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AN EXPANSION FOR THE EFFECTIVE ACTION OF AN INTERACTING QUANTUM FIELD THEORY IN CURVED SPACE*

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ABSTRACT

A natural modification of the DeWitt-Schwinger technique is applied to derive an expansion for the one loop effective action for $\lambda\phi^4$ theory in curved spacetime. Explicit calculation is performed through terms which are second order in the curvature or in derivatives of the mean field.

The effective action $\Gamma[\bar{\phi}]$ has come to play a central role in the study of quantum field theory in curved spacetime.¹ For interacting theories, $\Gamma[\bar{\phi}]$ is most usually calculated as a perturbation expansion about some effective free theory.^{2,3,4} In this way, at each order of perturbation theory, $\Gamma[\bar{\phi}]$ may be expressed in the form of an infinite series of local terms. In flat space these terms can be ordered by the number of spacetime derivatives of the mean field $\bar{\phi}$ they contain. Thus, $\Gamma[\bar{\phi}]$ may be written generically as

$$\Gamma[\bar{\phi}] = \int d^4x \left\{ -V(\bar{\phi}) - \frac{1}{2}Z(\bar{\phi})(\partial\bar{\phi})^2 + \text{higher derivative terms} \right\}. \quad (1)$$

In flat space, the calculation of $\Gamma[\bar{\phi}]$ is normally performed in momentum space and, indeed, if properly applied this provides an efficient means of evaluating the functions $V(\bar{\phi})$, $Z(\bar{\phi})$, etc.⁵ Momentum space techniques have also been applied to field theory in curved spacetime using a construction based on Riemann Normal coordinates.⁶ While this method fails to incorporate the global properties of the Green's functions it is still expected to provide a reasonable description if the background curvature is sufficiently mild. Hu and O'Connor have applied this representation to derive an expansion for $\Gamma[\bar{\phi}]$ which is accurate to second order in the curvature and in derivatives of $\bar{\phi}$.⁷ An alternative expansion for $\Gamma[\bar{\phi}]$ can be obtained by using the DeWitt-Schwinger proper time series representation for Green's functions.^{8,9} This series has proven itself very useful, in particular, for the purposes of renormalization, where only the first few terms are required. It provides an asymptotic expansion for $\Gamma[\bar{\phi}]$ in terms of increasing powers of the background curvature and derivatives of $\bar{\phi}$. The ratio of successive terms in the expansion is of order $(1/mr_0)^2$, where m is some mass scale of the theory and r_0 a characteristic curvature radius of the background spacetime.

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Thus, the expansion is expected to be valid whenever the "Compton wavelength" $1/m$ associated with m is small compared to r_0 . Massive noninteracting quantum fields in the background of a large mass (i.e., $\text{mass} \gg 1/m$) black hole have been studied by use of this series.¹⁰

In this paper we recast the DeWitt-Schwinger representation in a modified form which is well suited to the study of $\Gamma[\bar{\varphi}]$ for an interacting quantum field theory in curved space. In this way an expression is obtained for the renormalized one-loop effective action for $\lambda\varphi^4$ theory. Explicit calculation is carried out in a general coordinate system through terms which are second order in the curvature and in derivatives of $\bar{\varphi}$. These low order terms take a very simple form so that the effective action compares readily with the well known Minkowski limit. A simple derivation of the renormalization group equations is presented.

We consider, for simplicity, a scalar quantum field theory with $\lambda\varphi^4$ self-interaction in a fixed n -dimensional spacetime described by a metric $g_{\mu\nu}$. n will serve as a dimensional regulator which will be set equal to four when the renormalization is complete. In addition to its minimal coupling to gravity, a bare coupling of φ to the scalar curvature $\xi R\varphi^2$ must also be introduced if the theory is to be renormalizable. ξ is dimensionless for all values of n . Any other coupling to gravity involves a dimensional coupling constant when $n = 4$. The bare action $I[\varphi]$ describing the theory is now given by

$$I[\varphi] = \int d^n x \sqrt{-g} \left\{ -\frac{1}{2} \varphi (-\square + m^2 + \xi R) \varphi - \frac{\lambda \varphi^4}{4!} \right\} + I_g, \quad (2)$$

where

$$I_g = \int d^n x \sqrt{-g} \left\{ \Lambda + \kappa R + \alpha_1 R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} + \alpha_2 R^{\alpha\beta} R_{\alpha\beta} + \alpha_3 R^2 \right\}.$$

The bare gravity action I_g introduced here is necessary for renormalization purposes. The field φ and all coupling constants written here are bare quantities. Boundary effects are not considered. Thus, we drop all total divergence terms. In this context, note that in four dimensions the Gauss-Bonnet density $R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} - 4R^{\alpha\beta} R_{\alpha\beta} + R^2$ is a total divergence. Thus, in four and only in four dimensions, modulo a total divergence only two of the three curvature quadratic terms are independent.

