CERN-TH.4612/86

HIGH-ENERGY, LARGE-MOMENTUM-TRANSFER PROCESSES:

LADDER DIAGRAMS IN ϕ^3 THEORY II

P. Osland^{*)}

Gordon McKay Laboratory, Harvard University
Cambridge, Massachusetts 02138

and

T.T. Wu^{*)}

CERN - Geneva
and
Gordon McKay Laboratory, Harvard University
Cambridge, Massachusetts 02138

ABSTRACT

The scattering amplitude for the four-rung ladder diagram in ϕ^3 theory is evaluated at high energies and for large momentum transfers. The result takes the form of $s^{-1}|t|^{-3}$ multiplied by a homogeneous sixth-order polynomial in $\ln s$ and $\ln |t|$. The novel and unexpected feature is that this polynomial is different depending on whether $\ln |t|$ is larger or less than $\frac{1}{2}\ln s$. Thus the asymptotic formula is not analytic at $\ln |t| = \frac{1}{2}\ln s$, although the first five derivatives are continuous.

^{*)} Work supported in part by the U.S. Department of Energy under grant No. DE-FG02-84ER40158.

1. Introduction

In an attempt to acquire some understanding of scattering processes at high energies and for large momentum transfers, we initiated in paper I [1] a program of studying the asymptotic behavior of Feynman diagrams in this limit. Before considering gauge theories, we begin with the presumably simpler case of ladder diagrams in φ^3 theory. In paper I, results are obtained for the two-rung (fourth-order) and the three-rung (sixth-order) ladder diagrams:

$$I_2(s, t) \sim -2s^{-1}|t|^{-1} \ln s \ln |t|, \quad (1.1)$$

and

$$I_3(s, t) \sim 2s^{-1}|t|^{-2}(\ln |t|)^2[(\ln s)^2 - \frac{2}{3}(\ln s)(\ln |t|) + \frac{1}{6}(\ln |t|)^2]. \quad (1.2)$$

See eqs. (I.3.11) and (I.4.20). Here “ $\sim$ ” denotes equality, to leading order, asymptotically for

$$s \gg -t \gg m^2 = 1, \quad (1.3)$$

where m is the mass of the φ^3 theory. We do *not* require $\ln s \gg \ln |t|$.

It is the purpose of the present paper to show that the corresponding result for the four-rung (eighth-order) ladder diagram is

$$\begin{aligned} I_4(s, t) \sim & -\frac{4}{3}s^{-1}|t|^{-3}\{(\ln |t|)^3[(\ln s)^3 - 2(\ln s)^2(\ln |t|) + \frac{17}{10}(\ln s)(\ln |t|)^2 \\ & - \frac{17}{30}(\ln |t|)^3] + \frac{1}{120}(2 \ln |t| - \ln s)^6 \theta(2 \ln |t| - \ln s)\}, \end{aligned} \quad (1.4)$$

where $\theta(x)$ denotes, as usual, 1 for $x > 0$ and 0 otherwise.

The novel feature here is that this leading term in the asymptotic expansion is non-analytic at $\ln |t| = \frac{1}{2} \ln s$. More precisely, the right-hand side of (1.4) is continuous together with its first five derivatives at this point, but the sixth derivative (with respect to t , for example) is discontinuous. This is plotted in fig. 1. We expect all higher-order ladder diagrams to show such non-analytic behavior, presumably at various rational values of $(\ln |t|)/(\ln s)$ between 0 and 1. [It should be noted that it is not unusual for analytic functions to have non-analytic asymptotic formulas.]

This result (1.4) has been derived using several different methods, all starting with the basic approach of sect. 1.2. In general, the more straightforward methods are also more lengthy, sometimes by a sizable factor. We have chosen to present here one of the shortest derivations. For this reason, before launching into the problem of the four-rung ladder diagram in sect. 3, several useful manipulations are given first in sect. 2.

2. Useful manipulations

As already seen in paper I, we have to deal with various non-integer powers of polynomials. It is therefore often natural to use the Feynman-parameter representation [2], slightly generalized to non-integer powers:

$$A^{-a}B^{-b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 d\beta \beta^{a-1} (1-\beta)^{b-1} [A\beta + B(1-\beta)]^{-(a+b)}. \quad (2.1)$$

For more than two factors, it takes the form

$$\prod_{j=1}^n A_j^{-a_j} = \frac{\Gamma(\sum_j a_j)}{\prod_j \Gamma(a_j)} \int_0^1 \left(\prod_j d\beta_j \right) \delta\left(1 - \sum_j \beta_j\right) \left(\prod_j \beta_j^{a_j-1} \right) \left(\sum_j A_j \beta_j \right)^{-\sum_j a_j}. \quad (2.2)$$

The derivation of these formulas is straightforward, for example, by using the integral representation

$$A^{-a} = [\Gamma(a)]^{-1} \int_0^\infty dx x^{a-1} e^{-Ax}, \quad (2.3)$$

and a rescaling.

When the real parts of the a 's are all positive, (2.1) and (2.2) clearly hold. When this condition is not satisfied, they can still be used with suitable interpretations of the integrals, provided that none of the arguments of the gamma functions is zero or a negative integer. Alternatively, the path of integration can be replaced by a suitable contour. For example, when both $\text{Re } b$ and $\text{Re } (a+b)$ are positive, and a is not an integer, then

$$A^{-a}B^{-b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} (1 - e^{2\pi i a})^{-1} \int_C d\beta \beta^{a-1} (1-\beta)^{b-1} [A\beta + B(1-\beta)]^{-(a+b)}, \quad (2.4)$$

where the contour of integration C is shown in fig. 2.

A second useful manipulation is closely related to (2.1); more precisely it is the inverse Mellin-transform formula corresponding to eq. (I.2.1):

$$(A+B)^{-\tau} = \frac{1}{2\pi i} \frac{1}{\Gamma(\tau)} \int_{c_1-i\infty}^{c_1+i\infty} dz_1 \int_{c_2-i\infty}^{c_2+i\infty} dz_2 \Gamma(z_1) \Gamma(z_2) A^{-z_1} B^{-z_2} \times \delta_c(\tau - z_1 - z_2), \quad (2.5)$$

where δ_c is the delta function defined in sect. I.2. Here the values of c_1 and c_2 , which specify the integration paths that are parallel to the imaginary axes, are chosen so that the right-hand side has the required properties. For example, when $\text{Re } \tau > 0$, then both c_1 and c_2 may be chosen to be positive. Eq. (2.5) will be used in sect. 5.

A third useful manipulation involves the elimination of one of the factors in a multiple Feynman-parameter integration and is best illustrated by two examples. For ζ small, the following two multiple integrals agree to the leading order:

$$\begin{aligned} & \int_0^1 d\alpha_2 d\alpha_4 d\alpha_6 \alpha_4^{-1+\zeta(y_1+y_2)} \alpha_6^{-1+\zeta y_1} (\alpha_2 + \alpha_6)^{-1+\zeta y_2} (\alpha_2 + \alpha_4 + \alpha_6)^{-\zeta(y_1+y_2)} \\ & \sim 2 \int_0^1 d\alpha_2 d\alpha_4 d\alpha_6 \alpha_4^{-1+\zeta(y_1+y_2)} \alpha_6^{-1+\zeta y_1} (\alpha_2 + \alpha_6)^{-1+\zeta y_2}. \end{aligned} \quad (2.6)$$

The reason for the validity of (2.6) is that, if the standard scaling procedure is applied to the variables α_2 , α_4 and α_6 , both the right-hand and the left-hand sides are given by, to leading order in ζ ,

$$\begin{aligned} & [\zeta(y_1 + y_2)]^{-1} \int_0^1 d\alpha_2 d\alpha_4 d\alpha_6 \delta(1 - \alpha_2 - \alpha_4 - \alpha_6) \alpha_4^{-1+\zeta(y_1+y_2)} \\ & \times \alpha_6^{-1+\zeta y_1} (\alpha_2 + \alpha_6)^{-1+\zeta y_2}. \end{aligned} \quad (2.7)$$

The relation (2.6) will be used in sect. 4 in connection with the evaluation of K_{11} as defined by (4.9). As a second, somewhat more complicated example, consider an integral encountered in the evaluation of K_{15} as defined by (4.13). The relation is, in the same sense as (2.6),

$$\int_0^1 d\alpha_2 d\alpha_4 d\alpha_6 \alpha_4^{\zeta y_1} \alpha_6^{-1+\zeta y_1} (\alpha_2 \alpha_4)^{-1+\zeta y_2} (\alpha_2 + \alpha_4 + \alpha_6)^{-\zeta(y_1+y_2)}$$

$$\begin{aligned}
& \times \left[\ln \frac{\alpha_2 + \alpha_4 + \alpha_6}{\alpha_6(\alpha_4 + \alpha_6)} - \left(1 + \frac{\alpha_6}{\alpha_2}\right)^{\zeta y_2} \ln \frac{\alpha_2 + \alpha_6}{\alpha_6} \right] \\
& \sim 2 \int_0^1 d\alpha_2 d\alpha_4 d\alpha_6 \alpha_4^{\zeta y_1} \alpha_6^{-1+\zeta y_1} (\alpha_2 \alpha_4)^{-1+\zeta y_2} \\
& \times \left[\ln \frac{1}{\alpha_6(\alpha_4 + \alpha_6)} - \left(1 + \frac{\alpha_6}{\alpha_2}\right)^{\zeta y_2} \ln \frac{\alpha_2 + \alpha_6}{\alpha_6} \right]. \tag{2.8}
\end{aligned}$$

This is obtained in the same way as (2.6).

3. Four-rung ladder diagram

The four-rung, i.e., eighth-order, ladder diagram in φ^3 theory is shown in fig. 3. The scattering amplitude from this diagram is proportional to [3]

$$I_4(s, t) = 3! \int_0^1 \left(\prod_{j=1}^{10} d\alpha_j \right) \delta \left(1 - \sum_{j=1}^{10} \alpha_j \right) \Lambda^2 (D_s s + D_t t + D_m + i\epsilon)^{-4}, \tag{3.1}$$

where

$$\begin{aligned}
\Lambda &= (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9)(\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) \\
&\quad - \alpha_8^2(\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) - \alpha_9^2(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8), \tag{3.2}
\end{aligned}$$

$$\begin{aligned}
D_t &= \alpha_1 \alpha_4 [(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9)(\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) - \alpha_9^2] \\
&\quad + \alpha_2 \alpha_5 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) \\
&\quad + \alpha_3 \alpha_6 [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2] \\
&\quad + \alpha_8 (\alpha_1 \alpha_5 + \alpha_2 \alpha_4)(\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) + \alpha_9 (\alpha_2 \alpha_6 + \alpha_3 \alpha_5) \\
&\quad \times (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_8 \alpha_9 (\alpha_1 \alpha_6 + \alpha_3 \alpha_4), \tag{3.3}
\end{aligned}$$

$$D_s = \alpha_7 \alpha_8 \alpha_9 \alpha_{10}, \tag{3.4}$$

and

$$\begin{aligned}
D_m &= -\Lambda + \alpha_7 (\alpha_1 + \alpha_4) [(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9)(\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) - \alpha_9^2] \\
&\quad + \alpha_{10} (\alpha_3 + \alpha_6) [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2]
\end{aligned}$$

$$\begin{aligned}
& + \alpha_7 \alpha_8 (\alpha_2 + \alpha_5) (\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) + \alpha_9 \alpha_{10} (\alpha_2 + \alpha_5) \\
& \times (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_7 \alpha_8 \alpha_9 (\alpha_3 + \alpha_6) + \alpha_8 \alpha_9 \alpha_{10} (\alpha_1 + \alpha_4). \quad (3.5)
\end{aligned}$$

For non-negative values of α , D_s is zero when one α , namely α_7 , α_8 , α_9 or α_{10} , is zero. In order for D_t to be zero, it is necessary that at least three α 's be zero—for example, α_1 , α_2 and α_3 . Therefore, apart from logarithms, $I_4(s, t)$ is expected to be of the order of $s^{-1}|t|^{-3}$. Similar to eqs. (I.3.2) for the box diagram and (I.4.6) for the three-rung ladder diagram, define the Mellin transform by [4]

$$\tilde{I}_4(\zeta) = \int_0^\infty ds s^{3\gamma-\zeta} I_4(s, -s^\gamma). \quad (3.6)$$

By eq. (I.2.3), this is

$$\begin{aligned}
\tilde{I}_4(\zeta) &= (2\pi i)^{-1} \int_{c'_1-i\infty}^{c'_1+i\infty} dz_1 \int_{c'_2-i\infty}^{c'_2+i\infty} dz_2 \Gamma(z_1) \Gamma(z_2) \Gamma(4 - z_1 - z_2) \\
&\times \delta_c(1 + 3\gamma - \zeta - z_1 - \gamma z_2) \int_0^1 \left(\prod_{j=1}^{10} d\alpha_j \right) \delta\left(1 - \sum_{j=1}^{10} \alpha_j\right) \\
&\times (\alpha_7 \alpha_8 \alpha_9 \alpha_{10})^{-z_1} (-D_t + i\epsilon)^{-z_2} (D_m + i\epsilon)^{-4+z_1+z_2} \Lambda^2, \quad (3.7)
\end{aligned}$$

where the contours of integration are parallel to the imaginary axes, and the function δ_c is defined in sect. I.2. For the asymptotic limit (1.3), we are interested in small values of ζ .

