SUPER-KÄHLER GEOMETRY IN SUPERGRAVITY AND SUPERSTRINGS

S. Cecotti ^{*)}
Dipartimento di Fisica, Università di Pisa
and INFN, Sezione di Pisa

S. Ferrara ^{**)}
CERN - Geneva
and
Department of Physics,
University of California, Los Angeles

L. Girardello ⁺⁾
Dipartimento di Fisica, Università di Milano
and INFN, Sezione di Milano

and
M. Porrati ⁺⁺⁾
Department of Physics,
University of California, Los Angeles

A B S T R A C T

We show that only a restricted class of $N=1$ four-dimensional curvature terms are compatible with super-Kähler geometry. The scaling properties required from Kähler invariance in order to be compatible with local supersymmetry in higher-derivative interactions are shown in the case of superstrings to be related to the global scale invariance of the ten-dimensional effective action.

^{*)} Address after 1 November 1986: Theory Division, CERN, Geneva, Switzerland.

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⁺⁾ Partially supported by the Ministry of Public Instruction (MPI), Italy.

⁺⁺⁾ On leave of absence from Scuola Normale Superiore and INFN, Sezione di Pisa, Italy.

One of the basic geometric properties of $N=1$ four-dimensional σ -models, both in global and local supersymmetry, is their Kählerian structure [1, 2].

The Kähler structure, however, is a consequence of supersymmetry only when high-derivative terms are not present in the basic action.

In superstring effective actions for a $D=4$, $N=1$ supersymmetric compactification, higher-curvature terms of any power do arise from the α' expansion [3] (α' is the Regge slope related to the string tension).

The question arises whether such terms are compatible with the Kähler structure of the lowest-order standard supergravity Lagrangian [4].

In this paper we show that super-Kähler invariance imposes severe restrictions on possible higher-curvature terms. In particular, it requires certain scaling properties among the Kähler potential and polynomials in the Lorentz and Yang-Mills curvature of a given degree. These scaling properties, in the case of superstrings, turn out to be a consequence of the global scale covariance of the ten-dimensional effective action under a shift of the dilaton $\phi \rightarrow \phi + c$ [5].

Let us denote by $Z_i = (z_i, \chi_{Li}, h_i)$ a set of chiral multiplets: z_i are complex scalar fields; χ_{Li} are left-handed fermions; h_i are 'auxiliary fields'. In global supersymmetry the σ -model dynamics is described by a real scalar superfield $\Phi(z_i, \bar{z}_i)$, whose first component $\Phi(z_i, z_i^*)$ is the Kähler potential.

Because of supersymmetry, the Kähler invariance

$$\Phi(z_i, z_i^*) \rightarrow \Phi(z_i, z_i^*) + \Lambda(z_i) + \Lambda^*(z_i^*) \quad (1)$$

extends to a super-Kähler invariance

$$\Phi(z_i, \bar{z}_i) \rightarrow \Phi(z_i, \bar{z}_i) + \Lambda(z_i) + \bar{\Lambda}(\bar{z}_i), \quad (2)$$

where $\Lambda(z_i)$ is a super-Kähler (chiral) parameter.

The σ -model Lagrangian is nothing but the last (D-component) of the Φ superfield [1]. In superspace notation the action is

$$S_1 = \int d^4x d^4\theta \Phi \quad (3)$$

and is manifestly invariant under the transformation (2) since

$$\int d^4x d^4\theta (\lambda + \bar{\lambda}) = 0. \quad (4)$$

If we add to the previous action a 'superpotential term'

$$S_2 = \text{Re} \int d^4x d^2\theta g(Z_i), \quad (5)$$

the action is still Kähler-invariant, although S_2 can break some of the isometries of the Kähler manifold.

At this point it should be stressed that by Kähler invariance we mean invariance under the gauge transformation given by Eq. (2). In general, when higher-derivative interactions are present, the Kähler invariance will not imply that the geometry is Kählerian.

In local supersymmetry the situation is quite different. In this case the super-Kähler transformation also acts on the supergravity multiplet as well as on the superpotential [4].

Let us recall, for the reader's convenience, some basics of superspace geometry.

