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ORBIFOLDS AND WILSON LINES

L.E. Ibáñez, H.P. Nilles and F. Quevedo

CERN, 1211 Geneva 23, Switzerland

ABSTRACT

We study the consequences of the presence of gauge background fields on the torus underlying orbifold compactification of the heterotic string. It is pointed out that such Wilson lines provide us with a mechanism for controlling the number of chiral matter states both from twisted and untwisted sectors, as well as breaking the symmetry group. Starting from the Z orbifold, we can construct a variety of four-dimensional string models with three families of quarks and leptons and different gauge groups such as E_6 , $SU(3)^3$, $SU(6) \times U(1)$ or $SU(5) \times [SU(2) \times U(1)]^2$.

1. INTRODUCTION

A clarification of an eventual connection between superstring theories and physics necessitates a better understanding of four-dimensional string theories. Compactification of higher-dimensional string theories on a torus $T^n = \mathbb{R}^n/\Lambda$ is well understood [1] but seems to be unrealistic because of the absence of chiral fermions. The consideration of orbifolds [2, 3] [or, more precisely, of twisted tori $O^n = T^n/P^*$] seems to be more promising in this respect, still sharing the conceptual simplicity of torus compactification [4, 5]. In this direction the work has mainly been concentrated on the choice of an Abelian point group with standard embedding in the gauge group $E_8 \times E_8$ of the heterotic string [3]. Whilst this leads to encouraging results concerning the breakdown of E_8 and the existence of chiral fermions, it usually leads to an embarrassing proliferation of matter fields. Also, one example with non-standard embedding has been worked out [6].

In this paper we consider the inclusion of background gauge fields corresponding to non-contractible loops on T^n in the framework of orbifold compactification. Such Wilson lines can be equivalently understood as the embedding of the full space group S into the gauge group of the high-dimensional string theory. We show, as might have been expected [3], that this leads to more flexibility with respect to gauge symmetry breakdown. However, it will also be shown, as might not have been expected from the beginning, that such Wilson lines provide us with a mechanism for controlling the number of matter multiplets. In the framework of the Z orbifold [2] we will see that this method allows us to easily construct models with three generations of quarks and leptons with semi-realistic gauge groups such as E_6 , $SU(6) \times U(1)$, $SU(3)^3$, and $SU(5) \times [SU(2) \times U(1)]^2$. In a certain way our work is complementary to the attempts to classify all possible four-dimensional string models on the basis of modular invariant partition functions [7, 8]. Although here a very large number of such models can be classified, a mechanism for selecting the interesting ones is still lacking.

Let us therefore consider the motion of closed strings on the twisted torus $O^n = T^n/P \cong \mathbb{R}^n/S$. Consistency requires that the action of P leaves the torus T^n invariant. We denote the two-dimensional coordinates by $\sigma = \sigma_1 + \tau\sigma_2$, where τ is the modular parameter. We then observe the existence of several sectors of closed strings denoted by (g, h) , where g and h are elements of P :

$$\begin{aligned} X(\sigma_1 + \pi, \sigma_2) &= h X(\sigma_1, \sigma_2) \\ X(\sigma_1, \sigma_2 + \pi) &= g X(\sigma_1, \sigma_2) \end{aligned} \tag{1}$$

and g and h have to commute for the boundary conditions to be compatible. The contribution of the (g, h) sector to the partition function is given by $\text{Tr } g e^{2\pi i \tau H}$, with H being the Hamiltonian corresponding to the h -twist. The sector $h = 1$ is called the untwisted sector; it corresponds to

*) P is a discrete group called the point group. Equivalently, we can consider $O^n = \mathbb{R}^n/S$, where the space group S contains P as well as the shifts Λ of the lattice defining the torus T^n .

strings already closed on T^n . Because of the division of T^n by P , we can keep only those states in the Hilbert space of strings moving on T^n that are invariant under the action of P . This projection is actually given by the sum of the contributions of the $(g, 1)$ sectors with $g \in P^*$). The untwisted sector contains, in general as massless states, the graviton, possibly the gravitinos, the gauge bosons, and possibly also matter fields.

The sectors with $h \neq 1$ are called twisted sectors. Applying a modular transformation such as $\tau \rightarrow -1/\tau$ on the untwisted sector shows that they have to be included. These are the string configurations that are not closed on T^n but only on T^n/P . The number of these twisted sectors can be computed either by the consideration of conjugacy classes of P or by counting the fixed points of T^n under the action of P . The last statement can easily be seen by expanding $X(\sigma)$, and requiring $X(\pi) = h X(0)$ modulo a lattice vector. This needs the centre-of-mass coordinate to coincide with a fixed point. For the oscillators this generally implies the presence of non-integer modes depending on the order of P . In any of the twisted sectors, we again have to sum over all the sectors (g, h) with $gh = hg$ projecting on the P -invariant Hilbert space.

