gauge group is also assumed to be compact and simple. One assumes that the metric of M is block diagonal.

One defines symmetric gauge fields by demanding that the action of S on the gauge field A_μ is compensated by a gauge transformation. Given a coset space S/R , it is possible to determine explicitly both the compensating gauge transformation and the dependence of A_μ on the coset space coordinates. When A_μ is inserted into the Yang-Mills lagrangian on M , one can integrate trivially over the coset space coordinates and obtain a Yang-Mills-Higgs theory in four dimensions. The four-dimensional theory has the following characteristic features: (i) The gauge group, which we denote by H , is the centralizer in G of a subgroup R_G (isomorphic to R) of G . (ii) The Higgs fields can be determined, as well as the Higgs potential.

Formally, the Higgs potential takes the form:

$$V(\Phi) \propto \delta^{rr'} \delta^{ss'} \text{Tr}[(f_{rst} \Phi_t - [\Phi_s, \Phi_r])(f_{r's't'} \Phi_{t'} - [\Phi_{s'}, \Phi_{r'}])] \quad (2.2)$$

where $\Phi_s(x)$ is a scalar field taking values in the Lie algebra of the initial gauge group G , and the proportionality constant depends on the metric tensor of the coset space. In component form, $\Phi_s(x) = \Phi_s^a(x) t^a$, where t^a are generators of G . The indices s, s', r, r' correspond to generators of S and f 's are the structure constants of S .

The above expression for $V(\Phi)$ is only formal because Φ_s must satisfy the constraints⁽⁵⁾

$$f_{smt} \Phi_t - [\Phi_s, \Phi_m] = 0 \quad (2.3)$$

where s takes all values, but the index m is restricted to values corresponding to generators of R . These constraints imply that some components Φ_s^a are zero, some are constants, and the rest can be identified with components of the genuine Higgs fields. One can easily find the genuine Higgs content of the dimensionally reduced theory. The gauge fields, which become scalars in four dimensions, transform under R as a vector as specified by the embedding:

$$\text{ad } S = \text{ad } R + \mathbf{v} \quad (2.4)$$

and $\mathbf{v} = \sum s_k$, where each s_k is an irreducible representation of R . Then the multiplets under H of independent Higgs fields can be found by decomposing the adjoint representation of G under $R_G \times H$:

$$\text{ad } G = \sum (r_k, h_k) \quad (2.5)$$

where r_k is some irreducible representation of R_G and h_k is some irreducible representation of H . For each pair of (r_k, s_j) , where r_k and s_j are identical irreducible representations, there is a Higgs multiplet h_k in the four dimensional theory. The rule given above for finding which Higgs multiplets occur in four dimensions is a consequence of solving eq.(2.3). When the $V(\Phi)$ is expressed in terms of the unconstrained independent Higgs fields, it remains a quartic polynomial which is invariant under gauge transformations of the final gauge group H , and its minimum determines the vacuum expectation value of the Higgs fields.

It turns out that the minimization of this potential is the most difficult problem remaining in the analysis of such theories. However, if S has an isomorphic image S_G within G which contains R_G in a consistent way, then it is possible to allow the Φ_s to become generators of S_G . In this case the potential $V(\Phi)$ is zero, which is clearly the absolute minimum value that it can take. These non-zero vacuum expectation values of the Higgs fields break the symmetry from H to K , where all elements of K commute with the generators $\langle \Phi_s \rangle$ of S_G . Thus one deduces that the final unbroken symmetry group K , is just the centralizer of S_G in G .^(7,17) Recently it was shown⁽¹⁴⁾ that the fermions of such a theory become all very massive after spontaneous symmetry breaking. Therefore this attractive possibility is ruled out in searches to obtain the standard model from C.S.D.R. of a

gauge group in higher dimensions. Thus we return to the original problem, namely to derive the scalar potential of the four dimensional theory in terms of the genuine Higgs fields and to minimize it.

In many examples (including some which we will consider here) the full Higgs potential will prove too complicated to minimize analytically. This tends to be the case when the Higgs fields appear in large H-multiplets. For example we will find ourselves faced with the case $H=E_6$ and $H=E_6 \times U_1$ in which the Higgs fields lie in 27-plets of E_6 . In the first we have 54 independent real components and in the second 108 with respect to which the potential must be minimized.

To help deal with this problem, we will invoke the following argument based on the existence of a subgroup T of G for which $K=C_G(T)$. That such a group exists follows from the fact that elements k of K satisfy:

$$\begin{aligned} k^{-1} \cdot \Phi_m \cdot k &= \Phi_m \\ k^{-1} \cdot \langle \Phi_s \rangle \cdot k &= \langle \Phi_s \rangle \end{aligned} \quad (2.6)$$

where Φ_m are the generators of R_G G and $\langle \Phi_s \rangle$ are the vacuum expectation values of the Higgs fields. Clearly, since K commutes with Φ_m and $\langle \Phi_s \rangle$ it must also commute with any commutator of the Φ_m and $\langle \Phi_s \rangle$. If we keep taking commutators $[\Phi_s, \Phi_t], [\Phi_s, [\Phi_t, \Phi_u]]$, etc. until closure is obtained then we will find ourselves with a closed Lie algebra of generators of the subgroup T of G, having the property that $K=C_G(T)$.

Now, suppose we think that H, the four dimensional gauge group, breaks to some subgroup $K' \subset H \subset G$ when the genuine Higgs acquire vacuum expectation values. We may have in mind a selection of possible K_i' . In each case, we would require the Higgs fields to obtain vacuum expectation values within a subgroup T_i' of G where $T_i' = C_G(K_i')$.

Next we can calculate the minimum value which can possibly be obtained for the potential $V(\Phi)$ consistent with this breaking by the simple expedient of constraining the Higgs fields to lie within $T_i' \subset G$ in each case.

The problem of calculating $V_{\min}^i(\Phi)$, the minimum corresponding to each candidate unbroken gauge group K_i' is much simpler for two reasons. Firstly, within $T_i' \subset G$ only, we will find far fewer solutions to the constraint

equations (2.3) for the scalar fields. Thus the potential $V^i(\Phi)$ will depend on far fewer independent scalar components and will be easier to minimize. Secondly, in order to construct the potential we need only the structure constants of T_i' instead of all the structure constants of G . Thus $V^i(\Phi)$ is also easier to evaluate than the full $V(\Phi)$. Finally, the constant term in $V(\Phi)$ (corresponding to setting $\Phi=0$) will be unaltered when going to a subgroup T_i' of G in this way. In order to find out which of the various K_i' is the preferred candidate for the unbroken symmetry group, we simply compare the various $V_{\min}^i(\Phi)$, and the preferred candidate is the one giving the smallest value for this.

In principle, to discover the actual symmetry breaking pattern we have to calculate $V_{\min}^i(\Phi)$ for all possible groups $T_i' \subset G$. By exploiting the gauge symmetry group H , we can see that we do not have to consider T_i' as large as G itself. Unfortunately, in cases where H is exceptional, it is not known how large T may be for all multiplets which might contain Higgs fields (in other words we do not know all the strata of symmetry breaking patterns in these cases). Nevertheless, we can use the comparison of different $V_{\min}^i(\Phi)$ in order to rule out particular candidates for K , which may be of use in avoiding difficult calculations in cases where we believe that only particular candidates are consistent with realistic four dimensional theories.