Curved superspace is best described in terms of superconformal geometry [6, 7]. The key ingredients are the Weyl gauge supermultiplet containing the (unconstrained) gauge fields of the $N=1$ superconformal algebra and a chiral compensator S_0 , whose last component is the (complex) scalar auxiliary field of the 'minimal' formulation of $N=1$ supergravity.

Conformal supersymmetry requires any D-density to have Weyl weight two (and chiral weight zero) and any F-density to have Weyl weight three.

In the absence of a superpotential, the standard supergravity action coupled to a non-linear σ -model can be written as

$$\int d^8 \Omega \, \Phi(z_i, \bar{z}_i) S_0 \bar{S}_0, \quad (6)$$

where the Z_i (physical) multiplets are assigned to have Weyl weight zero (the compensator S_0 has Weyl weight one); $d^8 \Omega$ is an appropriate superspace measure.

The action (6) has a superconformal (i.e. dilation + R) invariance,

$$\begin{aligned} \Phi &\longrightarrow e^{-\frac{1}{3}(\Lambda + \bar{\Lambda})} \Phi \\ S_0 &\longrightarrow S_0 e^{\Lambda/3} \quad \bar{S}_0 \longrightarrow \bar{S}_0 e^{\bar{\Lambda}/3}. \end{aligned} \quad (7)$$

These equations have a simple meaning: if we think of S_0 as an analytic object living on the Kähler manifold, the 'compensating field' S_0 is not a field at all-- rather, it is a superfield-valued section of a given line bundle L . The 'function' Φ , which must be positive in the physical domain, is then the corresponding fibre metric on L .

The usual form representing the Chern class of the bundle L^3 ,

$$\frac{i}{2\pi} \partial \bar{\partial} (3 \ln \Phi), \quad (8)$$

turns out to be proportional to the Kähler form of the σ -model, which then is a Hodge manifold [2].

If we include a superpotential term

$$\text{Re} \int d^6 \Omega \, g(z_i) S_0^3, \quad (9)$$

we see that g must transform as

$$g(z_i) \longrightarrow g(z_i) e^{-\Lambda}, \quad (10)$$

that is, $g(z_i)$ is a superfield-valued section

$$g(z_i) \in \Gamma(\mathcal{O}(L^{-3})). \quad (11)$$

In the case of canonical supergravities (no higher curvature term), since the Lagrangian is globally defined it must depend only on the global functions. The only such function is the square-norm of the section g , i.e. $\|g\|_L^2 = \phi^{-3} |g|^2$; then the Lagrangian can be written in terms of the function

$$G = \ln \|g\|_L^2 \quad (12)$$

alone. In string theory, the situation is more subtle. In the compactification of the superstring [18] the σ -model is a *product* manifold,

$$\Delta \otimes B_{1+n}$$

(Δ is the unit disk and B_{1+n} is the open unit ball in C^{n+1} *) and the function ϕ is also a product,

$$\Phi = \Phi_1^{1/3} \tilde{\Phi},$$

where ϕ_1 and $\tilde{\phi}$ are exactly the two functions which cause Δ and B_{1+n} , separately, to be flat models. Now, let us consider the natural projection

*) In the language of Ref. (18), Δ refers to the S field and B_{1+n} to the T and C charged fields, respectively.

$$\pi : \Delta \otimes B_{1+n} \longrightarrow B_{1+n}.$$

It is easy to see that there is a line bundle

$$\tilde{L} \longrightarrow B_{1+n}$$

such that

$$L = \pi^* \tilde{L}.$$

There are geometrical reasons for requiring that the section of L given by the superpotential g is also the pullback of a section of $\tilde{L}$ on B_{1+n} ; so now we require Kähler invariance only with respect to changes of local trivializations in B_{1+n} .