Similar to eqs. (I.3.5) and (I.4.8), define y_1 and y_2 by

$$z_1 = 1 - \zeta y_1$$

and

$$z_2 = 3 - \zeta y_2. \quad (3.8)$$

Then, to leading order in ζ ,

$$\Gamma(z_1) = \Gamma(1 - \zeta y_1) \sim 1,$$

$$\Gamma(z_2) = \Gamma(3 - \zeta y_2) \sim 2,$$

$$\Gamma(4 - z_1 - z_2) = \Gamma[\zeta(y_1 + y_2)] \sim \frac{1}{\zeta(y_1 + y_2)},$$

$$\delta_c(1 + 3\gamma - \zeta - z_1 - \gamma z_2) = \delta_c[-\zeta(1 - y_1 - \gamma y_2)] = \frac{1}{\zeta} \delta_c(1 - y_1 - \gamma y_2), \quad (3.9)$$

and hence

$$\begin{aligned}
\tilde{I}_4(\zeta) \sim & -(\pi i)^{-1} \int_{c_1-i\infty}^{c_1+i\infty} dy_1 \int_{c_2-i\infty}^{c_2+i\infty} dy_2 (y_1 + y_2)^{-1} \delta_c(1 - y_1 - \gamma y_2) \\
& \times \int_0^1 \left(\prod_{j=1}^{10} d\alpha_j \right) \delta \left(1 - \sum_{j=1}^{10} \alpha_j \right) (\alpha_7 \alpha_8 \alpha_9 \alpha_{10})^{-1+\zeta y_1} \\
& \times D_t^{-3+\zeta y_2} (-D_m)^{-\zeta(y_1+y_2)} \Lambda^2,
\end{aligned} \tag{3.10}$$

where c_1 and c_2 are both positive and, by the definition of δ_c , related by $1 - c_1 - \gamma c_2 = 0$.

Similar to the simpler cases of $\tilde{I}_2$ and $\tilde{I}_3$ of paper I, the dominating contributions to (3.10) for small ζ are expected to come from the regions where all the α 's except one are small, i.e., the corners α_j near 1, $j = 1, 2, \dots, 10$. [Actually, the two corners α_8 near 1 and α_9 near 1 turn out not to contribute to the leading term; see sect. 6.] Since all the α 's are non-negative, one α being close to 1 implies that all the other nine α 's are small.

It may be instructive to give a heuristic argument about the dominating importance of the corners. From (3.10) the integrations over α_7 near 0, α_8 near 0, α_9 near 0, and α_{10} near 0 each yield a factor ζ^{-1} . Suppose we set $\alpha_7 = \alpha_8 = \alpha_9 = \alpha_{10}$; then it follows from (3.2), (3.3) and (3.5) that

$$\Lambda = -D_m = (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_5)(\alpha_3 + \alpha_6), \tag{3.11}$$

$$\begin{aligned}
D_t = & \alpha_1 \alpha_4 (\alpha_2 + \alpha_5)(\alpha_3 + \alpha_6) + \alpha_2 \alpha_5 (\alpha_3 + \alpha_6)(\alpha_1 + \alpha_4) \\
& + \alpha_3 \alpha_6 (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_5).
\end{aligned} \tag{3.12}$$

The substitution into (3.10) leads to a factor

$$\begin{aligned}
& \int_0^1 \left(\prod_{j=1}^6 d\alpha_j \right) \delta \left(1 - \sum_{j=1}^6 \alpha_j \right) [(\alpha_1 + \alpha_4)(\alpha_2 + \alpha_5)(\alpha_3 + \alpha_6)]^{2-\zeta(y_1+y_2)} \\
& \times [\alpha_1 \alpha_4 (\alpha_2 + \alpha_5)(\alpha_3 + \alpha_6) + \alpha_2 \alpha_5 (\alpha_3 + \alpha_6)(\alpha_1 + \alpha_4) \\
& + \alpha_3 \alpha_6 (\alpha_1 + \alpha_4)(\alpha_2 + \alpha_5)]^{-3+\zeta y_2},
\end{aligned} \tag{3.13}$$

in addition to the $(\zeta y_1)^{-4}$ from the integrations over $\alpha_7, \alpha_8, \alpha_9$ and α_{10} . By symmetry, in (3.13) the six corners α_j near 1 are all equivalent. Without loss of generality, consider α_1

near 1. Then the integration over $\alpha_2 + \alpha_5$ near 0 yields a factor of ζ^{-1} , as does that over $\alpha_3 + \alpha_6$ near 0, and finally that over α_4 near 0 yields a third factor of ζ^{-1} . Consequently,

$$\tilde{I}_4(\zeta) = O(\zeta^{-7}), \quad (3.14)$$

for small ζ . If more than one of the ten α 's were allowed to be not small, one would lose at least one power of ζ^{-1} , and hence such a region of integration fails to contribute to the leading term.

Although this heuristic argument correctly gives the order of magnitude (3.14), it is not precise enough to yield the coefficient of ζ^{-7} . The reason is that, in writing down (3.13), it is implicitly assumed that $\alpha_7, \alpha_8, \alpha_9$ and α_{10} are smaller than $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and α_6 . Indeed, it is not difficult to evaluate (3.13), leading to the approximation

$$-\frac{1}{30} s^{-1} |t|^{-3} \{(\ln s)^6 - \frac{1}{3} [\ln(s/|t|)]^6\} \quad (3.15)$$

for $I_4(s, t)$. This formula has no resemblance to either (1.4) or the known result [5] for large s and fixed t . This demonstrates that great care must be exercised in evaluating the asymptotic behavior of Feynman diagrams for large s and large $|t|$.

We now return to (3.10). Since nine of the ten α 's are small, the D_m of (3.5) can always be approximated by $-\Lambda$. By symmetry, there are four types of contributions:

- (i) $\alpha_1, \alpha_3, \alpha_4$, or α_6 is near 1,
- (ii) α_2 or α_5 is near 1,
- (iii) α_8 or α_9 is near 1, and
- (iv) α_7 or α_{10} is near 1. (3.16)

Therefore,

$$\tilde{I}_4(\zeta) \sim 4\tilde{I}_{41}(\zeta) + 2\tilde{I}_{42}(\zeta) + 2\tilde{I}_{43}(\zeta) + 2\tilde{I}_{44}(\zeta), \quad (3.17)$$

where, for $i = 1, 2, 3, 4$,

$$\tilde{I}_{4i}(\zeta) = -(\pi i)^{-1} \int_{c_1 - i\infty}^{c_1 + i\infty} dy_1 \int_{c_2 - i\infty}^{c_2 + i\infty} dy_2 (y_1 + y_2)^{-1} \delta_c(1 - y_1 - \gamma y_2) J_i(\zeta, y_1, y_2). \quad (3.18)$$

If we define $k = k(i)$ as 6, 5, 9 and 10 respectively for $i = 1, 2, 3, 4$ [see (3.16) above], then the J_i 's of (3.18) are

$$J_i(\zeta, y_1, y_2) = \int_0^1 \left(\prod_{\substack{j=1 \\ j \neq k}}^{10} d\alpha_j \right) [(\alpha_7 \alpha_8 \alpha_9 \alpha_{10})^{-1+\zeta y_1} D_t^{-3+\zeta y_2} \Lambda^{2-\zeta(y_1+y_2)}]_{\alpha_k=1}. \quad (3.19)$$

These four cases are dealt with in the next four sections. More explicit formulas for $J_i(\zeta, y_1, y_2)$ are to be found in (4.3), (5.3), (6.3) and (7.3).

4. Evaluation of $J_1(\zeta, y_1, y_2)$

For J_1 , α_6 is near 1 and hence all the other nine α 's are small. However, it should be kept in mind that these nine small α 's need not be of the same order of magnitude. We have from eqs. (3.2) and (3.3)

$$\Lambda \sim (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2, \quad (4.1)$$

and

$$\begin{aligned} D_t &\sim \alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_2 \alpha_5 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) \\ &\quad + \alpha_3 [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2] \\ &\quad + \alpha_8 (\alpha_1 \alpha_5 + \alpha_2 \alpha_4) + \alpha_2 \alpha_9 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_1 \alpha_8 \alpha_9 \\ &= \alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_2 (\alpha_5 + \alpha_9) (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) \\ &\quad + \alpha_3 [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2] \\ &\quad + \alpha_8 [\alpha_1 (\alpha_5 + \alpha_9) + \alpha_2 \alpha_4]. \end{aligned} \quad (4.2)$$

The terms which have been neglected are all small compared to those that are kept. For example, in D_t we have dropped the terms $\alpha_3 \alpha_5 \alpha_9 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)$ and $\alpha_3 \alpha_4 \alpha_8 \alpha_9$ (and others). The first of these is less than part of the third term that is retained, $\alpha_3 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) \alpha_5$, by a factor $\alpha_9 \ll 1$. And the second one is less than part of the third term that is retained, $\alpha_3 \alpha_4 \alpha_8$, again by a factor $\alpha_9 \ll 1$.

With (4.1) and (4.2), the definition (3.19) for J_1 is

$$\begin{aligned}
J_1(\zeta, y_1, y_2) = & \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_7 d\alpha_8 d\alpha_9 d\alpha_{10} (\alpha_7 \alpha_8 \alpha_9 \alpha_{10})^{-1+\zeta y_1} \\
& \times \{ \alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_2 (\alpha_5 + \alpha_9) (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) \\
& + \alpha_3 [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2] \\
& + \alpha_8 [\alpha_1 (\alpha_5 + \alpha_9) + \alpha_2 \alpha_4] \}^{-3+\zeta y_2} \\
& \times [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2]^{2-\zeta(y_1+y_2)}. \quad (4.3)
\end{aligned}$$

Since the leading-order contributions come from the lower limits of integration, the upper limit has somewhat arbitrarily been set to 1. [See the discussion following eq. (I.4.14).] In eq. (4.3), the integration over α_{10} is trivial, yielding a factor $(\zeta y_1)^{-1}$. Furthermore, the integration over α_3 is of the form

$$\int_0^1 d\alpha (A\alpha + B)^{-n-1+\zeta} \sim \frac{1}{n} A^{-1} B^{-n+\zeta}; \quad n \geq 1, \quad (4.4)$$

since $B \ll A (= \Lambda)$, and we are left with

$$\begin{aligned}
J_1(\zeta, y_1, y_2) \sim & \frac{1}{2\zeta y_1} \int_0^1 d\alpha_1 d\alpha_2 d\alpha_4 d\alpha_5 d\alpha_7 d\alpha_8 d\alpha_9 (\alpha_7 \alpha_8 \alpha_9)^{-1+\zeta y_1} \\
& \times \{ \alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_2 (\alpha_5 + \alpha_9) (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) \\
& + \alpha_8 [\alpha_1 (\alpha_5 + \alpha_9) + \alpha_2 \alpha_4] \}^{-2+\zeta y_2} \\
& \times [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2]^{1-\zeta(y_1+y_2)}. \quad (4.5)
\end{aligned}$$