In order to incorporate fermions one has to add a Dirac lagrangian to the pure Yang-Mills Lagrangian. One then demands that the action of S on the spinors is compensated by a gauge transformation and a local Lorentz rotation in the tangent space. After dimensional reduction, the entire theory becomes four dimensional with spinors being in some multiplets of the group H . In order to identify the four-dimensional fermion multiplets after dimensional reduction, one first notes that R is naturally embedded in the Lorentz group $SO(N)$ of S/R in such a way that the N of $SO(N)$ has the branching rule:

$$N = \nu \quad (2.7)$$

where ν is given by (2.4).

This follows from the way R acts on tangent vectors at the point which R leaves fixed. In this way the embedding of R into $SO(N)$ is uniquely determined. Next one takes the spinor σ of $SO(N)$ and decomposes it into irreps of R :

$$\sigma = \sum \sigma_k \quad (2.8)$$

Then one has to decompose the representation F of the gauge group G to which the fermions are assigned under $R_G \times H$:

$$G \supset R_G \times H \quad ; \quad F = \sum (r_k, \sigma_k) \quad (2.9)$$

It turns out that for each pair of (r_k, σ_k) where r_k and σ_k are equivalent irreps, there is an h_k multiplet of spinor fields in the four-dimensional theory.

It is possible to obtain parity violation in the dimensionally reduced theory provided one starts with Weyl spinors and $\text{rank } S = \text{rank } R$.^(6,8) It turns out that one must take the Weyl spinors to be in a complex multiplet of G , unless the total number of dimensions is $D=4n+2$ when one can impose Weyl and Majorana conditions simultaneously and so halve the number of fermion states. To do this it is necessary for the fermions to be in a real (resp. pseudoreal) representation of G if $D=8n+2$ (resp. $D=8n+6$).⁽⁸⁾

3. Examples based on $G=SU_5$ in higher dimensions.

In this section we present a couple of examples of C.S.D.R. starting with the gauge group $G=SU_5$ in higher dimensions. We shall use the general method established in ref.(7) to obtain the four dimensional potential and we are going to demonstrate in specific examples that the faster method using the group T , which was described in section 2, gives the same result for the spontaneous symmetry breaking.

3.1. $G=SU_5$, $S/R=G_2/SU_3$.

The first model is a pure Yang-Mills with $G=SU_5$ in ten dimensions where the extra dimensions is the non-symmetric six-dimensional coset space G_2/SU_3 . We are going to identify R_G as the SU_3 subgroup of SU_5 appearing in the decomposition

$$\begin{aligned} SU_5 &\supset SU_3 \times SU_2 \times U_1 \\ 24 &= (8,1)(0) + (1,3)(0) + (\bar{3},2)(1) + (3,2)(-1) + (1,1)(0) \end{aligned} \quad (3.1)$$

From eq.(3.1) one can immediately see that the centralizer of SU_3 in SU_5 is $SU_2 \times U_1$, which according to chapter 2 is the gauge group of the effective theory in four dimensions. The decomposition of the adjoint of G_2 under SU_3 is:

$$\begin{aligned} G_2 &\supset SU_3 \\ 14 &= 8 + 3 + \bar{3} \end{aligned} \quad (3.2)$$

and therefore from eqs(3.1) and (3.2) we find that the genuine Higgs fields β appearing in the four-dimensional theory transform as the representation $(2_1 + 2_{-1})$ of $H = SU_2 \times U_1$. In order to construct the Higgs potential we shall need the commutation relations in SU_5 . The decomposition (3.1) suggests the introduction of the following generators of SU_5 :

$$Q_{SU(5)} = \{Q^p(0), Q^i(0), Q_{r\nu}(1), Q^{r\nu}(-1), Q(0)\} \quad (3.3)$$

where $p=1, \dots, 8$; $i=1, 2, 3$; $r=1, 2, 3$; $\nu=1, 2$. The generators $Q_{SU(5)}$ are in full correspondence with the decomposition (3.1). Their commutation relations are given in appendix 1. For the construction of the Higgs potential we shall need also the commutation relations of G_2 . According to the decomposition (3.2) we introduce the following generators of G_2 :

$$T_{G2} = \{T^p, T^r, T_r\} \quad (3.4)$$

where $p=1, \dots, 8$; $r=1, 2, 3$.

The commutation relations of T_{G2} 's are given in appendix 2.

Finally in order to express the formal Higgs potential of eq.(2.2) in terms of the genuine Higgs fields, we need explicit expressions for the scalar fields Φ_s . Once more the decomposition (3.2) suggests the following change in the notation of the scalar fields Φ_s :

$$\{\Phi_s; s=1, \dots, 14\} \rightarrow \{\Phi^p, \Phi^r, \Phi_r\} \quad (3.5)$$

Then the solution of the constraints (2.3) in terms of the genuine Higgs fields β is:

$$\begin{aligned}
 \Phi^p &= Q^p(0) \\
 \Phi^r &= \beta_\mu(1) Q^{r\mu}(-1) \\
 \Phi_r &= \beta^\mu(-1) Q_{r\mu}(1)
 \end{aligned}
 \tag{3.6}$$

We can now rewrite the potential (2.2) as:

$$\begin{aligned}
 V(\Phi) \propto & \text{Tr}_{\text{SU}(5)} \{ f^r_{tp} \Phi^p - [\Phi^r, \Phi_t] \} \{ f^t_{rs} \Phi^s - [\Phi_r, \Phi^t] \} + \\
 & \text{Tr}_{\text{SU}(5)} \{ f^{rt}_p \Phi_p - [\Phi^r, \Phi^t] \} \{ f_{rtv} \Phi^v - [\Phi_r, \Phi_t] \}
 \end{aligned}
 \tag{3.7}$$

where f 's are the structure constants of G_2 in the decomposition under SU_3 (see appendix 2). Substituting in eq.(3.7) the Φ 's from eq.(3.6) as a function of genuine Higgs β we find:

$$V(\beta) \propto \{ 8 - 8(\beta^+ \beta) + 12(\beta^+ \beta)^2 \}
 \tag{3.8}$$

which obviously breaks the gauge symmetry $\text{SU}_2 \times \text{U}_1$ of the effective four dimensional theory to $\text{U}_{1\text{e.m.}}$.

It is worth noting that if we introduce a 75-plet of fermions in the ten dimensional theory then by looking at the decomposition:

$$\begin{aligned}
 \text{SU}_5 &\supset \text{SU}_3 \times \text{SU}_2 \times \text{U}_1 \\
 75 &= (1,1)(0) + (3,1)(2) + (3,2)(-1) + (3,1)(-2) + (3,2)(1) + \\
 &\quad (6,2)(-1) + (6,2)(1) + (8,1)(0) + (8,3)(0)
 \end{aligned}
 \tag{3.9}$$

we find, according to chapter 2, in four dimensions fermions that transform as $1(0) + 1(2) + 2(-1)$ under $\text{SU}_2 \times \text{U}_1$, i.e. as a lepton family⁽⁸⁾. The bosonic part of the model has been mentioned also in ref.(6) but has not been worked out there.