To proceed we first define $W[J]$, the generating functional for connected Green's functions

$$W[J] = -i \ln Z[J],$$

where

$$Z[J] = \text{constant} \int [d\varphi] \exp i \left(I[\varphi] + \int d^n x \sqrt{-g} J \varphi \right). \quad (3)$$

The effective action or generating functional for one particle irreducible Green's functions is now defined through the functional Legendre transformation

$$\Gamma[\bar{\varphi}] = W[J] - \int d^n x \sqrt{-g} J \bar{\varphi}, \quad (4)$$

where $\bar{\varphi}(x) = \frac{1}{\sqrt{-g}} \frac{\delta W}{\delta J(x)} [J]$ is the functional average or mean field in the presence of the source J .

Now $\Gamma[\bar{\varphi}]$ satisfies

$$\frac{1}{\sqrt{-g}} \frac{\delta \Gamma[\bar{\varphi}]}{\delta \bar{\varphi}(x)} = -J(x), \quad (5)$$

so solutions to $\frac{\delta \Gamma[\bar{\varphi}]}{\delta \bar{\varphi}} = 0$ represent field configurations which can exist in the absence of a source.

To generate the loop expansion for Γ we set $\varphi = \bar{\varphi} + \hat{\varphi}$, where $\hat{\varphi}$ represents the quantum fluctuation in φ . Now, if we Taylor expand $I[\varphi]$ about $\varphi = \bar{\varphi}$ we obtain the equation

$$\Gamma[\bar{\varphi}] = I[\bar{\varphi}] + \Gamma_1[\bar{\varphi}] \quad (6)$$

where

$$\Gamma_1[\bar{\varphi}] = -i \ln \int [d\hat{\varphi}] \exp \{A\}$$

and

$$A = i \int d^n x \sqrt{-g} \left\{ -\frac{1}{2} \hat{\varphi} (-\square + \tilde{m}^2(\bar{\varphi}) + \xi R) \hat{\varphi} - \left(\frac{\lambda}{3!} \bar{\varphi} \hat{\varphi}^3 + \frac{\lambda}{4!} \hat{\varphi}^4 + \frac{\delta \Gamma_1[\bar{\varphi}]}{\delta \bar{\varphi}} \hat{\varphi} \right) \right\}, \quad (7)$$

$$\tilde{m}^2(\bar{\varphi}) = m^2 + \frac{\lambda}{2} \bar{\varphi}^2.$$

To one loop accuracy, only the term quadratic in $\hat{\varphi}$ contributes to give

$$\begin{aligned} \Gamma_1[\bar{\varphi}] &= i \ln \text{Det } \hat{G}^{1/2} \\ &= \frac{i}{2} \text{tr} \ln \hat{G}, \end{aligned} \quad (8)$$

where the Green's function satisfies

$$(-\square + \tilde{m}^2(\bar{\varphi}) + \xi R - i\epsilon) \hat{G}(x, x') = -\delta(x, x'), \quad (9)$$

with $\delta(x, x')$ denoting the biscalar δ -function $\delta(x - x')/\sqrt{-g}$.

The DeWitt-Schwinger proper time representation is now applied to $\hat{G}$ so that it takes the form

$$\hat{G}(x, x') = \int_0^\infty i ds K(x, x', s), \quad (10)$$

where the kernel K satisfies the Schrödinger equation

$$(-\square + \tilde{m}^2(\bar{\varphi}) + \xi R) K = i \partial_s K \quad (11)$$

with the boundary condition

$$K(x, x', 0) = \delta(x, x'). \quad (12)$$

We introduce a modification of the standard DeWitt-Schwinger technique by rewriting K in the form