This integrand is homogeneous and a rescaling according to eq. (I.A2) yields a factor $[\zeta(y_1 + y_2)]^{-1}$. In order to facilitate comparison with the results of sect. I.4, we change notation,

$$\begin{aligned}
\alpha_1 & \rightarrow \alpha_1, & \alpha_7 & \rightarrow \alpha_5, \\
\alpha_2 & \rightarrow \alpha_2, & \alpha_8 & \rightarrow \alpha_6, \\
\alpha_4 & \rightarrow \alpha_3, & \alpha_9 & \rightarrow \alpha_7, \\
\alpha_5 & \rightarrow \alpha_4, & &
\end{aligned} \quad (4.6)$$

and get

$$\begin{aligned}
J_1(\zeta, y_1, y_2) &\sim \frac{1}{2}\zeta^{-2}[y_1(y_1 + y_2)]^{-1} \int_0^1 \left(\prod_{j=1}^7 d\alpha_j \right) \delta\left(1 - \sum_{j=1}^7 \alpha_j\right) (\alpha_5 \alpha_6 \alpha_7)^{-1+\zeta y_1} \\
&\quad \times \{ \alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2 (\alpha_4 + \alpha_7) (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) \\
&\quad + \alpha_6 [\alpha_1 (\alpha_4 + \alpha_7) + \alpha_2 \alpha_3] \}^{-2+\zeta y_2} \\
&\quad \times [(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) - \alpha_6^2]^{1-\zeta(y_1+y_2)}. \quad (4.7)
\end{aligned}$$

The integrand here is very similar to that of (I.4.9), the only difference being that two factors of α_4 in the D_t of eq. (I.4.4) are here replaced by $\alpha_4 + \alpha_7$. Therefore the procedure of paper I can be used for this $J_1(\zeta, y_1, y_2)$. However, since the integrand here has no symmetry, the seven contributions from α_j near 1, $j = 1, 2, \dots, 7$, must be calculated separately. In particular, the case of α_2 near 1 is especially tedious to treat this way.

We choose to present here a slightly different approach. On the right-hand side of (4.7), α_4 appears only in the combination $\alpha_4 + \alpha_7$. If we let $\alpha'_4 = \alpha_4 + \alpha_7$, and use α'_4 instead of α_4 , then the α_7 integration is trivial. After carrying out this α_7 integration, and dropping the prime on α'_4 , we get

$$\begin{aligned}
J_1(\zeta, y_1, y_2) &\sim \frac{1}{2}\zeta^{-3}[y_1^2(y_1 + y_2)]^{-1} \int_0^1 \left(\prod_{j=1}^6 d\alpha_j \right) \delta\left(1 - \sum_{j=1}^6 \alpha_j\right) \\
&\quad \times \alpha_4^{\zeta y_1} (\alpha_5 \alpha_6)^{-1+\zeta y_1} [\alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6) \\
&\quad + \alpha_2 \alpha_4 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)]^{-2+\zeta y_2} \\
&\quad \times [(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_4 + \alpha_6) - \alpha_6^2]^{1-\zeta(y_1+y_2)} \\
&\sim \frac{1}{2}\zeta^{-3}[y_1^2(y_1 + y_2)]^{-1} \sum_{i=1}^6 K_{1i}(\zeta, y_1, y_2), \quad (4.8)
\end{aligned}$$

where K_{1i} is the contribution from the corner α_i near 1. These are explicitly

$$\begin{aligned}
K_{11} &= \int_0^1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 \alpha_4^{\zeta y_1} (\alpha_5 \alpha_6)^{-1+\zeta y_1} \\
&\quad \times [\alpha_3 (\alpha_2 + \alpha_4 + \alpha_6) + \alpha_4 (\alpha_2 + \alpha_6)]^{-2+\zeta y_2}
\end{aligned}$$

$$\times (\alpha_2 + \alpha_4 + \alpha_6)^{1-\zeta(y_1+y_2)}, \quad (4.9)$$

$$\begin{aligned} K_{12} = & \int_0^1 d\alpha_1 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 \alpha_4^{\zeta y_1} (\alpha_5 \alpha_6)^{-1+\zeta y_1} \\ & \times [\alpha_4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_3(\alpha_1 + \alpha_6)]^{-2+\zeta y_2} \\ & \times (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{1-\zeta(y_1+y_2)}, \end{aligned} \quad (4.10)$$

$$\begin{aligned} K_{13} = & \int_0^1 d\alpha_1 d\alpha_2 d\alpha_4 d\alpha_5 d\alpha_6 \alpha_4^{\zeta y_1} (\alpha_5 \alpha_6)^{-1+\zeta y_1} \\ & \times [\alpha_1(\alpha_2 + \alpha_4 + \alpha_6) + \alpha_2(\alpha_4 + \alpha_6)]^{-2+\zeta y_2} \\ & \times (\alpha_2 + \alpha_4 + \alpha_6)^{1-\zeta(y_1+y_2)}, \end{aligned} \quad (4.11)$$

$$\begin{aligned} K_{14} = & \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_5 d\alpha_6 (\alpha_5 \alpha_6)^{-1+\zeta y_1} \\ & \times [\alpha_2(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_1(\alpha_3 + \alpha_6)]^{-2+\zeta y_2} \\ & \times (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{1-\zeta(y_1+y_2)}, \end{aligned} \quad (4.12)$$

$$\begin{aligned} K_{15} = & \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_6 \alpha_4^{\zeta y_1} \alpha_6^{-1+\zeta y_1} \\ & \times [\alpha_2 \alpha_4 + \alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6) + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)]^{-2+\zeta y_2} \\ & \times (\alpha_2 + \alpha_4 + \alpha_6)^{1-\zeta(y_1+y_2)}, \end{aligned} \quad (4.13)$$

and

$$\begin{aligned} K_{16} = & \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 \alpha_4^{\zeta y_1} \alpha_5^{-1+\zeta y_1} \\ & \times [(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)]^{-2+\zeta y_2} (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5)^{1-\zeta(y_1+y_2)}. \end{aligned} \quad (4.14)$$

These six integrals need to be evaluated for small ζ . For K_{11} , carrying out the α_3 and α_5 integrations leads to the left-hand side of (2.6) divided by ζy_1 . Since the right-hand side of (2.6), being the product of two separate integrals, one over α_4 and the other over α_2 and α_6 , is easy to evaluate, we get the desired answer for K_{11} . The results are:

$$K_{11}(\zeta, y_1, y_2) \sim \frac{2}{\zeta^4 y_1^2 (y_1 + y_2)^2}, \quad (4.15)$$

$$K_{12}(\zeta, y_1, y_2) \sim \frac{2}{\zeta^4 y_1^2 (y_1 + y_2)(2y_1 + y_2)}, \quad (4.16)$$

$$K_{13}(\zeta, y_1, y_2) \sim \frac{2}{\zeta^4 y_1^2 y_2 (2y_1 + y_2)}, \quad (4.17)$$

$$K_{14}(\zeta, y_1, y_2) \sim \frac{2}{\zeta^4 y_1^2 y_2 (y_1 + y_2)}, \quad (4.18)$$

$$K_{15}(\zeta, y_1, y_2) \sim \frac{2(3y_1 + 2y_2)}{\zeta^4 y_1 y_2 (y_1 + y_2)^2 (2y_1 + y_2)}, \quad (4.19)$$

and

$$K_{16}(\zeta, y_1, y_2) = O(\zeta^{-3}). \quad (4.20)$$

The evaluations of K_{13} and K_{14} are entirely similar to that of K_{11} as discussed above. K_{16} is small because, if we rescale to the variable $\rho = \sum_{j=1}^5 \alpha_j$, the integration over ρ fails to give a factor of ζ^{-1} . The derivations of (4.16) and (4.19) are comparatively more complicated, and are relegated to appendix A.

It is important to note the appearance of a factor $2y_1 + y_2$ in the denominator for some of the terms. This factor, which is not cancelled when all the contributions are added together, has the interesting consequence that, as shown explicitly in (1.4), the asymptotic behavior of the four-rung amplitude is given by two different formulas, depending on whether $(\ln |t|)/(\ln s)$ is greater or less than $\frac{1}{2}$.

The substitution of (4.15)–(4.20) into (4.8) gives finally

$$J_1(\zeta, y_1, y_2) \sim 2\zeta^{-7} \frac{3y_1 + 2y_2}{y_1^4 y_2 (y_1 + y_2)^2 (2y_1 + y_2)}, \quad (4.21)$$

asymptotically for small ζ .

5. Evaluation of $J_2(\zeta, y_1, y_2)$

For α_5 near 1, we have from (3.2) and (3.3), by neglecting smaller terms,

$$\Lambda \sim (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}), \quad (5.1)$$

and

$$\begin{aligned} D_t \sim & \alpha_2(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) + \alpha_1(\alpha_4 + \alpha_8) \\ & \times (\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) + \alpha_3(\alpha_6 + \alpha_9)(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8). \end{aligned} \quad (5.2)$$

Thus the definition (3.19) for J_2 is

$$\begin{aligned}
J_2(\zeta, y_1, y_2) = & \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_6 d\alpha_7 d\alpha_8 d\alpha_9 d\alpha_{10} (\alpha_7 \alpha_8 \alpha_9 \alpha_{10})^{-1+\zeta y_1} \\
& \times \{ \alpha_2 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) (\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) \\
& + \alpha_1 (\alpha_4 + \alpha_8) (\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) \\
& + \alpha_3 (\alpha_6 + \alpha_9) (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) \}^{-3+\zeta y_2} \\
& \times [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) (\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10})]^{2-\zeta(y_1+y_2)}. \quad (5.3)
\end{aligned}$$

Integration over α_2 using (4.4) gives

$$\begin{aligned}
J_2(\zeta, y_1, y_2) \sim & \frac{1}{2} \int_0^1 d\alpha_1 d\alpha_3 d\alpha_4 d\alpha_6 d\alpha_7 d\alpha_8 d\alpha_9 d\alpha_{10} (\alpha_7 \alpha_8 \alpha_9 \alpha_{10})^{-1+\zeta y_1} \\
& \times [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) (\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10})]^{1-\zeta(y_1+y_2)} \\
& \times [\alpha_1 (\alpha_4 + \alpha_8) (\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10}) \\
& + \alpha_3 (\alpha_6 + \alpha_9) (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)]^{-2+\zeta y_2}. \quad (5.4)
\end{aligned}$$

Since the integrand is homogeneous in the α 's, it is possible to proceed with the standard rescaling technique. In this case, however, it is more efficient to use eq. (2.5) with $\tau = 2 - \zeta y_2$ for the last factor in the integrand of (5.4). We wish to interchange the α and the z integrations; to insure the convergence of the α_1 and α_3 integrations, we must choose

$$c_1 < 1, \quad c_2 < 1. \quad (5.5)$$

Since by the definition of δ_c [1],

$$z_1 + z_2 = 2 - \zeta y_2, \quad (5.6)$$

eq. (5.5) implies that c_1 and c_2 are both close to 1. Define y_3 and y_4 both of order 1 so that

$$z_1 = 1 - \zeta y_3 \quad \text{and} \quad z_2 = 1 - \zeta y_4. \quad (5.7)$$

If c_3 and c_4 are such that

$$c_1 = 1 - \zeta c_3 \quad \text{and} \quad c_2 = 1 - \zeta c_4, \quad (5.8)$$

then it follows from (5.4) and (2.5) that

$$J_2(\zeta, y_1, y_2) \sim \frac{1}{2\pi i} \zeta \int_{c_3-i\infty}^{c_3+i\infty} dy_3 \int_{c_4-i\infty}^{c_4+i\infty} dy_4 \delta_c(y_2 - y_3 - y_4) K_2(\zeta, y_1, y_3, y_4), \quad (5.9)$$

where

$$\begin{aligned} K_2(\zeta, y_1, y_3, y_4) &= \frac{1}{2} \int_0^1 d\alpha_1 d\alpha_3 d\alpha_4 d\alpha_6 d\alpha_7 d\alpha_8 d\alpha_9 d\alpha_{10} (\alpha_7 \alpha_8 \alpha_9 \alpha_{10})^{-1+\zeta y_1} \\ &\quad \times [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10})]^{1-\zeta(y_1+y_3+y_4)} \\ &\quad \times [\alpha_1(\alpha_4 + \alpha_8)(\alpha_3 + \alpha_6 + \alpha_9 + \alpha_{10})]^{-1+\zeta y_3} \\ &\quad \times [\alpha_3(\alpha_6 + \alpha_9)(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)]^{-1+\zeta y_4}. \end{aligned} \quad (5.10)$$