Next we are going to use this example in order to illustrate that the structure $T \times K$ within G , which was discussed in section 2 can help in minimizing the potential. We know that $G \supset T \supset R_G$, so that since $G = \text{SU}_5$ and $R = \text{SU}_3$ the only possibilities for T are SU_3, SU_4 or SU_5 . At the vacuum, the Higgs fields construct generators $\langle \Phi_t \rangle$ of T . Therefore one can replace G with some subgroup T^1 of G and evaluate the potential for this simpler case. The number of unconstrained Higgs field components using the procedure outlined in section 2 is smaller than obtained

when considering the full group G . The potential $V^1(\Phi)$, as was noted in section 2, is easier to evaluate and minimize. However, $V^1(\Phi)$ can now be regarded precisely as the full potential $V(\Phi)$ for the model based on a gauge group G under the restriction that the Higgs fields lie in the subgroup T^1 of G . This is equivalent to evaluating $V(\Phi)$ under the restriction that all Higgs fields be K^1 invariant (where $K^1 = C_G(T^1)$). Now by comparing values of $V^1(\Phi)$ at their minimum for different T^1 's allows us to locate which of the corresponding $K^1 \subset H$ would be the most favoured pattern of symmetry breaking.

In the previous model we had three possible T^1 's : SU_3 , SU_4 and SU_5 . The last corresponds in considering the whole of G , which has been done in deriving (3.8). If we take $T^1 = SU_3$, we note that there are no non trivial solutions to the constraint equation (2.3) for the scalar fields within this group $T^1 \subset G$. Thus:

$$V^1(\Phi) \propto \text{Tr } f^\alpha_{\gamma\rho} \Phi^\rho f^\delta_{\beta\sigma} \Phi^\sigma \propto 8 \quad (3.10)$$

which is also the minimum for this T^1 .

If we take $T^2 = SU_4$ then the constraints yield a single unconstrained complex scalar component $\beta'(-1)$, where in parentheses is the charge under $U(1) = C_{SU(4)}(SU(3))$. The solution of the constraints in terms of this field β' is:

$$\begin{aligned} \Phi^\rho &= Q^\rho \\ \Phi^\alpha &= \beta'^*(1) Q^\alpha(-1) \\ \Phi_\alpha &= \beta'(-1) Q_\alpha(1) \end{aligned} \quad (3.11)$$

Now the potential has the form of eq.(3.7) with the difference that the scalar fields Φ 's take values in SU_4 instead of SU_5 . Expressing the Φ 's according to eq. (3.11) we find:

$$V^2(\beta') \propto \{8 - 8(\beta'^* \beta') + 12(\beta'^* \beta')^2\} \quad (3.12)$$

which has a minimum at $|\beta'|^2 = 1/3$, giving:

$$V^2_{\min} \propto 20/3 \quad (3.13)$$

with the same proportionality constant as in eq.(3.10). Thus we conclude that $T^2 = SU_4$, with $K^2 = U(1)$, is preferred over $T^1 = SU_3$, which corresponds to no symmetry breaking.

Notice that $V^2(\beta')$ in eq.(3.12) and $V(\beta)$ in eq. (3.8) are essentially the same. This is due to the well known fact that a complex Higgs doublet of $SU_2 \times U_1$ can always be gauge transformed such that one component vanishes, i.e. $(\beta_1, \beta_2) \rightarrow (\beta', 0)$. In our example, this ensures that the Φ_α and Φ^α lie within $T = SU_4$, in other words that by gauge transformations the Higgs fields in SU_5 can be brought into a chosen SU_4 subgroup which contains $R_G = SU_3$. In the present example we do not gain very much by exploiting possible subgroups $T^i \times K^i$, although with a little thought we could have seen that we only needed the structure constants of $SU_4 \subset SU_5$ in order to locate the minimum of the potential. Later, when G will be much bigger compared to R , we will find it very difficult to minimize the potential without using such subgroup structures.

3.2. $G = SU_5 : S/R = SU_2/U_1$.

Another model is based on a pure Yang-Mills in six dimensions with $G = SU_5$ while the extra dimensions belong to the symmetric coset space SU_2/U_1 i.e. the S^2 . We have chosen the $R = U_1$ to be the U_1 appearing in the decomposition (3.1). Therefore the four dimensional gauge group is $H = SU_3 \times SU_2 \times U_1$. On the other hand the decomposition of the adjoint of SU_2 under U_1 :

$$\begin{aligned} SU_2 &\supset U_1 \\ 3 &= 1, 0, -1 \end{aligned} \quad (3.14)$$

suggests according to section 2 that the complex genuine Higgs fields β transform under H as follows:

$$\begin{aligned} &SU_3 \times SU_2 \times U_1 \\ &(3, 2)(1) \end{aligned} \quad (3.15)$$

In order to construct the Higgs potential we shall need the commutation relations of the generators (3.3), which can be found in appendix 1. We further

need the commutation relations of SU_2 . According to the decomposition (3.14) we introduce the generators:

$$T_{SU(2)} = \{T_+, T_0, T_-\} \quad (3.16)$$

and their commutation relations are given in appendix 2.

Then in order to express the formal Higgs potential of eq.(2.2) in terms of the genuine Higgs fields, we express the scalar fields Φ_s according to the decomposition (3.14) as:

$$\{\Phi_s; s=1,2,3\} \rightarrow \{\Phi_+, \Phi_0, \Phi_-\} \quad (3.17)$$

The solution of the constraints (2.3) are now in terms of the genuine Higgs fields β :

$$\begin{aligned} \Phi_0 &= 2\sqrt{(3/5)}Q(0) \\ \Phi_+ &= \beta^{\alpha\mu}(-1)Q_{\alpha\mu}(1) \\ \Phi_- &= \beta_{\alpha\mu}(1)Q^{\alpha\mu}(-1) \end{aligned} \quad (3.18)$$

Now the Higgs potential with respect to the genuine Higgs fields is:

$$V(\beta) \propto \{(48/5) - 8\beta^+ \beta + 4(\beta_{\alpha\mu}\beta^{\alpha\mu})(\beta_{\alpha\nu}\beta^{\alpha\nu})\} \quad (3.19)$$

which breaks⁽¹⁸⁾ the gauge group $H = SU_3 \times SU_2 \times U_1$ to $K = SU_2 \times U_1$.

4. Examples based on $G = E_8$ in ten dimensions.

Let us now discuss a couple of models based on a Yang-Mills theory with gauge group E_8 in ten dimensions. Similar attempts using other coset spaces and embeddings could be useful for dimensional reduction of theories that might emerge in more general context than the present, such as superstrings.⁽³⁾ In fact applications of C.S.D.R. in superstring theories can be found in ref.(19).