$$K(x, x', s) = \frac{1}{(4\pi i s)^{n/2}} \exp \left[\frac{i\sigma(x, x')}{2s} - i\tilde{m}^2(\varphi) s \right] \Psi(x, x') \Omega(x, x', s), \quad (13)$$

with the spacetime dependent $\tilde{m}^2(\varphi(x))$ in the exponent as opposed to the constant m^2 which appears in the standard description. Here $\sigma(x, x')$ denotes half the squared geodesic distance between x and x' . For later convenience, we have written $\Psi(x, x') = \Delta^{1/2}(x, x')$ where the biscalar $\Delta(x, x') = (-g(x))^{-1/2} D(x, x') (-g(x'))^{-1/2}$ and $D(x, x') = -\det(-\sigma_{;\mu\nu})$ is the Van Vleck-Morette determinant. As a result of the boundary condition (12) on K , Ω satisfies $\Omega(x, x', 0) = 1$. We now substitute Eq. (13) into Eq. (11). By making use of the following two results of geodesic theory⁹

$$\sigma_{;\alpha} \sigma^{;\alpha} = 2\sigma$$

$$\square \sigma - n = -\sigma^{;\alpha} (\ln \Delta)_{;\alpha}$$

the result is

$$\begin{aligned} i\partial_s \Omega + \frac{1}{\Psi} \square(\Psi \Omega) + \frac{i}{s} \sigma^{;\alpha} \Omega_{;\alpha} - \xi R \Omega + \sigma_{;\alpha} \tilde{m}^{2;\alpha} \Omega \\ - (is) \{ 2\tilde{m}^{2;\alpha} (\ln \Psi)_{;\alpha} \Omega + \Omega_{;\alpha} + (\square \tilde{m}^2) \Omega \} \\ - s^2 \tilde{m}^{2;\alpha} \tilde{m}_{;\alpha}^2 \Omega = 0. \end{aligned} \quad (14)$$

Note that the choice of the spacetime dependent $\tilde{m}^2$ in the exponent of Eq. (13) has the advantage that it leads to an Ω which depends on $\tilde{m}^2$ only through its derivatives. It might appear that a high price has been paid for this in that Eq. (14) is rather more complicated than it would be if instead m^2 had been used in the exponent. In that case, Eq. (14) would read

$$i\partial_s \Omega + \frac{1}{\Psi} \square(\Psi \Omega) + i\sigma^{;\alpha} \Omega_{;\alpha} - \left(\frac{\lambda \varphi^2}{2} + \xi R \right) \Omega = 0. \quad (14')$$

However, the complexity of Eq. 14 is not a serious problem. We will discover that the terms involving $\tilde{m}^{2;\alpha}$ require little extra calculational effort.

Recently, it has been proven that if $\exp \left[-i \left(\xi - \frac{1}{6} \right) R s \right]$ is factored out of $K(x, x, s)$, then the remaining factor depends on the scalar curvature R only through its derivatives.¹¹ Thus, an alternative procedure would be to use the quantity $\tilde{m}^2 + (\xi - \frac{1}{6})R$ in the exponent of Eq. (13), but we have not pursued this possibility.

A power series solution for Ω

$$\Omega(x, x', s) = \sum_{k=0}^{\infty} (is)^k \Omega_k(x, x')$$

reduces Eq. (13) to the following recursion relationships

$$\sigma^{;\alpha} \Omega_{0;\alpha} = 0 \quad (15)$$

$$\begin{aligned} - (k+1) \Omega_{k+1} + \frac{1}{\Psi} \square(\Psi \Omega_k) - \sigma^{;\alpha} \Omega_{k+1;\alpha} - \xi R \Omega_k + \sigma_{;\alpha} \tilde{m}^{2;\alpha} \Omega_k \\ - (2\tilde{m}^{2;\alpha} [(\ln \Psi)_{;\alpha} \Omega_{k-1} + \Omega_{k-1;\alpha}] + \Omega_{k-1} \square \tilde{m}^2) + \tilde{m}_{;\alpha}^2 \tilde{m}^{2;\alpha} \Omega_{k-2} = 0. \end{aligned} \quad (16)$$

It is understood that $\Omega_k = 0$ for all $k < 0$.