The advantage of using this representation is that the $\alpha_1, \alpha_4, \alpha_7, \alpha_8$ and the $\alpha_3, \alpha_6, \alpha_9, \alpha_{10}$ integrations separate. Therefore

$$K_2(\zeta, y_1, y_3, y_4) = \frac{1}{2} L_2(\zeta, y_1, y_3) L_2(\zeta, y_1, y_4), \quad (5.11)$$

where

$$\begin{aligned} L_2(\zeta, y_1, y_2) &= \int_0^1 d\alpha_1 d\alpha_4 d\alpha_7 d\alpha_8 (\alpha_7 \alpha_8)^{-1+\zeta y_1} \\ &\quad \times (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)^{-\zeta(y_1+y_2)} [\alpha_1(\alpha_4 + \alpha_8)]^{-1+\zeta y_2}. \end{aligned} \quad (5.12)$$

But this is precisely the integral studied in appendix A of paper I:

$$L_2(\zeta, y_1, y_2) \sim \zeta^{-4} \frac{2}{y_1^2 y_2 (y_1 + y_2)}, \quad (5.13)$$

and hence

$$K_2(\zeta, y_1, y_3, y_4) \sim 2\zeta^{-8} \frac{1}{y_1^4 y_3 y_4 (y_1 + y_3)(y_1 + y_4)}. \quad (5.14)$$

Finally the substitution of (5.14) into (5.9) gives, after eliminating y_4 with the δ_c and closing the contour of integration in the y_3 plane on the poles at $y_3 = 0$ and $y_3 = -y_1$,

$$J_2(\zeta, y_1, y_2) \sim 4\zeta^{-7} \frac{1}{y_1^4 y_2 (y_1 + y_2)(2y_1 + y_2)}. \quad (5.15)$$

This is the desired answer. Note the appearance of the factor $2y_1 + y_2$ once more in the denominator.

6. Evaluation of $J_3(\zeta, y_1, y_2)$

In this section, we show that J_3 does not contribute to the leading order. For α_9 near 1, and hence all the other nine α 's small, it follows from (3.2) and (3.3) that

$$\Lambda \sim (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_8 + \alpha_{10}) - \alpha_8^2, \quad (6.1)$$

and

$$\begin{aligned} D_t &\sim \alpha_1 \alpha_4 (\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_8 + \alpha_{10}) + (\alpha_2 + \alpha_3)(\alpha_5 + \alpha_6) \\ &\times (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_8 [\alpha_1(\alpha_5 + \alpha_6) + \alpha_4(\alpha_2 + \alpha_3)]. \end{aligned} \quad (6.2)$$

Thus the definition (3.19) for J_3 is

$$\begin{aligned} J_3(\zeta, y_1, y_2) &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 d\alpha_7 d\alpha_8 d\alpha_{10} (\alpha_7 \alpha_8 \alpha_{10})^{-1+\zeta y_1} \\ &\times \{ \alpha_1 \alpha_4 (\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_8 + \alpha_{10}) \\ &+ (\alpha_2 + \alpha_3)(\alpha_5 + \alpha_6)(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) \\ &+ \alpha_8 [\alpha_1(\alpha_5 + \alpha_6) + \alpha_4(\alpha_2 + \alpha_3)] \}^{-3+\zeta y_2} \\ &\times [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_8 + \alpha_{10}) \\ &- \alpha_8^2]^{2-\zeta(y_1+y_2)}. \end{aligned} \quad (6.3)$$

We note that the variables α_2 and α_3 only appear in the combination $\rho = \alpha_2 + \alpha_3$, and likewise that α_5 and α_6 only appear as $\bar{\rho} = \alpha_5 + \alpha_6$. With $C_1 = O(1)$, we thus get

$$\begin{aligned} J_3(\zeta, y_1, y_2) &\sim C_1 \int_0^1 d\alpha_1 d\rho d\alpha_4 d\bar{\rho} d\alpha_7 d\alpha_8 d\alpha_{10} \rho \bar{\rho} (\alpha_7 \alpha_8 \alpha_{10})^{-1+\zeta y_1} \\ &\times [\alpha_1 \alpha_4 (\rho + \bar{\rho} + \alpha_8 + \alpha_{10}) + \rho \bar{\rho} (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) \\ &+ \alpha_8 (\alpha_1 \bar{\rho} + \alpha_4 \rho)]^{-3+\zeta y_2} \\ &\times [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\rho + \bar{\rho} + \alpha_8 + \alpha_{10}) - \alpha_8^2]^{2-\zeta(y_1+y_2)}. \end{aligned} \quad (6.4)$$

There is no simple expression for C_1 ; and furthermore C_1 would be different if the upper limit of integration in (6.3) were changed to a value different from 1. This is already a

clear indication that J_3 is sub-dominant. If the standard rescaling procedure is applied to (6.4), it is readily verified that this is indeed the case,

$$J_3(\zeta, y_1, y_2) = O(\zeta^{-5}). \quad (6.5)$$

This result is analogous to (I.4.16).

7. Evaluation of $J_4(\zeta, y_1, y_2)$

The asymptotic evaluation of J_4 is by far the most difficult and the most interesting one, as expected from the simpler case of the three-rung ladder diagram. For α_{10} near 1, it follows from (3.2) and (3.3) that

$$\Lambda \sim (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2, \quad (7.1)$$

and

$$\begin{aligned} D_t \sim & \alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_2 \alpha_5 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_8 (\alpha_1 \alpha_5 + \alpha_2 \alpha_4) \\ & + \alpha_3 \alpha_6 [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2] \\ & + \alpha_9 (\alpha_2 \alpha_6 + \alpha_3 \alpha_5) (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_8 \alpha_9 (\alpha_1 \alpha_6 + \alpha_3 \alpha_4). \end{aligned} \quad (7.2)$$

Unlike the expressions (4.2), (5.2) and (6.2) for J_1 , J_2 and J_3 , respectively, the right-hand side of (7.2) is *not* homogeneous in the α 's. This is the cause for the complications, already encountered in a less severe way with the $\tilde{I}_{33}$ of (I.4.14) for the three-rung ladder diagram.

With (7.1) and (7.2), the definition (3.19) for J_4 is

$$\begin{aligned} J_4(\zeta, y_1, y_2) = & \int_0^1 \left(\prod_{j=1}^9 d\alpha_j \right) (\alpha_7 \alpha_8 \alpha_9)^{-1+\zeta y_1} \{ \alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) \\ & + \alpha_2 \alpha_5 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_8 (\alpha_1 \alpha_5 + \alpha_2 \alpha_4) \\ & + \alpha_3 \alpha_6 [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2] \\ & + \alpha_9 (\alpha_2 \alpha_6 + \alpha_3 \alpha_5) (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) \\ & + \alpha_8 \alpha_9 (\alpha_1 \alpha_6 + \alpha_3 \alpha_4) \}^{-3+\zeta y_2} \\ & \times [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2]^{2-\zeta(y_1+y_2)}. \end{aligned} \quad (7.3)$$

There are several ways to treat this J_4 . For example, we may use eq. (2.5) to recover the homogeneity in the α 's and then apply the standard rescaling procedure. This has been carried through but it is very lengthy. Alternatively, we may follow the procedure described in appendix C of paper I for the three-rung ladder diagram. This has also been carried through and it is much less lengthy. In this section, we choose to present a third method, where that of appendix C of paper I is followed initially in the sense of integrating over α_3 and α_6 first. After these integrations, a maneuver is carried out to simplify the integrand, as we shall see below.

The approximate formula (I.D1) can be readily generalized to the case where $(1 + Ax)^{-1+\varsigma}$ is replaced by $(1 + Ax)^{-n+\varsigma}$. For example, for $n = 2$, it is

$$\int_{\delta}^1 \frac{dx}{x} (1 + Ax)^{-2+\varsigma} \sim (1 + A\delta)^{\varsigma} \left[\ln \frac{1 + A\delta}{(1 + A)\delta} - \left(\frac{1}{1 + A\delta} - \frac{1}{1 + A} \right) \right]. \quad (7.4)$$

Actually, for application to J_4 , it is sufficient to keep the logarithmic term on the right-hand side. If, in (7.3), the α_3 integration is performed first and then (7.4) is used to carry out the α_6 integration, the result is

$$\begin{aligned} J_4(\varsigma, y_1, y_2) \sim & \frac{1}{2} \int_0^1 d\alpha_1 d\alpha_2 d\alpha_4 d\alpha_5 d\alpha_7 d\alpha_8 d\alpha_9 (\alpha_7 \alpha_8 \alpha_9)^{-1+\varsigma y_1} \\ & \times [(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2]^{1-\varsigma(y_1+y_2)} \\ & \times [\alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_2 \alpha_5 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) \\ & + \alpha_8 (\alpha_1 \alpha_5 + \alpha_2 \alpha_4)]^{-2+\varsigma y_2} \\ & \times \left\{ \ln \left[\frac{\alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_2 \alpha_5 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_8 (\alpha_1 \alpha_5 + \alpha_2 \alpha_4)}{\alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_2 (\alpha_5 + \alpha_9) (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_8 [\alpha_1 (\alpha_5 + \alpha_9) + \alpha_2 \alpha_4]} \right] \right. \\ & \times \left. \frac{(\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8)(\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) - \alpha_8^2}{\alpha_9 [\alpha_5 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_4 \alpha_8]} \right] \\ & - \left[\frac{\alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_5 (\alpha_2 + \alpha_9) (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_8 [\alpha_1 \alpha_5 + \alpha_4 (\alpha_2 + \alpha_9)]}{\alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_2 \alpha_5 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_8 (\alpha_1 \alpha_5 + \alpha_2 \alpha_4)} \right]^{\varsigma y_2} \\ & \times \ln \frac{\alpha_1 \alpha_4 (\alpha_2 + \alpha_5 + \alpha_8 + \alpha_9) + \alpha_5 (\alpha_2 + \alpha_9) (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_8 [\alpha_1 \alpha_5 + \alpha_4 (\alpha_2 + \alpha_9)]}{\alpha_9 [\alpha_5 (\alpha_1 + \alpha_4 + \alpha_7 + \alpha_8) + \alpha_4 \alpha_8]} \Big\}. \end{aligned} \quad (7.5)$$

This integrand can be simplified. The last logarithmic factor is important only when it is large, and this implies that the long factor multiplying this logarithm may be replaced by 1. Thus, the two logarithmic terms in the braces can be combined, after first making use of the symmetry under the exchanges $\alpha_1 \leftrightarrow \alpha_4$ and $\alpha_2 \leftrightarrow \alpha_5$. With the change of notation (4.6), we get

$$J_4(\zeta, y_1, y_2) \sim J_{41}(\zeta, y_1, y_2) - J_{42}(\zeta, y_1, y_2), \quad (7.6)$$

where

$$\begin{aligned} J_{41}(\zeta, y_1, y_2) = & -\frac{1}{2\zeta} \frac{\partial}{\partial y_2} \int_0^1 \left(\prod_{j=1}^7 d\alpha_j \right) (\alpha_5 \alpha_6 \alpha_7)^{-1+\zeta y_1} \\ & \times [(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) - \alpha_6^2]^{1-\zeta(y_1+y_2)} \\ & \times [\alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2 \alpha_4 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) \\ & + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)]^{-2+\zeta y_2}, \end{aligned} \quad (7.7)$$

and

$$\begin{aligned} J_{42}(\zeta, y_1, y_2) = & \int_0^1 \left(\prod_{j=1}^7 d\alpha_j \right) (\alpha_5 \alpha_6 \alpha_7)^{-1+\zeta y_1} \\ & \times [(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) - \alpha_6^2]^{1-\zeta(y_1+y_2)} \\ & \times [\alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2 \alpha_4 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) \\ & + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)]^{-2+\zeta y_2} \\ & \times \ln \frac{\alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2 (\alpha_4 + \alpha_7) (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 [\alpha_1 (\alpha_4 + \alpha_7) + \alpha_2 \alpha_3]}{\alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2 \alpha_4 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)}. \end{aligned} \quad (7.8)$$