4.1. $G=E_8$, $S/R=G_2/SU_3$

In the first model we start with a gauge group $G=E_8$ in ten dimensions and the extra dimensions belong to the non-symmetric six dimensional coset space G_2/SU_3 . The $R_G=SU_3$ is going to be identified with the one appearing in the decomposition:

$$\begin{aligned} E_8 &\supset SU_3 \times E_6 \\ 248 &= (8,1) + (1,78) + (3,27) + (\bar{3},\bar{27}) \end{aligned} \quad (4.1)$$

Therefore the four dimensional gauge group is $C_{E_8}(SU_3)=E_6$. From eqs(3.2) and (4.1) we find that the genuine Higgs field β is a complex 27.

In this model the group $S=G_2$ has an isomorphic image within $G=E_8$, unlike the corresponding case discussed in subsection 3.1. Therefore in the present case we know, which is the gauge group after spontaneous symmetry breaking due to the genuine Higgs fields. According to section 2, it is $C_{E_8}(G_2)=F_4$. However the actual derivation of this result can be used to check the procedure that we have followed in the previous simpler cases. Then one feels more confident to work out other difficult cases, one of which is examined in the next subsection.

In order to construct the Higgs potential, we introduce, in correspondence with eq.(4.1), the following generators of E_8 :

$$Q_{E_8} = \{Q^p, Q^a, Q^{ri}, Q_{ri}\} \quad (4.2)$$

where $p=1,\dots,8$; $a=1,\dots,78$; $r=1,2,3$; $i=1,\dots,27$. Their commutation relations are given in appendix 3. We need also the generators of G_2 given in eq.(3.4) and we redefine the scalar fields Φ_s as in eq.(3.5).

The solution of the constraints in terms of the genuine Higgs fields β is:

$$\begin{aligned} \Phi^p &= Q^p \\ \Phi^r &= \beta_i Q^{ri} \\ \Phi_r &= \beta^i Q_{ri} \end{aligned} \quad (4.3)$$

Then the Higgs potential is given by eq.(3.7), with the only difference that the

trace acts now on generators of E_8 instead of SU_5 . Substituting into the potential the Φ 's from eq.(4.3) as a function of the genuine Higgs β we find:

$$V(\beta) \propto \{16-16(\beta^\dagger \beta) + (198/7)(\beta^\dagger \beta)^2 - 12\sqrt{(10/21)}(d_{ijk}\beta^i \beta^j \beta^k + \text{h.c.})\} \quad (4.4)$$

where d_{ijk} are defined in appendices 3 and 5.

As a check we assume that β has only a non zero component in the direction which leaves the F_4 subgroup of E_6 invariant. According to section 2 this should be the minimum and moreover the potential, being positive definite, should vanish. Indeed with $\beta = (\beta_1, 0, \dots, 0)$ in the F_4 invariant direction of E_6 the potential becomes:

$$V(\beta_1) \propto \{1 - \beta_1 \beta_1^\dagger - \beta_1^\dagger \beta_1 \beta_1^\dagger - \beta_1 \beta_1^\dagger \beta_1 + 2(\beta_1 \beta_1^\dagger)^2\} \quad (4.5)$$

which vanishes for $\beta_1 = 1$ as expected.

4.2 $G=E_8 : S/R=Sp_4/SU_2 \times U_1$.

Our last example is a Yang-Mills theory with $G=E_8$ in ten dimensions and the extra dimensions is the non symmetric coset space $Sp_4/SU_2 \times U_1$. We identify the $R_6 = SU_2 \times U_1$ as the maximal subgroup of SU_3 appearing in the decomposition (4.1). Then the $S=Sp_4$ it has no isomorphic image within E_8 . The decomposition of the adjoint E_8 under $SU_2 \times U_1 \times E_6$ is:

$$\begin{aligned} E_8 \supset (SU_2 \times U_1) \times E_6 \\ 248 = (1(0), 1) + (2(3), 1) + (2(-3), 1) + (3(0), 1) + (1(0), 78) + \\ (1(-2), 27) + (2(1), 27) + (1(2), \bar{27}) + (2(-1), \bar{27}) \end{aligned} \quad (4.6)$$

while the decomposition of the adjoint of Sp_4 under $SU_2 \times U_1$ is:

$$\begin{aligned} Sp_4 \supset SU_2 \times U_1 \\ 10 = 3_0 + 1_{2,0,-2} + 2_{1,-1} \end{aligned} \quad (4.7)$$

From eqs(4.6),(4.7) we find that the four dimensional gauge group is $H=E_6 \times U_1$ and

the genuine Higgs fields are a complex $\beta=27(-2)$ and a complex $\gamma=27(1)$.

The decomposition (4.6) suggests the introduction of the following generators of E_8 :

$$Q_{E8}=\{Q(0), Q^a(3), Q_a(3), Q^p, Q^j(-2), Q^{aj}(1), Q_j(2), Q_{aj}(-1), Q^\alpha\} \quad (4.8)$$

where $a=1,2$; $p=1,2,3$; $j=1,\dots,27$; $\alpha=1,\dots,78$. Their commutation relations are given in appendix 4. Similarly the decomposition (4.7) suggests the introduction of the following generators of Sp_4 :

$$T=\{T^i, T_+, T_0, T_-, T^\alpha_+, T_{\alpha-}\} \quad (4.9)$$

with commutation relations given in appendix 6. Then we change the notation of the scalar fields Φ_s :

$$\{\Phi_s; s=1,\dots,10\} \rightarrow \{\Phi^i, \Phi_+, \Phi_0, \Phi_-, \Phi^\alpha_+, \Phi_{\alpha-}\} \quad (4.10)$$

The solutions of the constraints in terms of the genuine Higgs fields β, γ is:

$$\begin{aligned} \Phi^i &= Q^i \\ \Phi_0 &= \sqrt{3}Q(0) \\ \Phi_+ &= \beta^j(-2)Q_j(2) \\ \Phi_- &= \beta_j(2)Q^j(-2) \\ \Phi^\alpha_+ &= \gamma_i(-1)Q^{\alpha i}(1) \\ \Phi_{\alpha-} &= \gamma^i(1)Q_{\alpha i}(-1) \end{aligned} \quad (4.11)$$

Then we can express the Higgs potential in terms of the genuine Higgs fields. However since the potential to be minimized is a function of 108 variables we make use instead of the structures $T \times K$ within E_8 that we have discussed in section 2 and applied in section 3. Now since the gauge group G is very large as compared to R there exist in principle many possible $G=T \times K$ structures to be examined. However using Michel's conjecture⁽²⁰⁾ on spontaneous symmetry breaking by Higgs we eliminate many of them. Michel's conjecture suggests that a gauge group H would break always to a maximal little group of the representation

h of H containing the scalar fields, where this is irreducible. Then we find that only the SO_{10} and F_4 are suggested by this conjecture as final subgroup after spontaneous symmetry breaking. The corresponding $G=TxK$ structures are:

$$E_8 \supset SU_4 \times SO_{10}; \quad E_8 \supset G_2 \times F_4 \quad (4.12)$$