Obviously $\tilde{\Phi}^{-3}$ is the fibre metric for the bundle

$$\tilde{L}^{-3} \longrightarrow B_{1+n}.$$

So--in the string case--we have two Kähler-invariant functions, in this new sense, both of which can a priori appear in the Lagrangian:

$$\begin{aligned} G &= \ln \|g\|^2_{\pi^* \tilde{L}} \\ \tilde{G} &= \pi^* \ln \|g\|^2_{\tilde{L}}, \end{aligned} \tag{13}$$

where

$$\|g\|^2_{\tilde{L}} = \tilde{\Phi}^{-3} |g|^2.$$

Moreover, if we do not fix a conformal gauge, we have another two global functions:

$$\|s_0\|_L^2 = \bar{\Phi} |s_0|^2 \quad (14)$$

$$S_0^3 g(z;) . \quad (15)$$

In superstring theories the superpotential is generated by the ten-dimensional Yang-Mills fields and contains charged fields. It vanishes if these fields are set equal to zero. Only interactions which are non-singular in the $g \rightarrow 0$ limit are therefore acceptable.

Other possible super-Kähler invariants may in general be obtained using the super-Kähler Riemann tensor. However, in the particular case of a coset space Kähler manifold, there are no new invariants since the holomorphic sectional curvature is constant. We will not consider here the more general situation in which the projective curvature tensor is non-trivial.

This situation changes dramatically when we add higher derivative terms. In fact these terms have non-canonical Weyl and chiral weights which need to be compensated by the insertion of powers of S_0 and $\bar{S}_0$; that is, in order to be interpreted in Kählerian geometric terms, they must be sections of the bundles

$$\begin{array}{ll} L^{-1+(W+C)} \otimes \bar{L}^{-1+(W-C)} & \text{VECTOR DENSITIES} \\ L^{-3+W} & \text{CHIRAL DENSITIES} \end{array}$$

and so the number of would-be global 'functions', which *a priori* may enter the Lagrangian, grows enormously.

However, the odd thing about the situation is that these new global objects are not--in general--even locally functions of the σ -model coordinates, since their first components need not be coordinate functions. Then their interpretation as sections is inconsistent, and we lose the Hodge-Kähler geometric structure.

The moral of the story is that only appropriate combinations of the chiral

compensator, Kähler potential, and superpotential can appear in higher-curvature invariants.

Of particular interest are those interaction terms which come from the extension of the standard supergravity Lagrangian to higher-curvature terms such as one may obtain from the effective action of the massless modes of superstrings.

It is known that some of these terms come from the anomaly cancellation mechanism of 10D supergravity [9].

Higher curvature terms, up to four curvatures, have been computed using σ -model β -function calculations and S-matrix superstring techniques [5, 10].

The basic curvature multiplets in N=1 supergravity are the (chiral) Yang-Mills field strength W_α and the Weyl curvature multiplet $W_{\alpha\beta\gamma}$ [11]^{*} (α, β, γ are Weyl spinor indices). There is also a Kähler supercurvature,

$$W_\alpha^K = W_\alpha (V = \log \|S\|_L^2),$$

which contains the pullback of the Kähler form $ig_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}$ and the trace of the first fundamental form $g_{i\bar{j}} dz^i \otimes d\bar{z}^{\bar{j}}$.

This would give rise to higher-derivative interactions in the matter fields. Our later considerations will apply to this curvature as well.

The quadratic invariant expressions are [4, 13]

$$\text{Re} \int d^6 \Omega C_2^{YM} \text{Tr} W_\alpha^2 \quad (16)$$

$$\text{Re} \int d^6 \Omega C_2^W W_{\alpha\beta\gamma}^2, \quad (17)$$

where C_2^{YM} , C_2^W are chiral functions of Weyl weight zero. Super-Kähler invariance of formulae (16) and (17) follows from superconformal invariance.

For the fourth-order curvature terms, the super-Kähler-invariant expressions are

^{*}) Supermultiplets containing the Einstein tensor or the curvature scalar can be removed in high-curvature invariants by appropriate field redefinition of the metric [12].

$$\int d^8 \Omega C_4^{YH} W_\alpha^2 \bar{W}_{\dot{\alpha}}^2 (S_0 \bar{S}_0 \Phi)^{-2} \quad (18)$$

$$\int d^8 \Omega C_4^W W_{\alpha\beta\gamma}^2 \bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}^2 (S_0 \bar{S}_0 \Phi)^{-2} \quad (19)$$

$$\text{Re} \int d^8 \Omega C_4^{YH, W} W_\alpha^2 \bar{W}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}^2 (S_0 \bar{S}_0 \Phi)^{-2}, \quad (20)$$

where C_4 are general superfields of weight zero (which are Kähler-inert).