By (I.4.9), (I.4.2) and (I.4.4), this J_{41} is closely related to $\tilde{I}_3$ for the three-rung ladder diagram, and hence it is asymptotically given by, for small ζ ,

$$J_{41}(\zeta, y_1, y_2) \sim 4\zeta^{-7} \frac{2y_1 + 3y_2}{y_1^3 y_2^3 (y_1 + y_2)^2}. \quad (7.9)$$

The remaining part of this section is devoted to the much more difficult task of evaluating $J_{42}(\zeta, y_1, y_2)$ asymptotically, with much of the details relegated to the appendices B, C and D. As a first step, it seems useful in (7.8) to simplify the logarithmic factor, which is small unless $\alpha_7 \gg \alpha_4$. Thus, it is permissible to replace the factor $(\alpha_5 \alpha_6 \alpha_7)^{-1+\zeta y_1}$ by $[\alpha_5 \alpha_6 (\alpha_4 + \alpha_7)]^{-1+\zeta y_1}$. After this replacement, change the variable α_7 to

$$\sigma = \alpha_7 + \alpha_4, \quad (7.10)$$

and then integrate by parts with respect to α_4 :

$$\begin{aligned} J_{42} \sim & \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_5 d\alpha_6 d\sigma (\alpha_5 \alpha_6 \sigma)^{-1+\zeta y_1} \\ & \times [(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_6 + \sigma) - \alpha_6^2]^{1-\zeta(y_1+y_2)} \\ & \times \left\{ \frac{-1}{\alpha_2(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_1 \alpha_6} [\alpha_1 \alpha_3 (\alpha_2 + \alpha_6 + \sigma) \right. \\ & + \alpha_2 \alpha_4 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)]^{-1+\zeta y_2} \\ & \times \ln \frac{\alpha_1 \alpha_3 (\alpha_2 + \alpha_6 + \sigma) + \alpha_2 \sigma (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 (\alpha_1 \sigma + \alpha_2 \alpha_3)}{\alpha_1 \alpha_3 (\alpha_2 + \alpha_6 + \sigma) + \alpha_2 \alpha_4 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)} \Big|_{\alpha_4=0}^{\alpha_4=\sigma} \\ & - \int_0^\sigma [\alpha_1 \alpha_3 (\alpha_2 + \alpha_6 + \sigma) + \alpha_2 \alpha_4 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) \\ & \left. + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)]^{-2+\zeta y_2} d\alpha_4 \right\}. \end{aligned} \quad (7.11)$$

The important point here is that the last term, the only one with an α_4 integration, is negligible because it has one less logarithmic factor than the original integral on the right-hand side of (7.8). Therefore,

$$\begin{aligned} J_{42} \sim & \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_5 d\alpha_6 d\sigma (\alpha_5 \alpha_6 \sigma)^{-1+\zeta y_1} \\ & \times [(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_6 + \sigma) - \alpha_6^2]^{1-\zeta(y_1+y_2)} \bar{J}_{42}, \end{aligned} \quad (7.12)$$

where $\bar{J}_{42}$, which is a function of the α 's in addition to ζ, y_1, y_2 , and σ , is defined by

$$\begin{aligned} \bar{J}_{42} = & [\alpha_1 \alpha_3 (\alpha_2 + \alpha_6 + \sigma) + \alpha_2 \alpha_3 \alpha_6]^{-1+\zeta y_2} [\alpha_2 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_1 \alpha_6]^{-1} \\ & \times \ln \frac{\alpha_1 \alpha_3 (\alpha_2 + \alpha_6 + \sigma) + \alpha_2 \sigma (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 (\alpha_1 \sigma + \alpha_2 \alpha_3)}{\alpha_1 \alpha_3 (\alpha_2 + \alpha_6 + \sigma) + \alpha_2 \alpha_3 \alpha_6}. \end{aligned} \quad (7.13)$$

We first write the logarithm as an integral, and then apply the Feynman-parameter representation (2.1):

$$\begin{aligned}
\bar{J}_{42} &= [\alpha_1 \alpha_3 (\alpha_2 + \alpha_6 + \sigma) + \alpha_2 \alpha_3 \alpha_6]^{-1+\zeta y_2} \\
&\quad \times \int_0^\sigma d\tau [\alpha_1 \alpha_3 (\alpha_2 + \alpha_6 + \sigma) + \alpha_2 \tau (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) \\
&\quad + \alpha_6 (\alpha_1 \tau + \alpha_2 \alpha_3)]^{-1} \\
&= (1 - \zeta y_2) \int_0^\sigma d\tau \int_0^1 d\beta (1 - \beta)^{-\zeta y_2} [\alpha_1 \alpha_3 (\alpha_2 + \alpha_6 + \sigma) \\
&\quad + \alpha_2 \tau \beta (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 (\alpha_1 \tau \beta + \alpha_2 \alpha_3)]^{-2+\zeta y_2}. \tag{7.14}
\end{aligned}$$

It is natural to reintroduce α_4 as

$$\alpha_4 = \tau \beta. \tag{7.15}$$

Then the approximation

$$\begin{aligned}
\int_0^\sigma d\tau \int_0^1 d\beta (1 - \beta)^{-\zeta y_2} f(\beta \tau) &= \int_0^\sigma d\alpha_4 \int_{\alpha_4/\sigma}^1 d\beta \beta^{-1} (1 - \beta)^{-\zeta y_2} f(\alpha_4) \\
&\sim \int_0^\sigma d\alpha_4 \ln \frac{\sigma}{\alpha_4} f(\alpha_4) \tag{7.16}
\end{aligned}$$

gives

$$\begin{aligned}
J_{42}(\zeta, y_1, y_2) &\sim \int_0^1 \left(\prod_{j=1}^7 d\alpha_j \right) (\alpha_5 \alpha_6 \alpha_7)^{-1+\zeta y_1} \\
&\quad \times [(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) - \alpha_6^2]^{1-\zeta(y_1+y_2)} \\
&\quad \times [\alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2 \alpha_4 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) \\
&\quad + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)]^{-2+\zeta y_2} \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}. \tag{7.17}
\end{aligned}$$

In obtaining this result, we have also replaced $(\alpha_4 + \alpha_7)^{-1+\zeta y_1}$ by $\alpha_7^{-1+\zeta y_1}$ since the logarithm is small unless $\alpha_4 \ll \alpha_7$.

This formula (7.17) is very curious: it is the same as (7.8) except that the logarithmic factor has been simplified greatly. It may also be noted that, since the α 's are non-negative,

$$\ln \frac{\alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2 (\alpha_4 + \alpha_7) (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 [\alpha_1 (\alpha_4 + \alpha_7) + \alpha_2 \alpha_3]}{\alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2 \alpha_4 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)}$$

$$\leq \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}. \quad (7.18)$$

This inequality (7.18) implies that, to the leading order ζ^{-7} and after rescaling, the separate contributions from each corner of the α space are identical for (7.8) and (7.17). Let us, after a rescaling of (7.8) and (7.17), denote the contributions from the seven corners of the α space as A_j and B_j , respectively. Then

$$\sum_{j=1}^7 A_j \sim \sum_{j=1}^7 B_j, \quad (7.19)$$

by (7.8) and (7.17). On the other hand, it follows from (7.18) that

$$A_j \leq B_j, \quad \text{for } j = 1, \dots, 7, \quad (7.20)$$

and therefore

$$A_j \sim B_j, \quad (7.21)$$

for those j that contribute to the leading order.

Let

$$J_{42} = [\zeta(y_1 + y_2)]^{-1} \sum_{j=1}^7 K_{4j}(\zeta, y_1, y_2), \quad (7.22)$$

where K_{4j} is the contribution from the corner where the rescaled α_j ($= \alpha_j / \sum_{i=1}^7 \alpha_i$) is near 1. Clearly, K_{44} is negligible because the logarithm vanishes; so is K_{46} for reasons similar to the cases of $J_3(\zeta, y_1, y_2)$ in sect. 6 here and of $\tilde{I}_{32}(\zeta)$ of paper I. The other five K_{4j} 's are:

$$\begin{aligned} K_{41} = & \int_0^1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 d\alpha_7 (\alpha_5 \alpha_6 \alpha_7)^{-1+\zeta y_1} (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7)^{1-\zeta(y_1+y_2)} \\ & \times [\alpha_3(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_4(\alpha_2 + \alpha_6)]^{-2+\zeta y_2} \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}, \end{aligned} \quad (7.23)$$

$$\begin{aligned} K_{42} = & \int_0^1 d\alpha_1 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 d\alpha_7 (\alpha_5 \alpha_6 \alpha_7)^{-1+\zeta y_1} (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{1-\zeta(y_1+y_2)} \\ & \times [\alpha_4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_3(\alpha_1 + \alpha_6)]^{-2+\zeta y_2} \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}, \end{aligned} \quad (7.24)$$

$$\begin{aligned}
K_{43} &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_4 d\alpha_5 d\alpha_6 d\alpha_7 (\alpha_5 \alpha_6 \alpha_7)^{-1+\zeta y_1} (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7)^{1-\zeta(y_1+y_2)} \\
&\times [\alpha_1(\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_2(\alpha_4 + \alpha_6)]^{-2+\zeta y_2} \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}, \quad (7.25)
\end{aligned}$$

$$\begin{aligned}
K_{45} &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_6 d\alpha_7 (\alpha_6 \alpha_7)^{-1+\zeta y_1} (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7)^{1-\zeta(y_1+y_2)} \\
&\times [\alpha_2 \alpha_4 + \alpha_1 \alpha_3 (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7) + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)]^{-2+\zeta y_2} \\
&\times \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}, \quad (7.26)
\end{aligned}$$

and

$$\begin{aligned}
K_{47} &= \int_0^1 d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_5 d\alpha_6 (\alpha_5 \alpha_6)^{-1+\zeta y_1} (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{1-\zeta(y_1+y_2)} \\
&\times [\alpha_1 \alpha_3 + \alpha_2 \alpha_4 (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_6 (\alpha_1 \alpha_4 + \alpha_2 \alpha_3)]^{-2+\zeta y_2} \ln \frac{1}{\alpha_4}. \quad (7.27)
\end{aligned}$$

The integrals for K_{41} , K_{42} , K_{43} , and K_{45} are evaluated asymptotically in appendix B, and that for K_{47} in appendix C. The results are:

$$K_{41}(\zeta, y_1, y_2) \sim 2\zeta^{-6} \frac{1}{y_1^2 y_2^2 (y_1 + y_2)^2}, \quad (7.28)$$

$$K_{42}(\zeta, y_1, y_2) \sim 2\zeta^{-6} \frac{2y_1 + 3y_2}{y_1^2 y_2^2 (y_1 + y_2)^2 (2y_1 + y_2)}, \quad (7.29)$$

$$K_{43}(\zeta, y_1, y_2) \sim 2\zeta^{-6} \frac{1}{y_1^2 y_2 (y_1 + y_2)^2 (2y_1 + y_2)}, \quad (7.30)$$

$$K_{45}(\zeta, y_1, y_2) \sim 2\zeta^{-6} \frac{4}{y_1 y_2^3 (y_1 + y_2) (2y_1 + y_2)}, \quad (7.31)$$

and

$$K_{47}(\zeta, y_1, y_2) \sim 2\zeta^{-6} \frac{2y_1 + 3y_2}{y_1^2 y_2^3 (y_1 + y_2)^2}. \quad (7.32)$$

Substitution into (7.22) gives

$$J_{42}(\zeta, y_1, y_2) \sim 4\zeta^{-7} \frac{4}{y_1^2 y_2^3 (y_1 + y_2) (2y_1 + y_2)}, \quad (7.33)$$

which, together with (7.6) and (7.9), yields

$$J_4(\zeta, y_1, y_2) \sim 4\zeta^{-7} \frac{4y_1 + 3y_2}{y_1^3 y_2^2 (y_1 + y_2)^2 (2y_1 + y_2)}. \quad (7.34)$$

This is the desired result. Again the factor $2y_1 + y_2$ appears in the denominator.