So much for the action of P on the space-time degrees of freedom. Discussing the heterotic string, we could have the twist P accompanied by a gauge transformation in, for example, the $E_8 \times E_8$ gauge group. We then take a subgroup $P' \subset E_8 \times E_8$ homomorphic to P and act with the twist $P \times P'$. In the bosonic formulation, such an embedding can always be realized as transformations on the $E_8 \times E_8$ lattice. In the case of an Abelian point group, the action of P' can always be chosen to be a shift vector v^I , $I = 1, \dots, 16$, in the $E_8 \times E_8$ lattice accompanying the space-time twist. These shifts cannot be chosen arbitrarily, but they have to satisfy the group law and also restrictions from modular invariance leading to left-right level matching [3, 9]. The modular projection that previously led to the P -invariant Hilbert space now selects states that are invariant under the diagonal subgroup of $P \times P'$. In such a situation the gauge group $E_8 \times E_8$ will be broken down to the subgroup that commutes with P' . The twisted sectors contain only matter fields, whilst the graviton and the gauge bosons come exclusively from the untwisted sector.

2. WILSON LINES

We would now like to add constant background gauge fields (Wilson lines) corresponding to the non-contractible loops on T^n . In the interpretation $O^n = R^n/S$, this is nothing other than the embedding of the full space group S into the gauge group [3]. Contrary to the case of torus compactification, these background gauge field values have to satisfy certain constraints because of the non-trivial group structure of S . In particular, the gauge P' and Wilson-line W transformations must fulfil $(P'W)^n = 1$ for an order n moding [3]. As a consequence, the value of the Wilson lines are quantized, and it turns out that in general we cannot consider all the n possible values independently.

*) This is the generalization of, for example, the GSO projection in the spinning string obtained by a summation over spin structures.

A non-trivial consequence of the presence of Wilson lines shows up in the discussion of the twisted sectors. They are identified by the fixed points of certain elements of S . Whilst in the previous case only the subgroup $P \subset S$ was relevant, we now have to consider the full space group in order to properly treat the embedding in the twisted sectors. In group theoretic language, this comes from the fact that members belonging to the same conjugacy class of the point group might belong to different conjugacy classes of the space group. We have also to take into account that the space group is non-Abelian. This fact leads to complications, but in this paper we restrict ourselves to an Abelian embedding in the gauge group, and the non-Abelian structure of S will only affect the projections in the twisted sectors.

Let us now discuss the untwisted sector, which can be understood from torus compactification. We split the string into left and right movers, $X = \frac{1}{2}(X_L + X_R)$:

$$\begin{aligned} X_L^\Lambda &= q^\Lambda + 2 \tilde{L}^\Lambda u + \frac{i}{2} \sum_n \frac{\tilde{\alpha}_n^\Lambda}{n} e^{-2in u} \\ X_R^i &= \tilde{q}^i + 2 \tilde{L}^i v + \frac{i}{2} \sum_n \frac{\tilde{\alpha}_n^i}{n} e^{-2in v} \end{aligned} \quad (2)$$

with $i = 1, \dots, 6$ and $A = (i, I)$, $I = 1, \dots, 16$, for the heterotic string, in an obvious notation. The lattice $(L^\Lambda, \tilde{L}^i)$ for the compactified dimensions [1] should be Lorentzian, even, and self-dual, with signature $(22, 6)$. The mass spectrum can be obtained from

$$\begin{aligned} \frac{1}{4} m_L^2 &= N_L - 1 + \frac{1}{2} L^2 \\ \frac{1}{4} m_R^2 &= N_R + \frac{1}{2} \tilde{L}^2 \end{aligned} \quad (3)$$

with the constraint $m_L^2 = m_R^2$. The invariant Wilson line is given by

$$\int_i A_\mu^I dx^\mu \equiv 2\pi A_\mu^I e_i^\mu = 2\pi a_i^I$$

The vectors L including gauge background fields a_i^I are given by^{*)}

^{*)} Inclusion of a background value for the antisymmetric field B_{ij} does not change our results except for particular values of the compactification radii. Models with enlarged gauge groups have been considered in ref. [10].