Equations (18) to (20) can be generalized to any even power of the curvature multiplets. Let us introduce an operation Γ which converts a general (vector) multiplet V into a chiral multiplet [6, 7]:

$$\Sigma(V), \quad (21)$$

which is the generalization of the $\bar{D}\bar{D}$ projection to the case of local supersymmetry. In order for $\Gamma(V)$ to be a superconformal chiral multiplet (with equal Weyl and chiral weights $w + 1$), V must have chiral weight $w - 2$. This is because the Γ operator has Weyl weight one and chiral weight three.

Define now the following chiral multiplet:

$$\Sigma_{m+i\bar{n}} = \Sigma \left[\Sigma_m \otimes \bar{\Sigma}_{\bar{n}} (S_0 \bar{S}_0 \Phi)^{-2} \right], \quad (22)$$

where

$$\Sigma_1 = W^2 \quad (23)$$

and m and n are integral-valued complex numbers^{*}); $\Gamma_{m+i\bar{n}}$ is a chiral superfield with a discrete complex index $m + i\bar{n}$; Γ_{1+i} gives the quartic invariants in formulae (18) to (20).

A super-Kähler-invariant action is given as follows:

$$\int d^8 \Omega \sum_{m+i\bar{m} \in \mathbb{Z}[i]} (\Phi S_0 \bar{S}_0)^{-2} \sum_m \bar{\Sigma}_m C_{m+i\bar{m}}, \quad (24)$$

where $C_{m+i\bar{n}}$ are Kähler-invariant superfields of Weyl (and chiral) weight zero. Note that Γ_m contains all even powers of Yang-Mills and Weyl field strengths. Expression (24) has the property of having, among the bosonic components, terms of the form

$$C_p \Phi^{2-p} R^p \sqrt{g}, \quad (25)$$

where $p = 2(\text{Re } m + \text{Im } m)$; R^p generically denotes a polynomial of degree p in the Lorentz, matter and Yang-Mills curvatures.

We note that if $C_{m+i\bar{n}}$ is not a constant, then Eq. (24) is not unique since Eq. (22) can be generalized in an obvious way including non-trivial super-Kähler-invariant superfields under the Γ operation.

We notice that Eq. (25) has the special property that the Φ dependence can be eliminated altogether if we rescale the metric so that the Einstein ($p=1$) term has the canonical form

$$C_p R^p \sqrt{g}. \quad (26)$$

^{*}) Note that Γ_m is well defined by the recursive law (22) since

$$\Sigma [\Sigma_{m+p} \bar{\Sigma}_m (\Phi S_0 \bar{S}_0)^{-2}] = \Sigma [\Sigma_{m+i\bar{n}} \bar{\Sigma}_{i\bar{p}} (\Phi S_0 \bar{S}_0)^{-2}]$$

provided both sides make sense.

Equation (26) is a consequence of the four-dimensional Weyl-rescaling property

$$\sqrt{g} R^p \longrightarrow \sqrt{g} R^p \Omega^{2(2-p)}, \quad g_{\mu\nu} \longrightarrow \Omega^2 g_{\mu\nu}. \quad (27)$$

We want now to show that Eq. (26) just follows from the ten-dimensional scaling property of the superstring effective action. For this purpose let us define the S and T chiral multiplets such that [18]

$$\text{Re } S = e^{+3\sigma} e^{-\frac{1}{2}\phi} \quad (28)$$

$$\text{Re } t - cc^* = e^{\sigma} e^{\frac{1}{2}\phi} = \tilde{\Phi}. \quad (29)$$

Here σ is the compactification mode

$$g_{(10)\mu\nu} = e^{-3\sigma} g_{\mu\nu}$$

$$g_{(10)ij} = e^{\sigma} \delta_{ij}.$$

The scaling property of the ten-dimensional effective Lagrangian under $\phi \rightarrow \phi + c$ implies the form [5]

$$\mathcal{L}_{10} = e^{-2\phi} \left[\sqrt{g} R + \sum_p \sqrt{g} c_p R^p + \dots \right]$$

(where ... indicates terms which contain $\partial_\mu \phi$ and the antisymmetric field strength $H_{\mu\nu\rho}$).