8. Asymptotic behavior for large s and large $|t|$

By (3.17) and (3.18), the four-rung ladder diagram in φ^3 theory is described by

$$\begin{aligned} \tilde{I}_4(\zeta) \sim & -(\pi i)^{-1} \int_{c_1-i\infty}^{c_1+i\infty} dy_1 \int_{c_2-i\infty}^{c_2+i\infty} dy_2 (y_1 + y_2)^{-1} \\ & \times \delta_c(1 - y_1 - \gamma y_2) J(\zeta, y_1, y_2), \end{aligned} \quad (8.1)$$

with

$$J(\zeta, y_1, y_2) = 4J_1(\zeta, y_1, y_2) + 2J_2(\zeta, y_1, y_2) + 2J_3(\zeta, y_1, y_2) + 2J_4(\zeta, y_1, y_2). \quad (8.2)$$

By (4.21), (5.15), (6.5), and (7.34), this J is asymptotically

$$J(\zeta, y_1, y_2) \sim 8\zeta^{-7} \frac{4y_1 + 3y_2}{y_1^4 y_2^2 (y_1 + y_2)(2y_1 + y_2)} \quad (8.3)$$

for small ζ . As already noted in the next to the last paragraph of sect. 4, the factor $2y_1 + y_2$ is present in the denominator of this total J . The presence of this factor implies that the asymptotic behavior of the four-rung ladder amplitude for large s and large $|t|$ is given by two different formulas, one for $(\ln |t|)/(\ln s) \leq \frac{1}{2}$ and another one for $(\ln |t|)/(\ln s) \geq \frac{1}{2}$.

When (8.3) is substituted into (8.1), elimination of the y_1 integration by the delta function gives

$$\tilde{I}_4(\zeta) \sim -8\zeta^{-7} \frac{1}{\pi i} \int_{c_2-i\infty}^{c_2+i\infty} dy_2 \frac{4 + (3 - 4\gamma)y_2}{y_2^2 (1 - \gamma y_2)^4 [1 + (1 - \gamma)y_2]^2 [2 + (1 - 2\gamma)y_2]}. \quad (8.4)$$

The contour of integration, where $0 < c_2 < \gamma^{-1}$, is shown in fig. 4 together with the locations of the pole singularities. The aforementioned $2y_1 + y_2$, which equals $2 + (1 - 2\gamma)y_2$ when $1 - y_1 - \gamma y_2 = 0$, gives a simple pole at

$$y_2 = -\frac{2}{1 - 2\gamma}, \quad (8.5)$$

which is less than -2 when $0 < \gamma < \frac{1}{2}$, but larger than 2 when $\frac{1}{2} < \gamma < 1$. There are therefore these two distinct cases.

For $0 < \gamma \leq \frac{1}{2}$, it is seen from fig. 4a that we can close on the fourth-order pole at γ^{-1} to get, for small ζ ,

$$\tilde{I}_4(\zeta) \sim -8\zeta^{-7}\gamma^3(5! - 2 \cdot 5!\gamma + 12 \cdot 17\gamma^2 - 4 \cdot 17\gamma^3), \quad (8.6)$$

and hence by (3.6)

$$\begin{aligned} I_4(s, t) \sim & -\frac{4}{3}s^{-1}|t|^{-3}(\ln|t|)^3[(\ln s)^3 - 2(\ln s)^2(\ln|t|) \\ & + \frac{17}{10}(\ln s)(\ln|t|)^2 - \frac{17}{30}(\ln|t|)^3]. \end{aligned} \quad (8.7)$$

On the other hand, for $\frac{1}{2} < \gamma < 1$, it is seen from fig. 4b that there are two poles to each side of the contour of integration. Suppose we close the contour to the right; then (8.6) is modified with an additional term,

$$\tilde{I}_4(\zeta) \sim -8\zeta^{-7}[\gamma^3(5! - 2 \cdot 5!\gamma + 12 \cdot 17\gamma^2 - 4 \cdot 17\gamma^3) + (1 - 2\gamma)^6], \quad (8.8)$$

and hence

$$\begin{aligned} I_4(s, t) \sim & -\frac{4}{3}s^{-1}|t|^{-3}\{(\ln|t|)^3[(\ln s)^3 - 2(\ln s)^2(\ln|t|) + \frac{17}{10}(\ln s)(\ln|t|)^2 \\ & - \frac{17}{30}(\ln|t|)^3] + \frac{1}{120}[\ln s - 2\ln|t|]^6\} \\ = & -\frac{1}{90}s^{-1}|t|^{-3}[(\ln s)^6 - 12(\ln s)^5(\ln|t|) + 60(\ln s)^4(\ln|t|)^2 \\ & - 40(\ln s)^3(\ln|t|)^3 + 12(\ln s)(\ln|t|)^5 - 4(\ln|t|)^6]. \end{aligned} \quad (8.9)$$

Both (8.7) and (8.9) hold asymptotically when $s \gg -t \gg 1$. Together they are equivalent to (1.4).

9. Discussion

In order to gain some insight into this non-analyticity at $(\ln|t|)/(\ln s) = \frac{1}{2}$, let us take another look at the two-rung and three-rung ladder diagrams treated in paper I. The analogs of (8.1) and (8.4) are, from (I.3.9), (I.4.11), (I.A3) and (I.C6),

$$\begin{aligned} \tilde{I}_2(\zeta) \sim & -(\pi i)^{-1} \int_{c_1-i\infty}^{c_1+i\infty} dy_1 \int_{c_2-i\infty}^{c_2+i\infty} dy_2 \delta_c(1 - y_1 - \gamma y_2) \zeta^{-3} y_1^{-2} y_2^{-2} \\ = & -\zeta^{-3} \frac{1}{\pi i} \int_{c_2-i\infty}^{c_2+i\infty} dy_2 \frac{1}{y_2^2(1 - \gamma y_2)^2}, \end{aligned} \quad (9.1)$$

and

$$\begin{aligned}
\tilde{I}_3(\zeta) &\sim (\pi i)^{-1} \int_{c_1-i\infty}^{c_1+i\infty} dy_1 \int_{c_2-i\infty}^{c_2+i\infty} dy_2 \delta_c(1-y_1-\gamma y_2) 4\zeta^{-5} y_1^{-3} y_2^{-2} (y_1+y_2)^{-1} \\
&= \zeta^{-5} \frac{1}{\pi i} \int_{c_2-i\infty}^{c_2+i\infty} dy_2 \frac{4}{y_2^2(1-\gamma y_2)^3[1+(1-\gamma)y_2]}.
\end{aligned} \tag{9.2}$$

There is a basic difference between (9.1) and (9.2): in (9.1) for $\gamma > 0$ there is always one pole to the right and one to the left of the contour of integration, while in (9.2) for $\gamma > 0$ the pole at $y_2 = -(1-\gamma)^{-1}$ is on the left for $\gamma < 1$ but on the right for $\gamma > 1$.

It follows from (9.1) and (9.2) that

$$\tilde{I}_2(\zeta) \sim -4\gamma\zeta^{-3} \tag{9.3}$$

for all $\gamma > 0$, and

$$\tilde{I}_3(\zeta) \sim \begin{cases} 8\zeta^{-5}\gamma^2(6-4\gamma+\gamma^2) & \text{for } 0 < \gamma \leq 1, \\ 8\zeta^{-5}(-1+4\gamma) & \text{for } \gamma \geq 1. \end{cases} \tag{9.4}$$

The difference between the two forms on the right-hand side of (9.4) is

$$-8\zeta^{-5}(1-\gamma)^4. \tag{9.5}$$

Thus, the $\tilde{I}_3(\zeta)$ as given by (9.4) is non-analytic at $\gamma = 1$, although $\tilde{I}_3(\zeta)$ together with its first three derivatives are continuous there. A comparison of (9.4) with (1.4) shows the similarity between the three-rung and the four-rung ladder diagrams, the single but important difference being that, for the four-rung case, the non-analyticity occurs in the physical region.

For completeness, we summarize the results on the two-rung, three-rung, and four-rung ladder diagrams in φ^3 theory for large s and large $|t|$,

$$s \gg m^2 = 1, \quad -t \gg m^2 = 1, \tag{9.6}$$

including the unphysical region where $-t$ is larger than s :

$$I_2(s, t) \sim -2s^{-1}|t|^{-1} \ln s \ln |t|; \tag{9.7}$$

$$I_3(s, t) \sim \begin{cases} \frac{1}{3} s^{-1} t^{-2} (\ln |t|)^2 [6(\ln s)^2 - 4(\ln s)(\ln |t|) + (\ln |t|)^2], \\ \frac{1}{3} s^{-1} t^{-2} (\ln s)^3 [-\ln s + 4 \ln |t|], \end{cases} \tag{9.8}$$

where the upper line is for $\ln s \geq \ln |t|$, and the lower one for $\ln s \leq \ln |t|$; and

$$I_4(s, t) \sim \begin{cases} -\frac{1}{90} s^{-1} |t|^{-3} (\ln |t|)^3 [120 (\ln s)^3 - 240 (\ln s)^2 (\ln |t|) \\ \quad + 204 (\ln s) (\ln |t|)^2 - 68 (\ln |t|)^3], \\ -\frac{1}{90} s^{-1} |t|^{-3} [(\ln s)^6 - 12 (\ln s)^5 (\ln |t|) + 60 (\ln s)^4 (\ln |t|)^2 \\ \quad - 40 (\ln s)^3 (\ln |t|)^3 + 12 (\ln s) (\ln |t|)^5 - 4 (\ln |t|)^6], \\ -\frac{1}{90} s^{-1} |t|^{-3} (\ln s)^5 [-7 \ln s + 24 \ln |t|], \end{cases} \quad (9.9)$$

where the three formulas on the right-hand side hold, respectively, for $\ln |t| \leq \frac{1}{2} \ln s$, $\frac{1}{2} \ln s \leq \ln |t| \leq \ln s$, and $\ln |t| > \ln s$.

Aside from this striking feature of the lack of analyticity, there are also the following two peculiarities that have already been mentioned: (i) The integrands of (4.7) and (I.4.9) differ only in the appearance of $\alpha_4 + \alpha_7$ in (4.7) versus α_4 in (I.4.9); and (ii) the numerator and the denominator of the logarithmic factor in (7.8) again differ only in the appearance of $\alpha_4 + \alpha_7$ versus α_4 . These features are not accidental and, together with the lack of analyticity, are important clues that remain to be exploited.

Acknowledgments

Useful discussions with Professor Hung Cheng, Professor Raymond Gastmans, Professor Harry Lehmann, Professor Barry McCoy, and Professor Walter Troost are gratefully acknowledged. One of us (TTW) wishes to thank Professor John Ellis, Professor Maurice Jacob, and many other members of the Theory Division for their kind hospitality at CERN.

Appendix A

THE INTEGRALS K_{12} AND K_{15} OF EQS. (4.10) AND (4.13)

Rewrite the definition (4.10) in the form

$$K_{12} = \int_0^1 d\alpha_1 d\alpha_3 d\alpha_5 d\alpha_6 (\alpha_5 \alpha_6)^{-1+\zeta y_1} (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{-\zeta(2y_1+y_2)} \bar{K}_{12}, \quad (\text{A1})$$

where

$$\begin{aligned} \bar{K}_{12} = & (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) \int_0^1 d\alpha_4 [\alpha_4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)]^{\zeta y_1} \\ & \times [\alpha_4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \alpha_3(\alpha_1 + \alpha_6)]^{-2+\zeta y_2}. \end{aligned} \quad (\text{A2})$$

The Feynman-parameter representation (2.1) may be used to evaluate this $\bar{K}_{12}$:

$$\begin{aligned} \bar{K}_{12} = & (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) \int_0^1 d\alpha_4 \frac{\Gamma(2-\zeta y_1-\zeta y_2)}{\Gamma(-\zeta y_1)\Gamma(2-\zeta y_2)} \int_0^1 d\beta (1-\beta)^{-1-\zeta y_1} \beta^{1-\zeta y_2} \\ & \times [\alpha_4(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6) + \beta\alpha_3(\alpha_1 + \alpha_6)]^{-2+\zeta(y_1+y_2)} \\ \sim & \frac{\Gamma(2-\zeta y_1-\zeta y_2)}{\Gamma(-\zeta y_1)\Gamma(2-\zeta y_2)} \int_0^1 d\beta (1-\beta)^{-1-\zeta y_1} \beta^{1-\zeta y_2} [\beta\alpha_3(\alpha_1 + \alpha_6)]^{-1+\zeta(y_1+y_2)} \\ \sim & [\alpha_3(\alpha_1 + \alpha_6)]^{-1+\zeta(y_1+y_2)}. \end{aligned} \quad (\text{A3})$$

The result (4.16) follows immediately from (A1) and (A3). [Strictly speaking, (2.4) with the contour of fig. 2 should be used in (A3); see the discussion after (2.3).]