$$L_1 = n^i R_i e_i + \frac{m_i}{2 R_i} e^{*i} + \frac{a_i^I}{2 R_i} e^{*i} [p^I - \frac{1}{2} a_j^I n^j]$$

$$\tilde{L} = L - 2 n^i R_i e_i \quad (4)$$

$$L_2^I = p^I - a_i^I n^i$$

with $L = (L_1, L_2^I)$. The vectors e_i ($i = 1, \dots, 6$) define the lattice of the six-torus with corresponding radii R_i and we have chosen $(e_i^2) = 2$, whilst the e^{*i} define the dual lattice $e^{*i} e_j = \delta_j^i$; n and m are integers corresponding to the winding numbers and the quantized momenta on the torus. Finally, the p^I define the extra-dimensional lattice, e.g. $E_8 \times E_8$ root lattice. To illustrate these formulae, let us consider the Z orbifold [2], where the six-dimensional lattice consists of three copies of the $SU(3)$ root lattice. In Cartesian coordinates we have $e_1 = (\sqrt{2}, 0, 0, 0, 0, 0)$, $e_2 = (-1/\sqrt{2}, \sqrt{3}/\sqrt{2}, 0, 0, 0, 0)$, and similar expressions for the remaining e_i . The point group $P = Z_3$ consists of 120° rotations of the $SU(3)$ lattices simultaneously. It is useful to consider complex coordinates and represent the six real coordinates by three complex ones with lattice vectors such as $(1, 0, 0)$ and $(e^{2\pi i/3}, 0, 0)$. There are 27 fixed points of the Z_3 transformation given by $(p_1, p_2, p_3)d$, with $d = \sqrt{1/3} e^{i\pi/6}$ and $p_\alpha = 0, \pm 1$. The group law in S gives restrictions for the choice of the background values a_i^I , in that $3a_i^I$ has to be a vector in the $E_8 \times E_8$ lattice. Furthermore, here one can only choose three independent values, since $a_1 = a_2$, $a_3 = a_4$, and $a_5 = a_6$ have to be satisfied [3]. This is because $P e_i = e_{i+1}$ for $i = 1, 3, 5$.

In the untwisted sector, one starts with the string as it moves on T^6 , and then one has to project onto the S -invariant states in the Hilbert space, corresponding to the sum of the contributions from the $(g, 1)$ sectors. For $P = Z_3$ this leads to the well-known projections. Including the a_i gives new restrictions concerning a selection of special values of $a_i^I p^I$. To illustrate this, consider the massless sector which requires $\tilde{L} = 0$ and in turn implies $n^i = 0$ in the given example. Since m_i are integers, we observe that we have to project onto integer $a_i^I p^I$, as can be derived from formula (4). This is equivalent to projecting onto states which are Wilson-line singlets and generally leads to a further breakdown of the gauge group.

For the discussion of the twisted sectors, we should remember that there are 27 fixed points. In the usual Z orbifold all these 27 sectors lead to identical physics [2]. The oscillators corresponding to the six compact dimensions have third-integral moding and the twist $h = \theta = e^{2\pi i/3}$ is embedded in the gauge group, leading to a shift in the $E_8 \times E_8$ lattice $p^I \rightarrow p^I + v^I$. Again one has to project onto invariant states, summing over the (g, θ) sectors. This gives the states of positive helicity; the ones with negative helicity are obtained from the θ^{-1} sectors.

Inclusion of the Wilson lines leads to a completely different situation, since the different twisted sectors no longer show a universal behaviour. This comes from the fact that the fixed points

belong to different elements of the space group. The fixed point $(0, 0, 0)d$, for example, is a fixed point of the θ transformation; here we have to consider the lattice $p^I + v^I$ in the left-handed sector. The fixed point $(d, 0, 0)$, however, is not a fixed point of θ alone; a shift by e_1 on the $SU(3)$ lattice has to accompany this rotation. In this twisted sector the embedding in the gauge group requires the shift $p^I \rightarrow p^I + v^I + a_1^I$, and with non-trivial a_1^I different twisted sectors lead to different spectra. The twisted sectors can therefore be labelled by $h = (\theta, e_i)$ corresponding to a rotation and a shift. In each of these sectors we have to perform the projection on invariant states, and here we have to take into account the non-Abelian structure of the space group. In a given sector twisted by h , we have to sum the contributions of the sectors (g, h) , not for all $g \in S$ but only for those where g commutes with h .

The various shifts v^I and a_1^I have to satisfy further restrictions from modular invariance such as left-right level matching, and this will be discussed in the examples given below. Here we should state explicitly that in all our examples we have embedded P in an Abelian way in the gauge group. Together with the fact that we use only inner automorphisms acting on the lattices, this implies that we cannot reduce the rank of the gauge group. The rank can, however, be lowered by an explicit consideration of these complications.