Consider now the simplest dimensional reduction of a general curvature term in $D=10$:

$$e^{-2\phi} c_p R^p \sqrt{g}. \quad (30)$$

Under a Weyl rescaling which brings the Einstein term to a canonical form, we have

$$e^{-2\phi} \sqrt{g} R^P \longrightarrow e^{\frac{4(1-P)}{D-2}\phi} \sqrt{g} R^P \quad (31)$$

for $D=10$; using Eqs. (28) and (29) we get

$$e^{3\sigma} e^{\frac{1}{2}\phi(1-P)} \sqrt{\det g_{10\mu\nu}} R_4^P \quad (32)$$

and by further four-dimensional rescaling, $g_{10\mu\nu} \rightarrow e^{-3\sigma} g_{\mu\nu}$, we finally get

$$e^{\frac{1}{2}\phi(1-P)} e^{-3\sigma(1-P)} \sqrt{g} R_4^P = (\text{Res})^{P-1} \sqrt{g} R^P. \quad (33)$$

The fact that only $\text{Re } s$ (rather than ϕ) enters in Eq. (33) allows the super-Kähler completion of this term as given by Eq. (24).

In superstring theories the $\tilde{\Phi}$ function whose first component is given by Eq. (29) is related to the Kähler ϕ function as follows:

$$\Phi = \tilde{\Phi} (S + \bar{S})^{1/3}, \quad (34)$$

where S is the chiral multiplet whose first scalar component is given by Eq. (28). The pseudoscalar component is related to the antisymmetric tensor field $B_{\mu\nu}$ through a duality transformation^{*}). If we demand Kähler invariance under S -dependent super-Kähler transformations, the ϕ function should appear in Eq. (22). In the case where we require only invariance in the restricted sense of Eq. (13), the superinvariant expression giving rise to Eq. (25) is not uniquely defined because powers of the $S + \bar{S}$ superfield can be put inside or outside the Γ operation in Eq. (22).

^{*}) A supersymmetric duality transformation introduces the so-called 'linear multiplet' [14] which is the multiplet containing the field strength of the 4D antisymmetric tensor field $B_{\mu\nu}$ [15]. The duality-transformed theory has a restricted Kähler structure with respect to the $\tilde{\Phi}$ function.

If in Eq. (22) we choose the full Kähler function Φ , then the coefficient $C_{m+i\bar{n}}$ in Eq. (24) contains the power $(S + \bar{S})^{p-1}$.

Similarly we may consider terms involving the $B_{\mu\nu}$ curvature. In $D=10$ dimensions they are of the form

$$e^{-2\Phi} c_{pq} R^p H^{2q} \sqrt{g}. \quad (35)$$

We confine ourselves first to the four-dimensional spatial $B_{\mu\nu}$ curvature. After a dimensional reduction and the same four-dimensional rescaling as before, we get

$$c_{pq} (\text{Re } s)^{p-1+3q} \sqrt{g} R^p H^{2q}. \quad (36)$$

A super-Kähler-invariant expression containing, among the bosonic components, terms as in formula (36) is obtained by the same formulae as in (22) to (24), where now

$$\Sigma_{\perp} = \Sigma(\bar{\Phi} s_0 \bar{s}_0) \text{ or } W^2 \quad (37)$$

and $C_{m+i\bar{n}}$ is now $(S + \bar{S})^{p-1+q}$.

Let us finally consider the situation where superpotential terms are present.

If we add to Eq. (24) the superpotential term Eq. (10), the resulting theory is super-Kähler-invariant. However, because of formulae (14) and (15) we can construct higher-curvature interactions with powers of the super-Kähler-invariant combination $|g(z_i)|^{2/3}/\Phi$.

This is precisely what happens if we dimensionally reduce expression (35), but now with H being the internal curvature H_{ijk} . After dimensional reduction, we have [18]

$$H_{\text{INT}}^{2q} \sim \left(\frac{|g|^2}{\Phi^3} \right)^q, \quad (38)$$

so that we get

$$(Re s)^{p-1+q} e^{qG(z,z^*)} R^p \sqrt{g}, \quad (39)$$

which is a manifestly Kähler-invariant expression.