We should also note that in selecting $T=SU_4$ and $T=G_2$ as our candidate subgroups of E_8 to contain the Higgs, we have made an assumption concerning the 27-plets of Higgs components under E_6 . Specifically, we have taken the $\beta=27(-2)$ and $\gamma=27(1)$ of scalars under $H=E_6 \times U_1$ to have vacuum expectation values, which are aligned in the same direction within the 27 of E_6 , thus permitting us to regard the Higgs multiplet as an irreducible 27 so that Michel's conjecture can be invoked. This is not an unpalatable assumption if we recall from eq.(4.6) that the $R=SU_2 \times U_1$ is contained within an SU_3 factor of E_8 which commutes with E_6 . Then we can envisage making gauge transformations within G which are entirely within the SU_3 and so do not cause the components of the 27's to mix up. However in this SU_3 we see that the two 27's transform as $2_1 + 1_{-2}$ of R which lie within the irreducible 3 of SU_3 . This shows that the two R -irreducible representations are in some sense SU_3 -equivalent. Then we should be able to bring the two Higgs multiplets $(2_1, 27)$ and $(1_{-2}, 27)$ into the same direction corresponding to a configuration within $(3, 27)$ of $SU_3 \times E_6$, and this can be achieved using the SU_3 subgroup of gauge transformations within E_8 .^F

This argument, along with Michel's conjecture gives us some confidence that as a first guess for the symmetry breaking we can consider $K^1=SO_{10}$ and $K^2=F_4$ as the most likely candidates.

Following the procedure that we have repeated several times (the relevant commutation relations can be found in appendix 7) we find:

$$V^1(\beta, \gamma) \propto [24 - 8(\gamma^+ \gamma) - 4(\beta^+ \beta) + 12(\gamma^+ \gamma)^2 + 8(\gamma^+ \gamma)(\beta^+ \beta) + 4(\beta^+ \beta)^2] \quad (4.13)$$

which has a minimum at $|\beta|^2 = |\gamma|^2 = 1/4$, giving:

$$V^1_{\min} \propto 45/2 \quad (4.14)$$

Similarly we find that:

$$v^2(\phi) \propto [24-8(\gamma^+\gamma)-4(\beta^+\beta)+8\sqrt{3}(\gamma^2\beta^++\gamma^+2\beta)+12(\gamma^+\gamma)^2+(56/3)(\gamma^+\gamma)(\beta^+\beta)+(4/3)(\beta^+\beta)^2] \quad (4.15)$$

The minimum of this potential is at $\beta=-\sqrt{(3/4)}, \gamma=\sqrt{(3/4)}$, at which point:

$$v^2_{\min} \propto 15 \quad (4.16)$$

with the same proportionality constant as in eq.(4.14). Therefore comparing eqs(4.14) and (4.16) we conclude that F_4 is certainly preferred than SO_{10} . The latter, if it was preferred, it would have been a very interesting GUT with chiral 16-plet fermions. The methods outlined have enabled us to rule out this apparently promising model without performing the minimization of the full potential, a long and hard calculation.

5. Conclusions.

In the present paper we have examined the Higgs potential obtained after coset-space Dimensional Reduction of pure Yang-Mills theories in higher dimensions. We have applied a previously proposed method and developed a new one in order to derive and minimize the Higgs potential. Here we have considered cases without any restriction on S in contrast to most of the previous attempts, where it was assumed that S has an isomorphic image in G . This is necessary after a recent no-go theorem⁽¹⁴⁾ on the existence of light fermions when $S \subseteq G$. Even though we did not find a particularly appealing model after spontaneous symmetry breaking due to the Higgs potential, it seems to us that a breaking towards the low energy gauge group can have geometrical origin. Our work here can be useful in searches where the geometrical breaking is combined with other mechanisms, such as Wilson loop flux⁽²¹⁾ or dynamical breakings in order to obtain the standard model in the low energy limit.

Finally we note that finding the minimum of the potential in the general case ($S \not\subseteq G$) would also be very useful in generating stable, compactifying solutions of the coupled Einstein-Yang-Mills equations corresponding to the manifold M in eq.(2.1).

Acknowledgements.

We would like to thank A.Bais, K.J.Barnes, P.Forgacs and E.Papantonopoulos for discussions.

Footnote

We have found a justification of our argument in a simpler example using $G = SU_5$ and $S/R = Sp_4/SU_2 \times U_1$. Then $SU_5 \supset SU_3 \times SU_2 \times U_1 \supset (SU_2 \times U_1) \times SU_2 \times U_1$. In this example $H = SU_2 \times U_1$ plays the role of E_6 .

Appendix 1.

We present the non trivial commutation relations of SU_5 according to the decomposition (3.1).

$$[Q^p, Q^q] = 2if^{pqr}Q^r \quad ; \quad f's \text{ are the structure constants of } SU_3.$$

$$[Q^i, Q^j] = 2i\epsilon^{ijk}Q^k$$

$$[Q^{a\bar{a}}, Q_{b\bar{b}}] = -\sqrt{5/3}\delta^a_b\delta^{\bar{a}}_{\bar{b}}Q - (Q^p)^a_b\delta^{\bar{a}}_{\bar{b}}Q^p - (Q^j)^{\bar{a}}_{\bar{b}}\delta^a_bQ^j$$

$$[Q^p, Q^{a\bar{a}}] = -(Q^p)^a_bQ^{b\bar{a}}$$

$$[Q^p, Q_{a\bar{a}}] = (Q^p)^b_aQ_{b\bar{a}}$$

$$[Q^i, Q^{a\bar{a}}] = -(Q^i)^{\bar{a}}_{\bar{a}}Q^{a\bar{a}}$$

$$[Q^i, Q_{a\bar{a}}] = (Q^i)^{\bar{a}}_{\bar{a}}Q_{a\bar{a}}$$

$$[Q, Q^{a\bar{a}}] = -\sqrt{5/3}Q^{a\bar{a}}$$

$$[Q, Q_{a\bar{a}}] = \sqrt{5/3}Q_{a\bar{a}}$$

The normalization conventions are the following:

$$\text{Tr}(Q_m Q^n) = 2\delta_m^n; \quad \text{Tr}(Q_i Q^j) = 2\delta_i^j; \quad \text{Tr}(Q_{a\bar{a}} Q^{b\bar{b}}) = 2\delta_a^b\delta_{\bar{a}}^{\bar{b}}; \quad \text{Tr}(Q^2) = 2$$

Structure constants of $T=SU_4$ corresponding to a decomposition in terms of the subgroup $R=SU_3$:

$$SU_4 \supset SU_3 \times U_1$$

$$15 = 8_0 + 3_4 + 3_{-4} + 1_0$$

We write the generators of SU_4 as $\{Q^p, Q^\alpha, Q_\alpha, Q_0\}$ where $p=1, \dots, 8, \alpha=1, \dots, 3$ and Q^p are generators of SU_3, Q_0 of the U_1 factor within SU_4 and Q^α, Q_α lie in the 3_4 and 3_{-4} of $SU_3 \times U_1$ above.