We will need only the coincidence limit $x' \rightarrow x$ of each Ω_k . These limits we will denote by $[\Omega_k]$. The calculation of these coincidence limits without the mass terms in Eq. (16) is well-known.⁶ We will therefore give only a brief description emphasizing the new features introduced by the mass terms. For this, we will make repeated use of some purely geodesic coincidence limits which we list here.⁹ Using the notation $\sigma_\alpha \equiv \sigma_{;\alpha}$, $\sigma_{\alpha\beta} \equiv \sigma_{;\alpha\beta}$ etc., one has

$$\begin{aligned} [\sigma] &= 0 & [\sigma_\alpha] &= 0 \\ [\sigma_{\alpha\beta}] &= g_{\alpha\beta} & [\sigma_{\alpha\beta} \Psi] &= 0 \\ [\Psi] &= 1 & [\Psi_\alpha] &= 0 \\ [\square \Psi] &= \frac{1}{6} R \\ [\square^2 \Psi] &= \frac{1}{30} (R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} - R^{\alpha\beta} R_{\alpha\beta}) + \frac{1}{36} R^2 + \frac{1}{5} \square R. \end{aligned}$$

For $[\Omega_0]$ note that $\Omega(x, x', 0) = \Omega_0(x, x') = 1$ to give as immediate solution to Eq. (15)

$$[\Omega_0] = 1 \\ [\Omega_{0;\alpha\beta} \dots] = 0.$$

For $[\Omega_1]$ set $k = 0$ in Eq. (16). Now

$$-\Omega_1 + \frac{1}{\Psi} \square \Psi - \sigma^{;\alpha} \Omega_{1;\alpha} - \xi R + \sigma_{;\alpha} \tilde{m}^{2;\alpha} = 0. \quad (17)$$

On taking the coincidence limit:

$$[\Omega_1] = [\square \Psi] - \xi R = \left(\frac{1}{6} - \xi \right) R. \quad (18)$$

Not surprisingly $[\Omega_1]$ is independent of $\tilde{m}^{2;\alpha}$ because $[\Omega_1]$ has mass dimension two. For $[\Omega_2]$, set $k = 1$ in Eq. (16). Now

$$-2\Omega_2 + \frac{1}{\Psi} \square(\Psi\Omega_1) - \sigma^\alpha \Omega_{2;\alpha} - \xi R\Omega_1 + \sigma_\alpha \tilde{m}^{2;\alpha} \Omega_1 - 2\tilde{m}^{2;\alpha} (\ln \Psi)_{;\alpha} - \square \tilde{m}^2 = 0. \quad (19)$$

Again we take the coincidence limit to obtain for $[\Omega_2]$

$$[\Omega_2] = \frac{1}{2} \{ \square(\Psi\Omega_1) - \xi R[\Omega_1] - \square \tilde{m}^2 \}.$$

Here, $\square(\Psi\Omega_1) = [\Omega_1] \square \Psi + \square \Omega_1$ and $\square \Omega_1$ is obtained by operating on Eq. (17) with $\square$, and then taking the coincidence limit:

$$[\square \Omega_1] = \frac{1}{3} \{ \square^2 \Psi - [\square^2 \Psi] - [\square \Psi]^2 - \xi \square R + 2 \square \tilde{m}^2 \}. \quad (20)$$

Now $[\Omega_2]$ is expressed entirely in terms of known quantities

$$[\Omega_2] = \frac{1}{180} (R^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu} - R^{\alpha\beta} R_{\alpha\beta}) + \frac{1}{2} \left(\frac{1}{6} - \xi \right)^2 R^2 + \frac{1}{6} \left(\frac{1}{5} - \xi \right) \square R - \frac{1}{6} \square \tilde{m}^2. \quad (21)$$

The only new term in $[\Omega_2]$ is the last one. We might have anticipated its form – the product $\tilde{m}_{;\alpha}^2 \tilde{m}^{2;\alpha}$ or any contraction between $\tilde{m}^{2;\alpha}$ and the curvature tensor is ruled out on dimensional grounds. Of course, in Minkowski space the only surviving term is in fact the last one.

Now recall that our aim is to calculate $\Gamma[\tilde{\varphi}]$ through terms which are either second order in the curvature or in spacetime derivatives of $\tilde{\varphi}$. As will be seen below, the $\square R$ term of $[\Omega_2]$ appears in $\Gamma[\tilde{\varphi}]$ only through the combination $\ln \tilde{m}^2 \square R$, which is easily shown to be of higher order on performing an integration by parts and discarding surface terms.