It should be remembered that all the terms we have been writing respect the two symmetries [16]

$$g_{\mu\nu} \rightarrow t^2 g_{\mu\nu}, \quad S \rightarrow t^2 S, \quad \tilde{\Phi} \rightarrow \tilde{\Phi} \quad (40)$$

$$g_{\mu\nu} \rightarrow g_{\mu\nu}, \quad S \rightarrow r^{-1/2} S, \quad \tilde{\Phi} \rightarrow r^{1/2} \tilde{\Phi}, \quad k^2 \rightarrow r^{1/2} k^2, \quad c \rightarrow r^{1/4} c \quad (41)$$

under which the four-dimensional effective Lagrangian scales as

$$\mathcal{L} \rightarrow t^2 \mathcal{L} \quad \mathcal{L} \rightarrow r^{-1/2} \mathcal{L} \quad (42)$$

In conclusion, we have seen that super-Kähler geometry drastically restricts the possible form of higher-curvature invariants. In this note we have constructed, as an example, an infinite class of them, which contribute to invariants of arbitrary degree in the Weyl, matter, and Yang-Mills curvatures.

Other invariants, with different spinor and internal index contractions, are also possible.

The restriction of super-Kähler geometry is an example of a symmetry which may restrict higher-derivative interactions in superstring theories.

REFERENCES

- [1] B. Zumino, Phys. Lett. 87B (1979) 203.
- [2] E. Witten and I. Bagger, Phys. Lett. 115B (1982) 202.
- [3] I. Scherk and J.H. Schwarz, Nucl. Phys. B81 (1974) 118.
J.H. Schwarz, Phys. Rep. 89 (1982) 223.
- [4] E. Cremmer, B. Julia, J. Scherk, S. Ferrara, L. Girardello and
P. van Nieuwenhuizen, Nucl. Phys. B147 (1979) 105.
E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, Nucl. Phys. B212
(1983) 413.
- [5] E. Fradkin and A. Tsyelin, Phys. Lett. 158B (1985) 316; Nucl. Phys. B261
(1985) 1.
C.G. Callan, E. Martinec, D. Friedan and M. Perry, Nucl. Phys. B262 (1985) 593.
- [6] A. Van Proeyen, in Supersymmetry and supergravity 1983, ed. B. Milewsky (World
Scientific, Singapore, 1983).
T. Kugo and S. Uehara, Nucl. Phys. B226 (1983) 49.
- [7] S. Ferrara and P. van Nieuwenhuizen, Phys. Lett. 78B (1978) 573.
- [9] M. Green and J.H. Schwarz, Phys. Lett. 151B (1985) 21; Nucl. Phys. B255
(1985) 93.
P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. B258
(1985) 46.
- [10] M.T. Grisaru, A. van de Ven and D. Zanon, Phys. Lett. 173B (1986) 423;
Harvard preprints HUTP-86/A026 and HUTP-86/A027 (1986).
C.N. Pope, M.F. Sohnius and K.S. Stelle, preprint Imperial/TP/85-86/16
(1986).
M.D. Freeman and C.N. Pope, Phys. Lett. B174 (1986) 48.
M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, preprint
Imperial/TP/85-86/27 (1986).
D. Gross and E. Witten, Superstring modifications of Einstein's equations,
Princeton Univ. preprint (1986).

- [11] See, for example, J. Bagger and J. Wess, Supersymmetry and supergravity (Princeton University Press, Princeton, N.J., 1983).
- [12] S. Deser and A.N. Redlich, Brandeis Univ. preprint BRX-TH-200 (1986).
- [13] S. Cecotti, S. Ferrara, L. Girardello and M. Porrati, Phys. Lett. 164B (1985) 46.
- [14] S. Ferrara, J. Wess and B. Zumino, Phys. Lett. 51B (1974) 239.
- [15] W. Siegel, Phys. Lett. 85B (1979) 333.
- [16] C.P. Burgess, A. Font and F. Quevedo, Univ. Texas preprint UTTG/31-85/CTP-046 (1985).
H.P. Nilles, preprint CERN-TH.4522/86 (1986).
- [18] E. Witten, Phys. Lett. 155B (1985) 151.