The non vanishing commutators are then:

$$[Q^p, Q^q] = 2if^{pqr}Q^r$$

$$[Q^p, Q^\alpha] = -(Q^p)^\alpha_\beta Q^\beta$$

$$[Q^p, Q_\alpha] = (Q^p)^\beta_\alpha Q_\beta$$

$$[Q^\alpha, Q_\beta] = -(Q^\alpha)^\beta_\gamma Q^\gamma + \sqrt{8/3}\delta^\alpha_\beta Q_0$$

$$[Q_0, Q^\alpha] = \sqrt{8/3}Q^\alpha$$

$$[Q_0, Q_\alpha] = -\sqrt{8/3}Q_\alpha$$

Appendix 2

We present the commutation relations of G_2 according to the decomposition

(3.2)

$$[T^{\rho}, T^{\sigma}] = 2if\rho\sigma\tau T^{\tau}$$

$$[T^{\rho}, T^{\alpha}] = -(T^{\rho})^{\alpha}_{\beta} T^{\beta}$$

$$[T^{\rho}, T_{\alpha}] = (T^{\rho})^{\beta}_{\alpha} T_{\beta}$$

$$[T^{\alpha}, T^{\beta}] = 2\sqrt{(2/3)}\epsilon^{\alpha\beta\gamma} T_{\gamma}$$

$$[T_{\alpha}, T_{\beta}] = -2\sqrt{(2/3)}\epsilon^{\alpha\beta\gamma} T_{\gamma}$$

The normalization conditions are the following:

$$\text{Tr}(T_{\rho} T^{\sigma}) = 2\delta_{\rho}^{\sigma}; \text{Tr}(T_{\alpha} T^{\beta}) = 2\delta_{\alpha}^{\beta}.$$

We note also the commutation relations of SU_2 according to the decomposition (3.14)

$$[T_0, T_{+}] = 2T_{+}$$

$$[T_0, T_{-}] = -2T_{-}$$

$$[T_{+}, T_{-}] = 2T_0$$

The normalization conditions are:

$$\text{Tr}(T_{+} T_{-}) = 2; \text{Tr}(T_0^2) = 2.$$

Appendix 3

We present the commutation relations of E_8 according to the decomposition (4.2).

$$[Q^{\rho}, Q^{\sigma}] = 2if\rho\sigma\tau Q^{\tau}$$

$$[Q^{\alpha}, Q^{\beta}] = 2ig^{\alpha\beta\gamma} Q^{\gamma}$$

where g 's are the structure constants of E_6 .

$$[Q^{\rho}, Q^{\alpha}] = 0$$

$$[Q^{\rho}, Q^{ri}] = -(Q^{\rho})^r_s Q^{si}$$

$$[Q^{\rho}, Q_{ri}] = (Q^{\rho})^s_r Q_{si}$$

$$[Q^{\alpha}, Q^{ri}] = -(Q^{\alpha})^i_j Q^{rj}$$

$$[Q^{\alpha}, Q_{ri}] = (Q^{\alpha})^j_i Q_{rj}$$

$$[Q^{ri}, Q^{sj}] = \sqrt{(5/7)} \epsilon^{rst} d^{ijk} Q_{tk} ; d^{ijk} \text{'s are totally symmetric.}$$

$$[Q_{ri}, Q_{sj}] = -\sqrt{(5/7)} \epsilon_{rst} d_{ijk} Q^{tk}$$

$$[Q^{ri}, Q_{sj}] = -(Q^{\rho})^r_s \delta^i_j Q^{\rho} - \delta^r_s (Q^{\alpha})^i_j Q^{\alpha}$$

We note the normalization conventions used:

$$\text{Tr}(Q_{\rho} Q^{\sigma}) = 2\delta_{\rho}^{\sigma} ; \text{Tr}(Q_{\alpha} Q^{\beta}) = 12\delta_{\alpha}^{\beta} ; \text{Tr}(Q_{ri} Q^{sj}) = 2\delta_r^s \delta_i^j.$$

Appendix 4

Non trivial commutation relations of E_8 according to the decomposition (4.8):

$$\begin{aligned}
 [Q^P, Q^\sigma] &= 2i\epsilon^{P\sigma\tau} Q^\tau \\
 [Q^\alpha, Q^\beta] &= 2ig^{\alpha\beta\gamma} Q^\gamma \\
 [Q(0), Q^a(3)] &= \sqrt{3} Q^a(3) \\
 [Q(0), Q_a(-3)] &= -\sqrt{3} Q_a(-3) \\
 [Q^a(3), Q_b(-3)] &= \sqrt{3} \delta^a_b Q(0) - (Q^P)^a_b Q^P \\
 [Q^P, Q^a(3)] &= -(Q^P)^a_b Q^b(3) \\
 [Q^P, Q_a(-3)] &= (Q^P)^b_a Q_b(-3) \\
 [Q(0), Q^j(-2)] &= -(2/\sqrt{3}) Q^j(-2) \\
 [Q(0), Q_j(2)] &= (2/\sqrt{3}) Q_j(2) \\
 [Q(0), Q^{aj}(1)] &= (1/\sqrt{3}) Q^{aj}(1) \\
 [Q(0), Q_{aj}(-1)] &= -(1/\sqrt{3}) Q_{aj}(-1) \\
 [Q^a(3), Q^j(-2)] &= \sqrt{2} Q^{aj}(1) \\
 [Q_a(-3), Q_j(2)] &= -\sqrt{2} Q_{aj}(-1) \\
 [Q_a(-3), Q^{bj}(1)] &= \sqrt{2} \delta_a^b Q^j(-2) \\
 [Q^a(3), Q_{bj}(-1)] &= -\sqrt{2} \delta^a_b Q_j(2) \\
 [Q^P, Q^{aj}(1)] &= -(Q^P)^a_b Q^{bj}(1) \\
 [Q^P, Q_{aj}(-1)] &= (Q^P)^b_a Q_{bj}(-1) \\
 [Q^\alpha, Q^i(-2)] &= -(Q^\alpha)^i_j Q^j(-2) \\
 [Q^\alpha, Q_i(2)] &= (Q^\alpha)^j_i Q_j(2) \\
 [Q^\alpha, Q^{ai}(1)] &= -(Q^\alpha)^i_j Q^{aj}(1) \\
 [Q^\alpha, Q_{ai}(-1)] &= (Q^\alpha)^j_i Q_{aj}(-1) \\
 [Q^i(-2), Q^{aj}(1)] &= -\sqrt{(5/7)} \epsilon^{abd} ijk Q_{bk}(-1); \quad \epsilon^{12} = -\epsilon^{21} = 1. \\
 [Q_i(2), Q_{aj}(-1)] &= \sqrt{(5/7)} \epsilon^{abd} ijk Q^{bk}(1) \\
 [Q^{ai}(1), Q^{bj}(1)] &= \sqrt{(5/7)} \epsilon^{abd} ijk Q_k(2) \\
 [Q_{ai}(-1), Q_{bj}(-1)] &= -\sqrt{(5/7)} \epsilon^{abd} ijk Q^k(-2) \\
 [Q^i(-2), Q_j(2)] &= -(2/\sqrt{3}) \delta^i_j Q(0) - (Q^\alpha)^i_j Q^\alpha \\
 [Q^i(-2), Q_{aj}(-1)] &= \sqrt{2} \delta^i_j Q_a(-3) \\
 [Q_i(2), Q^{aj}(1)] &= -\sqrt{2} \delta^i_j Q^a(3) \\
 [Q^{ai}(1), Q_{bj}(-1)] &= (1/\sqrt{3}) \delta^i_j \delta^a_b Q(0) - \delta^a_b (Q^\alpha)^i_j Q^\alpha - \delta^i_j (Q^P)^a_b Q^P
 \end{aligned}$$