At this stage, we might suppose that we have exhausted all terms of the required order. To see that this is not so, consider $[\Omega_3]$ which has mass dimension six. Certainly all purely curvature terms are of third order. However, of the allowed mass terms, $\tilde{m}^{2;\alpha} \tilde{m}_{;\alpha}^2$ is quadratic and must be included in our sum. The calculation is done in Minkowski space for which $\Psi = 1$, $R_{\alpha\beta\mu\nu} = 0$. The algebra is a little tedious but nonetheless straightforward. The result is

$$[\Omega_3]_{2\text{nd order}} = \frac{1}{12} \tilde{m}_{;\alpha}^2 \tilde{m}^{2;\alpha}. \quad (22)$$

Armed with these coincidence limits, we return to the calculation of $\Gamma[\tilde{\varphi}]$. In $\mathcal{G}$ has a well-known proper time representation¹

$$\ln \mathcal{G}(x, x'; \tilde{m}^2) = \int_0^\infty K(x, x', s) \frac{ids}{s}$$

thus giving for the one loop effective action

$$\Gamma[\tilde{\varphi}] = I[\tilde{\varphi}] + \frac{1}{2(4\pi)^{n/2}} \sum_{k=0}^\infty \int d^n x \sqrt{-g} [\Omega_k] \times (\tilde{m}^2)^{\frac{n}{2}-k} \Gamma\left(k - \frac{n}{2}\right), \quad (23)$$

where we have performed the integration over s . The gamma function has poles at $k = 0, 1$ and 2 when $n = 4$ corresponding to the ultraviolet divergences of the theory. To remove these consistently in dimensional regularization we must first ensure that $\Gamma_1[\tilde{\varphi}]$ is dimensionless for all n . To do this we introduce a mass parameter μ , let

$$(\tilde{m}^2)^{\frac{n}{2}-k} \rightarrow \left(\frac{\tilde{m}^2}{\mu^2} \right)^{\frac{n-4}{2}} (\tilde{m}^2)^{2-k},$$

and expand about $n = 4$ to get

$$\begin{aligned} \Gamma[\tilde{\varphi}] = I[\tilde{\varphi}] &- \frac{1}{(4\pi)^2} \int d^n x \sqrt{-g} \left(\frac{1}{n-4} + \frac{1}{2} \left(\gamma + \ln \frac{\tilde{m}^2}{4\pi\mu^2} \right) \right) \\ &\times \left\{ \frac{4(\tilde{m}^2)^2}{n(n-2)} - \frac{2\tilde{m}^2}{n-2} [\Omega_1] + [\Omega_2] \right\} \\ &+ \sum_{k=3}^\infty \frac{(k-3)!}{2(4\pi)^2} \int d^n x \sqrt{-g} \frac{[\Omega_k]}{(\tilde{m}^2)^{k-2}} + \mathcal{O}(n-4), \end{aligned} \quad (24)$$

where γ is the Euler constant.

We now express the bare quantities appearing in $I[\tilde{\varphi}]$ in terms of renormalized ones plus counterterms. The counterterm is chosen to cancel the pole term in Eq. (24). We are left with the renormalized effective action in four dimensions

$$\begin{aligned} \Gamma_R[\tilde{\varphi}] = I_R[\tilde{\varphi}] &- \frac{1}{2(4\pi)^2} \int d^4 x \sqrt{-g} \ln \frac{\tilde{m}^2}{4\pi\mu^2} \left\{ \frac{(\tilde{m}^2)^2}{2} - \tilde{m}^2 [\Omega_1] + [\Omega_2] \right\} \\ &+ \text{higher order terms in (24) evaluated at } n = 4. \end{aligned} \quad (25)$$

$I_R[\tilde{\varphi}]$ here is the action (2) written in terms of renormalized parameters and γ has been absorbed in a redefinition of μ . There is no φ renormalization at one loop level. The expression (25) for I_R agrees to the order explicitly calculated with that obtained by Hu and O'Connor⁷ using momentum space techniques.

The familiar one loop effective action in Minkowski space is obtained by setting $R_{\alpha\beta\mu\nu} = 0$ in Eq. (25). Now $[\Omega_1] = 0$, $[\Omega_2] = -\frac{1}{6} \square \tilde{m}^2$ and $[\Omega_3]_{2\text{nd order}} = -\frac{1}{12} \tilde{m}_{;\alpha}^2 \tilde{m}^{2;\alpha}$.