The case of K_{15} is more complicated because the integrand in (4.13) is not homogeneous in the α 's. Following the procedure in appendix C of paper I, we first integrate over α_1 :

$$\begin{aligned} K_{15}(\zeta, y_1, y_2) \sim & \int_0^1 d\alpha_2 d\alpha_3 d\alpha_4 d\alpha_6 \alpha_4^{\zeta y_1} \alpha_6^{-1+\zeta y_1} (\alpha_2 + \alpha_4 + \alpha_6)^{1-\zeta(y_1+y_2)} \\ & \times [\alpha_3(\alpha_2 + \alpha_4 + \alpha_6) + \alpha_4\alpha_6]^{-1} \{ [\alpha_2(\alpha_4 + \alpha_3\alpha_6)]^{-1+\zeta y_2} \\ & - [\alpha_2\alpha_4 + \alpha_3(\alpha_2 + \alpha_4 + \alpha_6) + \alpha_4\alpha_6]^{-1+\zeta y_2} \}. \end{aligned} \quad (\text{A4})$$

Both terms in the braces must be kept because, in the notation of (4.4), we do not have $B \ll A$. Next, we shift the variable α_3 by defining

$$\alpha'_3 = \alpha_3 + \frac{\alpha_4 \alpha_6}{\alpha_2 + \alpha_4 + \alpha_6}, \quad (\text{A5})$$

and use the approximate formula (I.D1) to get that K_{15} is approximated by the left-hand side of (2.8). But the evaluation of the right-hand side of (2.8) is straightforward, and the result is (4.19).

Appendix B

THE INTEGRALS K_{41} , K_{42} , K_{43} AND K_{45} OF EQS. (7.23)–(7.26)

For K_{41} of eq. (7.23), the α_3 and α_5 integrations can be readily carried out:

$$\begin{aligned} K_{41} \sim & \frac{1}{\zeta y_1} \int_0^1 d\alpha_2 d\alpha_4 d\alpha_6 d\alpha_7 (\alpha_6 \alpha_7)^{-1+\zeta y_1} (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7)^{-\zeta(y_1+y_2)} \\ & \times [\alpha_4(\alpha_2 + \alpha_6)]^{-1+\zeta y_2} \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}. \end{aligned} \quad (\text{B1})$$

Application of an approximate formula similar to (2.6) then gives

$$\begin{aligned} K_{41} \sim & \frac{2}{\zeta y_1} \int_0^1 d\alpha_2 d\alpha_4 d\alpha_6 d\alpha_7 (\alpha_6 \alpha_7)^{-1+\zeta y_1} [\alpha_4(\alpha_2 + \alpha_6)]^{-1+\zeta y_2} \\ & \times \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}, \end{aligned} \quad (\text{B2})$$

which is a product of one double integral over α_2 and α_6 , and another one over α_4 and α_7 . This leads to (7.28).

The evaluation of K_{42} is less straightforward. Rewrite (7.24) in the form

$$K_{42} = \int_0^1 d\alpha_1 d\alpha_3 d\alpha_5 d\alpha_6 (\alpha_5 \alpha_6)^{-1+\zeta y_1} (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{-1-\zeta y_1} \bar{K}_{42}(\epsilon), \quad (\text{B3})$$

where

$$\epsilon = \frac{\alpha_3(\alpha_1 + \alpha_6)}{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6}, \quad (\text{B4})$$

and

$$\bar{K}_{42}(\epsilon) = \int_0^1 d\alpha_4 d\alpha_7 \alpha_7^{-1+\zeta y_1} (\alpha_4 + \epsilon)^{-2+\zeta y_2} \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}. \quad (\text{B5})$$

Scaling with respect to ϵ gives

$$\bar{K}_{42}(\epsilon) = \epsilon^{-1+\zeta(y_1+y_2)} \int_0^{1/\epsilon} d\alpha_4 d\alpha_7 \alpha_7^{-1+\zeta y_1} (\alpha_4 + 1)^{-2+\zeta y_2} \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}. \quad (\text{B6})$$

Since the ϵ of (B4) is small, this is readily approximated by

$$\bar{K}_{42}(\epsilon) \sim \epsilon^{-1+\zeta(y_1+y_2)} \int_0^{1/\epsilon} d\alpha_7 \alpha_7^{-1+\zeta y_1} \int_0^\infty d\alpha_4 (\alpha_4 + 1)^{-2} \ln \frac{\alpha_4 + \alpha_7}{\alpha_4}. \quad (\text{B7})$$

Evaluation of the α_4 integral gives

$$\bar{K}_{42}(\epsilon) \sim \epsilon^{-1+\zeta(y_1+y_2)} \int_0^{1/\epsilon} d\alpha_7 \alpha_7^{\zeta y_1} (\alpha_7 - 1)^{-1} \ln \alpha_7. \quad (\text{B8})$$

Since ζ is small, the leading contribution comes from large α_7 , and hence

$$\begin{aligned} \bar{K}_{42}(\epsilon) &\sim \epsilon^{-1+\zeta(y_1+y_2)} \int_1^{1/\epsilon} d\alpha_7 \alpha_7^{-1+\zeta y_1} \ln \alpha_7 \\ &= \zeta^{-2} y_1^{-2} \epsilon^{-1+\zeta y_2} [\epsilon^{\zeta y_1} - 1 - \zeta y_1 \ln \epsilon]. \end{aligned} \quad (\text{B9})$$

By (B4), the substitution of (B9) into (B3) gives

$$\begin{aligned} K_{42} &\sim \frac{1}{\zeta^2 y_1^2} \int_0^1 d\alpha_1 d\alpha_3 d\alpha_5 d\alpha_6 (\alpha_5 \alpha_6)^{-1+\zeta y_1} \\ &\quad \times (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{-\zeta(y_1+y_2)} [\alpha_3(\alpha_1 + \alpha_6)]^{-1+\zeta y_2} \\ &\quad \times \left\{ \left[\frac{\alpha_3(\alpha_1 + \alpha_6)}{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6} \right]^{\zeta y_1} - 1 + \zeta y_1 \ln \frac{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6}{\alpha_3(\alpha_1 + \alpha_6)} \right\}, \end{aligned} \quad (\text{B10})$$

which simplifies to, by a formula similar to (2.8),

$$\begin{aligned} K_{42} &\sim \frac{2}{\zeta^2 y_1^2} \int_0^1 d\alpha_1 d\alpha_3 d\alpha_5 d\alpha_6 (\alpha_5 \alpha_6)^{-1+\zeta y_1} [\alpha_3(\alpha_1 + \alpha_6)]^{-1+\zeta y_2} \\ &\quad \times \left\{ [\alpha_3(\alpha_1 + \alpha_6)]^{\zeta y_1} - 1 + \zeta y_1 \ln \frac{1}{\alpha_3(\alpha_1 + \alpha_6)} \right\}. \end{aligned} \quad (\text{B11})$$

This leads to (7.29).

The derivation of (7.30) from (7.25) for K_{43} is entirely similar to that of K_{41} . For the K_{45} of (7.26), the α_1 and the α_3 integrations can be carried out by (D1):

$$K_{45} \sim \int_0^1 d\alpha_2 d\alpha_4 d\alpha_6 d\alpha_7 (\alpha_6 \alpha_7)^{-1+\zeta y_1} (\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7)^{-\zeta(y_1+y_2)} \\ \times (\alpha_2 \alpha_4)^{-1+\zeta y_2} \ln \frac{\alpha_4 + \alpha_7}{\alpha_4} \ln \frac{\alpha_2 + \alpha_4 + \alpha_6 + \alpha_7}{(\alpha_2 + \alpha_6)(\alpha_4 + \alpha_6)}. \quad (\text{B12})$$

By an approximation analogous to (2.8), this is simplified to

$$K_{45} \sim 2 \int_0^1 d\alpha_2 d\alpha_4 d\alpha_6 d\alpha_7 (\alpha_6 \alpha_7)^{-1+\zeta y_1} (\alpha_2 \alpha_4)^{-1+\zeta y_2} \\ \times \ln \frac{\alpha_4 + \alpha_7}{\alpha_4} \ln \frac{1}{(\alpha_2 + \alpha_6)(\alpha_4 + \alpha_6)}. \quad (\text{B13})$$

If the last factor is rewritten as the two parts

$$-\ln(\alpha_2 + \alpha_6) - \ln(\alpha_4 + \alpha_6),$$

then the first term splits into the products of two double integrals, one over α_2 and α_6 , and the other over α_4 and α_7 , while the second term becomes a triple integral after the α_2 integration is carried out trivially to give $(\zeta y_2)^{-1}$. The rest is straightforward and leads to (7.31).

Appendix C

THE INTEGRAL K_{47} OF EQ. (7.27)

In the four cases of K_{41} , K_{42} , K_{43} and K_{45} as treated in appendix B, the logarithmic factor $\ln[(\alpha_4 + \alpha_7)/\alpha_4]$ does not play a role in the first stage of carrying out two of the six α integrations, i.e., this factor is dealt with only after a reduction to a four-fold integral. This is not the case for the K_{47} as defined by eq. (7.27), where the logarithmic factor $\ln(1/\alpha_4)$ must be treated after carrying out only one integration, the one over α_2 :

$$K_{47} \sim \int_0^1 d\alpha_1 d\alpha_3 d\alpha_5 d\alpha_6 (\alpha_5 \alpha_6)^{-1+\zeta y_1} (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{-\zeta(y_1+y_2)}$$

$$\begin{aligned}
& \times \int_0^1 d\alpha_4 \left[\alpha_4 + \frac{\alpha_3 \alpha_6}{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6} \right]^{-1} \left\{ \left[\alpha_1 \alpha_3 \left(1 + \frac{\alpha_6}{\alpha_3} \alpha_4 \right) \right]^{-1+\zeta y_2} \right. \\
& \left. - \left[\alpha_3 (\alpha_1 + \alpha_6) \left(1 + \frac{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6}{\alpha_3 (\alpha_1 + \alpha_6)} \alpha_4 \right) \right]^{-1+\zeta y_2} \right\} \ln \frac{1}{\alpha_4}. \quad (C1)
\end{aligned}$$

The α_4 integration is then carried out with the approximate formula (D9), where the first case is the relevant one, because $\epsilon_2 = 0$ here:

$$\begin{aligned}
K_{47} & \sim \frac{1}{2} \int_0^1 d\alpha_1 d\alpha_3 d\alpha_5 d\alpha_6 (\alpha_5 \alpha_6)^{-1+\zeta y_1} \\
& \times (\alpha_1 \alpha_3)^{-1+\zeta y_2} (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^{-\zeta(y_1+y_2)} \\
& \times \left[\ln \frac{\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6}{\alpha_6 (\alpha_3 + \alpha_6)} \ln \frac{(\alpha_3 + \alpha_6)(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)}{\alpha_3^2 \alpha_6} \right. \\
& \left. - \ln \frac{\alpha_1 + \alpha_6}{\alpha_6} \ln \frac{(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6)^2}{\alpha_3^2 \alpha_6 (\alpha_1 + \alpha_6)} \right]. \quad (C2)
\end{aligned}$$

In the last term, we have neglected a factor $(1 + \alpha_6/\alpha_1)^{\zeta y_2}$. Because of the third logarithm, this factor must be ~ 1 for the last term to contribute.

A variant of (2.8) gives

$$\begin{aligned}
K_{47} & \sim \int_0^1 d\alpha_1 d\alpha_3 d\alpha_5 d\alpha_6 (\alpha_5 \alpha_6)^{-1+\zeta y_1} (\alpha_1 \alpha_3)^{-1+\zeta y_2} \\
& \times \left\{ \ln \frac{1}{\alpha_6 (\alpha_3 + \alpha_6)} \ln \frac{\alpha_3 + \alpha_6}{\alpha_3^2 \alpha_6} - \ln \frac{\alpha_1 + \alpha_6}{\alpha_6} \ln \frac{1}{\alpha_3^2 \alpha_6 (\alpha_1 + \alpha_6)} \right\}. \quad (C3)
\end{aligned}$$

The α_5 integration gives a factor of $(\zeta y_1)^{-1}$, and the rest of the calculation is straightforward, leading to (7.32).