The normalization conventions are:

$$\text{Tr}(Q_p Q^\sigma) = 2\delta_p^\sigma; \text{Tr}(Q_\alpha Q^\beta) = 12\delta_\alpha^\beta; \text{Tr}(Q_i(2) Q^j(-2)) = 2\delta_i^j; \text{Tr}(Q_{ai}(1) Q^{bj}(-1)) = 2\delta_i^j.$$

Appendix 5

We note certain relations concerning the Clebsch-Gordan coefficients of E_6 . The following relations hold⁽²²⁾ for d^{ijk} appearing in eq.(4.4) and in the appendices 3 and 4:

$$d^{ijk}d_{k1j}=\alpha\delta^j_1 \quad (A5.1)$$

$$d^{ijk}d_{k1m}=\beta(\delta^i_1\delta^j_m+\delta^i_m\delta^j_1) \quad (A5.2)$$

Consistency of equations (A5.1) and (A5.2) requires that:

$$\alpha=28\beta \quad (A5.3)$$

while there is an arbitrariness in the normalization. We recall⁽²²⁾ the relation:

$$\Sigma(1/\alpha)(Q^p)^i_k(Q^p)^j_1=(1/6)\delta^i_1\delta^j_k+(1/18)\delta^i_k\delta^j_1-(5/3\alpha)d^{ijm}d_{mk1} \quad (A5.4)$$

which, using equations (A5.2) and (A5.3) and with $\alpha=12$, gives:

$$\Sigma(Q^p)^i_k(Q^p)^j_1=(9/7)(\delta^i_1\delta^j_k-(1/27)\delta^i_k\delta^j_1) \quad (A5.5)$$

Appendix 6

We present the non trivial commutation relations of the generators of Sp_4 according to the decomposition:

$$Sp_4 \supset SU_2 \times U_1$$

$$10 = 3(0) + 1(0) + 2(1) + 2(-1) + 1(2) + 1(-2)$$

The above decomposition suggests the introduction of the following generators of Sp_4 :

$$T = \{T^i, T_0, T_{a+}, T^{a-}, T_+, T_-\} ; i=1,2,3; a=1,2$$

$$[T^i, T^j] = 2i\epsilon^{ijk}T^k$$

$$[T^j, T^{a-}] = -(T^j)_a^b T^{b-}$$

$$[T^j, T_{a+}] = (T^j)_a^b T_{b+}$$

$$[T_0, T_{a+}] = (1/\sqrt{2})T_{a+}$$

$$[T_0, T^{a-}] = -(1/\sqrt{2})T^{a-}$$

$$[T_0, T_+] = \sqrt{2}T_+$$

$$[T_0, T_-] = -\sqrt{2}T_-$$

$$[T_{a+}, T^{b-}] = (T^i)_a^b T^i + (1/\sqrt{2})\delta_a^b T_0$$

$$[T_{a+}, T_-] = -\epsilon_{ab}T^{b-}$$

$$[T_{a+}, T_{b+}] = \epsilon_{ab}T_+$$

$$[T^{a-}, T^{b-}] = -\epsilon^{ab}T_-$$

$$[T^{a-}, T_+] = \epsilon^{ab}T_{b+}$$

$$[T_+, T_-] = \sqrt{2}T_0$$

The normalization conventions are as follows:

$$\text{Tr}(T_i T_j) = \delta_{ij}; \text{Tr}(T_0^2) = 2; \text{Tr}(T_{a+} T^{b-}) = \delta_a^b; \text{Tr}(T_+ T_-) = 1.$$

Appendix 7

We present the non trivial commutation relations of SU_4 according to the decomposition:

$$SU_4 \supset (SU_2 \times U_1) \times U_1$$

$$15 = 3(0,0) + 2(3,0) + 2(-3,0) + 1(0,0) + 2(1,4) + 1(-2,4) + 2(-1,-4) + 1(2,-4) + 1(0,0)$$

The above decomposition suggests the introduction of the following generators of SU_4 :

$$Q = \{Q^j, Q_a(3), Q^a(-3), Q_l(0), Q_a(+), Q(-), Q^a(-), Q(+), Q_{ll}(0)\} ; j=1,2,3; a=1,2.$$

$$[Q^i, Q^j] = 2i\epsilon^{ijk}Q^k$$

$$[Q^j, Q_a(3)] = (Q^j)^b_a Q_b(3)$$

$$[Q^j, Q^a(-3)] = -(Q^j)^a_b Q^b(-3)$$

$$[Q^j, Q_a(+)] = (Q^j)^b_a Q_b(+)$$

$$[Q^j, Q^a(-)] = -(Q^j)^a_b Q^b(-)$$

$$[Q_l(0), Q_a(3)] = 3(1/\sqrt{2})Q_a(3)$$

$$[Q_l(0), Q^a(-3)] = -3(1/\sqrt{2})Q^a(-3)$$

$$[Q_l(0), Q_a(+)] = (1/\sqrt{2})Q_a(+)$$

$$[Q_l(0), Q^a(-)] = -(1/\sqrt{2})Q^a(-)$$

$$[Q_l(0), Q(+)] = \sqrt{2}Q(+)$$

$$[Q_l(0), Q(-)] = -\sqrt{2}Q(-)$$

$$[Q_{ll}(0), Q_a(+)] = \sqrt{(4/3)}Q_a(+)$$

$$[Q_{ll}(0), Q^a(-)] = -\sqrt{(4/3)}Q^a(-)$$

$$[Q_{ll}(0), Q(+)] = -\sqrt{(4/3)}Q(+)$$

$$[Q_{ll}(0), Q(-)] = \sqrt{(4/3)}Q(-)$$

$$[Q_a(3), Q^b(-3)] = (Q^j)^b_a Q^j + (1/\sqrt{2})\delta^b_a Q_l(0)$$

$$[Q_a(3), Q^b(-)] = -\delta^b_a Q(+)$$

$$[Q_a(3), Q(-)] = Q_a(+)$$

$$[Q^a(-3), Q_b(+)] = \delta^a_b Q(-)$$

$$[Q^a(-3), Q(+)] = -Q^a(-)$$

$$[Q_a(+), Q^b(-)] = (Q^j)^b_a Q^j + \sqrt{(1/18)}\delta^b_a Q_l(0) + \sqrt{(4/3)}\delta^b_a Q_{ll}(0)$$

$$[Q_a(+), Q(+)] = Q_a(3)$$

$$[Q^a(-), Q(-)] = -Q^a(-3)$$

$$[Q(+), Q(-)] = \sqrt{(2/9)}Q_1(0) - \sqrt{(4/3)}Q_{11}(0)$$

The normalization conditions are as follows:

$$\text{Tr}(Q_i Q_j) = \delta_{ij}; \text{Tr}\{(Q_1(0))^2\} = 1; \text{Tr}\{(Q_{11}(0))^2\} = 1; \text{Tr}(Q_a(+))Q^b(-) = \delta_a^b;$$

$$\text{Tr}(Q(+))Q(-) = 1; \text{Tr}(Q_a(3))Q^b(-3) = \delta_a^b.$$

We now present the commutation relations of G_2 in the $SU_2 \times U_1$ basis. The decomposition.