The result up to terms quadratic in derivatives is

$$V(\varphi) = \frac{1}{2} m^2 \varphi^2 + \frac{\lambda}{4!} \varphi^4 + \frac{\tilde{m}^4}{64\pi^2} \ln \frac{\tilde{m}^2}{4\pi\mu^2} \quad (26)$$

$$Z(\varphi) = 1 + \frac{1}{(4\pi)^2} \frac{\lambda}{6} \frac{\lambda \varphi^2}{2\tilde{m}^2}, \quad (27)$$

where for the evaluation of $Z(\varphi)$ we have used a parts integration on $\ln \tilde{m}^2 \square \tilde{m}^2$ to add the $[\Omega_2]$ and $[\Omega_3]$ contributions.

The renormalization group equations for the coupling constants are readily obtained using the fact that the effective action is invariant under a change of renormalization point, *viz* $\mu d_\mu \Gamma = 0$. Thus, in one loop approximation

$$\mu d_\mu I_R + \frac{1}{(4\pi)^2} \int d^4x \sqrt{-g} \left\{ \frac{\tilde{m}^4}{2} - \tilde{m}^2 [\Omega_1] + [\Omega_2] \right\} = 0,$$

from which it follows that

$$\begin{aligned} d_r m^2 &= -2m^2 \lambda \\ d_r \xi &= 2\lambda \left(\frac{1}{6} - \xi \right) \\ d_r \lambda &= -6\lambda^2 \\ d_r \Lambda &= m^4 \\ d_r \kappa &= -2 \left(\frac{1}{6} - \xi \right) \\ d_r \alpha_1 &= \frac{1}{90} \\ d_r \alpha_2 &= -\frac{1}{90} \end{aligned}$$

and

$$d_r \alpha_3 = \left(\frac{1}{6} - \xi \right)^2$$

where $\tau = -2(4\pi)^2 \ln \mu$ and we have used the fact that there is no φ renormalization at this level in $\lambda\varphi^4$ theory. The solution of this set of equations is, of course, well-known.¹² The appeal of this derivation is that it makes no reference to the counterterms which were introduced to subtract the ultraviolet divergences.

To conclude, in this paper we have modified the DeWitt-Schwinger technique to obtain an expansion for the effective action of an interacting quantum field theory in curved space analogous to the derivative expansion in flat space. The method can easily be applied to other theories and it is possible to extend it to higher loop

processes. However, as is normal beyond one loop, the renormalization procedure will be considerably more involved.¹³

The expansion is local and so other resources must be applied to probe the effects of global structure on $\Gamma[\varphi]$. Exact expressions for $V(\varphi)$ as defined in Eq. (1) have been obtained at the one-loop level in the case of some highly symmetric spacetimes.¹⁴

As noted in the introduction the expansion of $\Gamma[\varphi]$ in derivatives is expected to be valid only if $1/\tilde{m}$ is small compared to the characteristic curvature radius r_0 . If $\tilde{m}^2 = 0$, however, the series does not even exist—the integrals which must be performed in obtaining Eq. (23) diverge at the upper limit of integration. On the other hand the effective action is expected to have a massless limit.¹⁵ Thus the breakdown of the series apparently represent the inability of a local expansion to approximate the essentially non-local massless effective action. Of course, this breakdown can be seen even in Minkowski space. For example, in $\lambda\varphi^4$ theory the one-loop contribution to $Z(\varphi)$ in Eq. (1) is proportional to $\lambda^2 \varphi^2 / \tilde{m}^2(\varphi)$, where the mass parameter $\tilde{m}^2 = m^2 + \lambda\varphi^2/2$. For the case of negative m^2 (describing a theory with spontaneous symmetry breaking), $\tilde{m}^2$ vanishes at the inflection points of the classical potential. Thus $Z(\varphi)$ and (as we might expect on dimensional grounds) the coefficients of all higher derivative terms blow up. The problem in Minkowski space, however, does not seem to have been addressed thus far. Attention has focused on situations where φ is constant and thus $\Gamma[\varphi]$ can be replaced by the effective potential $V(\varphi)$, which is well behaved in the troublesome region. The issue seems worth pursuing.

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