Appendix D

TWO USEFUL APPROXIMATE FORMULAS

The following simple approximate formula is often useful:

$$\begin{aligned}
& \int_0^1 d\alpha d\alpha' (A\alpha\alpha' + B\alpha + C\alpha' + D)^{-2+\zeta} \\
& \sim A^{-1} D^{-1+\zeta} \ln \frac{AD}{(B+D)(C+D)}, \quad (D1)
\end{aligned}$$

valid when $|\zeta|$ is small and

$$A \gg B, \quad A \gg C, \quad A \gg D, \quad AD \gg BC. \quad (\text{D2})$$

To get (D1), we first integrate over α' , denoting the left-hand side as I_0 ,

$$I_0 \sim \int_0^1 d\alpha (A\alpha + C)^{-1} [(B\alpha + D)^{-1+\zeta} - (A\alpha + C + D)^{-1+\zeta}], \quad (\text{D3})$$

where a term $B\alpha$ has been dropped from inside the last set of parentheses because of (D2).

Let

$$\bar{\alpha} = \alpha + \delta, \quad (\text{D4})$$

where

$$\delta = C/A \quad (\text{D5})$$

is small, again because of (D2). Then

$$I_0 \sim A^{-1} \int_{\delta}^1 d\bar{\alpha} \bar{\alpha}^{-1} \left[D^{-1+\zeta} \left(1 + \frac{B}{D} \bar{\alpha} \right)^{-1+\zeta} - D^{-1+\zeta} \left(1 + \frac{A}{D} \bar{\alpha} \right)^{-1+\zeta} \right], \quad (\text{D6})$$

where, in the first term, $B\delta$ has been neglected compared with D because of (D2). In this form, application of (I.D1) gives

$$I_0 \sim A^{-1} D^{-1+\zeta} \left[\ln \frac{1 + BC/AD}{(1 + B/D)C/A} - \ln \frac{1 + C/D}{(1 + A/D)C/A} \right], \quad (\text{D7})$$

which, by (D2), leads immediately to (D1).

Actually, the smallness of $|\zeta|$ is in no way needed. The more general formula is, for $\tau > 1$,

$$\begin{aligned} & \int_0^1 d\alpha d\alpha' (A\alpha\alpha' + B\alpha + C\alpha' + D)^{-\tau} \\ & \sim \frac{1}{\tau - 1} A^{-1} D^{-(\tau-1)} \ln \frac{AD}{(B + D)(C + D)} \end{aligned} \quad (\text{D8})$$

when (D2) holds.

A second useful approximate formula is

$$\int_0^1 \frac{dx}{x + \epsilon_1} (1 + Ax)^{-1+\zeta} \ln \frac{1}{x + \epsilon_2} \sim \begin{cases} \frac{1}{2}[1 - (1 + A)\epsilon_1]^{-1} \ln[(1 + A)\epsilon_1] \ln[\epsilon_1/(1 + A)], & \text{for } \epsilon_2 \leq \epsilon_1 \text{ and } A\epsilon_2 \leq 1, \\ \frac{1}{2}(1 - A\epsilon_1)^{-1} \{(\ln \epsilon_1)^2 + (\ln \epsilon_2)^2 - 2[\ln(1/A)](\ln \epsilon_2)\}, & \text{for } \epsilon_2 \leq \epsilon_1 \text{ and } A\epsilon_2 \geq 1, \\ \frac{1}{2}(1 - A\epsilon_1)^{-1} \{2(\ln \epsilon_1)(\ln \epsilon_2) - (\ln \epsilon_2)^2 - [\ln(1 + A)]^2\}, & \text{for } \epsilon_2 \geq \epsilon_1 \text{ and } A\epsilon_2 \leq 1, \\ (1 - A\epsilon_1)^{-1}(\ln \epsilon_2)[\ln(A\epsilon_1)], & \text{for } \epsilon_2 \geq \epsilon_1 \text{ and } A\epsilon_2 \geq 1. \end{cases} \quad (\text{D9})$$

This (D9) holds for ϵ_1 positive and small, ϵ_2 non-negative and small, $|\zeta|$ small, and A positive and not much larger than ϵ_1^{-1} . Note that ζ does not appear on the right-hand side.

Before studying this integral on the left-hand side of (D9), let us consider the simpler case of

$$\int_0^1 \frac{dx}{x + \epsilon_1} (1 + Ax)^{-1+\zeta}. \quad (\text{D10})$$

Since ϵ_1 is small, a shift of x by ϵ_1 shows that

$$(\text{D10}) \sim (1 - A\epsilon_1)^{-1+\zeta} \int_{\epsilon_1}^1 \frac{dx}{x} \left(1 + \frac{A}{1 - A\epsilon_1} x\right)^{-1+\zeta}. \quad (\text{D11})$$

Application of (I.D1) gives

$$\int_0^1 \frac{dx}{x + \epsilon_1} (1 + Ax)^{-1+\zeta} \sim (1 - A\epsilon_1)^{-1} \ln \frac{1}{(1 - A\epsilon_1 + A)\epsilon_1}. \quad (\text{D12})$$

This holds when ϵ_1 is positive and small, $|\zeta|$ small, and A positive and not much larger than ϵ_1^{-1} . The right-hand side is finite when $A\epsilon_1 = 1$. A slightly more elegant form of (D12) is

$$\int_0^1 \frac{dx}{x + \epsilon_1} (1 + Ax)^{-1+\zeta} \sim [1 - (1 + A)\epsilon_1]^{-1} \ln \frac{1}{(1 + A)\epsilon_1}. \quad (\text{D13})$$

That the right-hand side is independent of ζ can be easily understood. The important range of contribution is x of the order of ϵ_1 . Since $A\epsilon_1$ is not much larger than 1, in this range $(1 + Ax)^\zeta$ is approximately 1.

We are now ready to derive (D9) by partial fraction:

$$\begin{aligned} \text{Left-hand side of (D9)} &\sim \int_0^1 \frac{dx}{x + \epsilon_1} (1 + Ax)^{-1} \ln \frac{1}{x + \epsilon_2} \\ &= (1 - A\epsilon_1)^{-1} [F(\epsilon_1, \epsilon_2) - F(A^{-1}, \epsilon_2)], \end{aligned} \quad (\text{D14})$$

where

$$F(\epsilon_1, \epsilon_2) = \int_0^1 \frac{dx}{x + \epsilon_1} \ln \frac{1}{x + \epsilon_2}. \quad (\text{D15})$$

This $F(\epsilon_1, \epsilon_2)$ satisfies the exact relation

$$F(\epsilon_1, \epsilon_2) + F(\epsilon_2, \epsilon_1) = (\ln \epsilon_1)(\ln \epsilon_2) - \ln(1 + \epsilon_1) \ln(1 + \epsilon_2). \quad (\text{D16})$$

Since ϵ_2 is small, this reduces to

$$F(\epsilon_1, \epsilon_2) + F(\epsilon_2, \epsilon_1) \sim (\ln \epsilon_1)(\ln \epsilon_2), \quad (\text{D17})$$

which implies in particular that

$$F(\epsilon_2, \epsilon_2) \sim \frac{1}{2}(\ln \epsilon_2)^2 \quad (\text{D18})$$

for small ϵ_2 .

We next consider the case $0 < \epsilon_2 \leq \epsilon_1 \ll 1$. Then

$$\begin{aligned} F(\epsilon_1, \epsilon_2) - F(\epsilon_1, \epsilon_1) &= \int_0^1 \frac{dx}{x + \epsilon_1} \ln \frac{x + \epsilon_1}{x + \epsilon_2} \\ &= \int_{\epsilon_1}^{1+\epsilon_1} \frac{dx}{x} \ln \frac{1}{1 - (\epsilon_1 - \epsilon_2)/x}. \end{aligned} \quad (\text{D19})$$

Since

$$\frac{\epsilon_1 - \epsilon_2}{x} \leq \frac{\epsilon_1}{x} \leq 1 \quad \text{for } x > \epsilon_1,$$

we get

$$\begin{aligned} 0 < F(\epsilon_1, \epsilon_2) - F(\epsilon_1, \epsilon_1) &< \int_{\epsilon_1}^{\infty} \frac{dx}{x} \ln \frac{1}{1 - \epsilon_1/x} \\ &= \int_0^1 \frac{dy}{y} \ln \frac{1}{1 - y} = \frac{\pi^2}{6} = O(1). \end{aligned} \quad (\text{D20})$$

Neglecting terms of $O(1)$, we therefore have

$$F(\epsilon_1, \epsilon_2) \sim \frac{1}{2} (\ln \epsilon_1)^2, \quad (\text{D21})$$

for $0 < \epsilon_2 \leq \epsilon_1 \ll 1$. By (D17), we also get

$$F(\epsilon_1, \epsilon_2) \sim \ln \epsilon_1 \ln \epsilon_2 - \frac{1}{2} (\ln \epsilon_2)^2, \quad (\text{D22})$$

for $0 < \epsilon_1 \leq \epsilon_2 \ll 1$.

These results are adequate for $F(\epsilon_1, \epsilon_2)$ of (D14), but not quite for $F(A^{-1}, \epsilon_2)$ there, because $1/A$ may be large. If $1/A$ is of the order of 1 or larger, i.e., if $A = O(1)$, then $F(1/A, \epsilon_2)$ does not contribute to the integral (D9). Therefore, it is sufficient to make sure that, if $A = O(1)$, $F(1/A, \epsilon_2)$ is negligible. This is most simply accomplished by a suitable replacement of $\ln(1/A)$ by $\ln[1/(1+A)]$. In other words, we use

$$F(1/A, \epsilon_2) \sim \begin{cases} \frac{1}{2} [\ln(1+A)]^2, & 1/A \geq \epsilon_2 > 0, \\ [\ln(1/A)](\ln \epsilon_2) - \frac{1}{2} (\ln \epsilon_2)^2, & \epsilon_2 \geq 1/A > 0. \end{cases} \quad (\text{D23})$$

Note that, strictly speaking, this (D23) is not consistent because the right-hand sides are not quite the same if $1/A = \epsilon_2$. But the discontinuity is small, and either formula may be used for $A^{-1} = \epsilon_2$.

The substitution of (D21)–(D23) into (D14) yields (D9).

REFERENCES

- [1] P. Osland and T. T. Wu, DESY Preprint (1986)
- [2] R. P. Feynman, Phys. Rev. 76 (1949) 769
- [3] K. Symanzik, Progr. Theor. Phys. 20 (1958) 690
- [4] J. D. Bjorken and T. T. Wu, Phys. Rev. 130 (1963) 2566
- [5] D. Amati, A. Stangellini, and S. Fubini, Nuovo Cimento 26 (1962) 896;
B. W. Lee and R. F. Sawyer, Phys. Rev. 127 (1962) 2266;
M. Gell-Mann and M. L. Goldberger, Phys. Rev. Letters 9 (1962) 275;
P. G. Federbush and M. T. Grisaru, Ann. Phys. 22 (1963) 263 and 299;
J. C. Polkinghorne, J. Math. Phys. 4 (1963) 503, 5 (1964) 431 and 1491;
R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge University Press, 1966), Chapter 3

Fig. 1. The asymptotic behavior of the amplitude $I_4(s, t)$ for the four-rung ladder diagram when s and $-t$ are both large. The curve is analytic except at $\gamma = \frac{1}{2}$. The dotted lines show the analytic continuation of the function valid in the region $\gamma > \frac{1}{2}$ ($\gamma < \frac{1}{2}$) to the region $\gamma < \frac{1}{2}$ ($\gamma > \frac{1}{2}$).

Fig. 2. The contour of integration for eq. (2.4).

Fig. 3. The four-rung, or eighth-order, ladder diagram in φ^3 theory.

Fig. 4. The contour of integration in the y_2 plane for the right-hand side of (8.4), together with the location of the pole singularities.