$$G_2 \supset SU_2 \times U_1$$

$$14 = 3(0) + 2(3) + 2(-3) + 1(0) + 2(1) + 1(-2) + 2(-1) + 1(2)$$

suggests writing the G_2 generators as:

$$Q = \{Q^j, Q_a(3), Q^a(-3), Q(0), Q_a(1), Q(-), Q^a(-1), Q(+)\} ; j=1,2,3, a=1,2.$$

The non vanishing commutators are:

$$[Q^i, Q^j] = 2i\epsilon^{ijk}Q^k$$

$$[Q^j, Q_a(1)] = (Q^j)^b_a Q_b(1)$$

$$[Q^j, Q^a(-1)] = -(Q^j)^a_b Q^b(-1)$$

$$[Q^j, Q_a(3)] = (Q^j)^b_a Q_b(3)$$

$$[Q^j, Q^a(-3)] = -(Q^j)^a_b Q^b(-3)$$

$$[Q(0), Q_a(1)] = (1/\sqrt{2})Q_a(1)$$

$$[Q(0), Q^a(-1)] = -(1/\sqrt{2})Q^a(-1)$$

$$[Q(0), Q_a(3)] = 3(1/\sqrt{2})Q_a(3)$$

$$[Q(0), Q^a(-3)] = -3(1/\sqrt{2})Q^a(-3)$$

$$[Q(0), Q(+)] = \sqrt{2}Q(+)$$

$$[Q(0), Q(-)] = -\sqrt{2}Q(-)$$

$$[Q_a(1), Q^b(-1)] = (Q^j)^b_a Q^j + \sqrt{(1/18)}\delta_a^b Q(0)$$

$$[Q_a(1), Q_b(1)] = -\sqrt{(4/3)}\epsilon_{ab}Q(+)$$

$$[Q_a(1), Q(+)] = Q_a(3)$$

$$[Q_a(1), Q(-)] = \sqrt{(4/3)}\epsilon_{ab}Q^b(-1)$$

$$[Q_a(1), Q^b(-3)] = -\delta_a^b Q(-)$$

$$[Q^a(-1), Q^b(-1)] = \sqrt{(4/3)}\epsilon^{ab}Q(-)$$

$$[Q^a(-1), Q(-)] = -Q^a(-3)$$

$$[Q^a(-1), Q(+)] = -\sqrt{(4/3)}\epsilon^{ab}Q_b(1)$$

$$[Q_a(3), Q^b(-1)] = -\delta_a^b Q(+)$$

$$[Q_a(3), Q(-)] = Q_a(1)$$

$$[Q(+), Q(-)] = \sqrt{2/9} Q(0)$$

Our normalization conditions are, in this case:

$$\text{Tr}(Q_i Q_j) = \delta_{ij}; \quad \text{Tr}((Q(0))^2) = 3; \quad \text{Tr}(Q_a(3) Q^b(-3)) = \delta_{ab}; \quad \text{Tr}(Q_a(1) Q^b(-1)) = \delta_{ab};$$

$$\text{Tr}(Q(+) Q(-)) = 1.$$

References

1. Th.Kaluza, Sber., Preuss. Acad. Wiss. K1(1921)966;
O.Klein, Zeit. Phys. 37(1926)895.
2. Z.Horvath, L.Palla, E.Cremmer, J.Scherk, Nucl. Phys. B127(1977)205.
3. M.Green, J.Schwarz, Phys. Lett. 149B(1984)117.
4. E.Witten, Phys. Rev. Lett. 38(1977)121.
5. P.Forgacs and N.S.Manton, Comm. Math. Phys. 72(1980)15.
6. N.S.Manton, Nucl. Phys. B193(1981)502.
7. G.Chapline and N.S.Manton, Nucl. Phys. B184(1981)391.
8. G.Chapline and R.Slansky, Nucl. Phys. B209(1982)461.
9. D.Olive and P.West, Nucl. Phys. B217(1983)248.
10. P.Forgacs and G.Zoupanos, Phys. Lett. 148B(1984)99.
11. F.A.Bais, K.J.Barnes, P.Forgacs and G.Zoupanos, Nucl. Phys. B263(1986)557;
proceedings of H.E.P. '85(Bari).
12. D.Lüst and G.Zoupanos, Phys. Lett. 165B(1985) 309.
13. K.J.Barnes, M.Surridge, Southampton Univ. preprint SHEP 85/86-7(1985), to
be published in Nucl. Phys. B;
K.J.Barnes, R.C.King, M.Surridge, Southampton Un. preprint SHEP 85/86-1(1985),
to be published in Z.f. Phys. C.
14. K.J.Barnes, P.Forgacs, M.Surridge and G.Zoupanos, "On fermion masses in a
dimensional reduction scheme", Southampton Univ. preprint, 85/86-23,
to be published in Z.f. Physik C.
15. A.N.Schellekens, Nucl. Phys. B248(1984)706;
S.Randjbar-Daemi, A.Salam, J.Strathdee, Phys. Lett. 124B(1983)345;
P.Forgacs, Z.Horvath, L.Palla, Phys. Lett. 147B(1984)311; Z. Phys. C30(1986)261;
E.Cremmer and J.Scherk, Nucl. Phys. B103(1976)339; B108(1976)409;
B118(1977)61;
J.F.Luciani, Nucl. Phys. B135(1978)111;
P.Forgacs and Z.Horvath, Gen. Rel. Grav. 10(1979)93; 11(1979)20.
16. D.Lüst, P.Forgacs and G.Zoupanos, proceedings of 2nd Hellenic School in H.E.P.
(Corfu 1985).
17. J.Harnad, S.Schnider and J.Tafel, Lett. Math. Phys. 4(1980)41.
18. L.F.Li, Phys. Rev. D9(1974)1723.
19. D.Lüst, Caltech preprint 68-1329, to be published in Nucl. Phys. B. and 68-1345;
L.Castellani and D.Lüst, Caltech preprint 68-1353;

- B.P.Dolan,D.Dunbar,R.G.Moorhouse and A.B.Henriques,"Torsion in Modified Ten Dimensional Supergravity",Glasgow Univ. preprint,1986.
- B.P.Dolan and D.Dunbar,"Compactifying Solutions to an extended Chapline-Manton Lagrangian", Glasgow Univ. preprint '86.
20. C.Michel,CERN-TH-2716(1979),contribution to A.Visconti seminar.
21. Y.Hosotani,Phys.Lett.129B(1983)193;
J.D.Breit,B.Ovrut and G.Segré,Phys.Lett.158B(1985)33;
F. del Aguila,G.Blair,M.Daniel and G.G.Ross,Nucl.Phys.B272(1986)413.
22. P.Cvitanovic,Phys.Rev.D14(1976)1536;
T.W.Kephart and M.T.Vaughn,Ann.Phys.145(1983)162.