3. EXAMPLES

All the above points may be conveniently illustrated by considering the existence of a specific Wilson line on the Z orbifold. There are four non-trivial embeddings of the point group operation θ into the $E_8 \times E_8$ gauge group consistent with modular invariance constraints [3]. These are represented by the four different shift vectors v^I shown in table 1. The corresponding unbroken gauge groups were given in ref. [3]. We also give here the matter superfield content from untwisted and twisted sectors. One can add Wilson lines a_1^I on any of the four possible embeddings (i.e. on top of any of the modings generated by the four v^I 's). Only the standard embedding 1 in the table has been studied in detail in the literature [2], and it is also the only one which contains complete standard quark-lepton generations in the twisted sector. One version of the second embedding in the table has also been studied [6]. Later we will show that all the non-canonical embeddings can lead to interesting models when one adds Wilson lines. As we remarked above, modular invariance restricts the possible Wilson-line shifts a_1^I in a given twisted sector of the theory [3, 9]. In the Z orbifold case, one has the condition

$$\sum_{a=1}^3 q r_a^2 = 6 = \sum_I q (v_I + a_I)^2 \mod 3 \quad (5)$$

where the r_a 's indicate the shift in the space-time sector. A simple way to find a_1^I 's verifying this equation is to choose them equal to any of the shifts of table 1 but without any overlap with the v^I shift of the moding. However, this is not a necessary condition (see example 1 of tables 2 and 3).

Notice that since different twisted sectors may get different a^I shifts, one has to check that condition (5) is respected in all the twisted sectors.

Let us consider one example in detail. Take the usual canonical embedding with the P' moding generated by the shift $v^I = (1/3, 1/3, 2/3, \dots)$ and add a Wilson line on one of the six compactified dimensions $a_1^I = (0, 0 \dots 1/3, 1/3) (2/3 \dots)$, in an obvious notation. Since $\theta e_1 = e_2$, we have necessarily to add another (equal) Wilson line on the other dimension, $a_2^I = a_1^I \equiv a^I$. We assume that the other four dimensions have no Wilson line, $a_3^I = a_4^I = a_5^I = a_6^I = 0$. Notice that $3a^I$ is a vector in the $E_8 \times E_8$ lattice, as it should be. It is interesting to remark that a Wilson line such as $(0, 0 \dots 1/3, 1/3) (0 \dots 0)$ would have induced a non-modular invariant result, which is why we included an action of the a^I shift on E_8 . Let us start by considering the massless sector of the *untwisted sector*. The states in the left-handed sector, before adding the Wilson line, can be classified according to their eigenvalue with respect to $P' = \exp(i2\pi p^I v^I)$. There are the 16 $U(1)$'s corresponding to $\alpha^I |0\rangle_L$ and the states which complete the adjoint of $SU(3) \times E_6 \times E_8$, all of these with $P' = 1$. Then there are states with quantum numbers $(\underline{3}, \underline{27})$ and $P' = \alpha^2$ ($\alpha = e^{2\pi i/3}$), and states with quantum numbers $(\underline{\bar{3}}, \underline{\bar{27}})$ and $P' = \alpha$. After addition of the Wilson lines, the GSO-like projection will project on states ($P_1^2 = 2$ roots) which verify $p^I \cdot a^I = m$, $m = \text{integer}$. In the $P' = 1$ sector these states will correspond to the E_8 roots $(0, 0, 0, \dots, \pm 1, \dots, \pm 1, 0, 0), \pm(0, 0, \dots, 1, -1), \pm(1, -1, 0, \dots), \pm(1, 0, 1, \dots), \pm(0, 1, 1, 0, \dots), \pm(-1/2, -1/2, 1/2, \pm 1/2, \pm 1/2, \pm 1/2, 1/2, -1/2), \pm(-1/2, -1/2, 1/2, \pm 1/2, \pm 1/2, \pm 1/2, -1/2, 1/2)$. These correspond to the non-vanishing roots of an $SU(3) \times SU(6) \times U(1)$ group. The second E_8 gives in the $P' = 1$ sector the gauge group $SO(14)' \times U(1)'$. All these $P' = 1$ left movers combine with the $P = 1$ right-handed ground states from $|i\rangle_R, |a\rangle_R$ to form gauge bosons and gauginos with gauge group $SU(3) \times SU(6) \times U(1) \times SO(14)' \times U(1)'$.

Doing a similar exercise with the $P' = \alpha^2$ untwisted sector, one easily verifies that from the original $(\underline{3}, \underline{27})$ states there survive only the $p^I a^I = m$ projected states which transform as a $(\underline{3}, \underline{15}, 0)$ under $SU(3) \times SU(6) \times U(1)$. There are no states with $SO(14)' \times U(1)'$ quantum numbers. The surviving $P' = \alpha^2$ left movers combine with the three $P = \alpha$ right-moving states $|i\rangle_R, |a\rangle_R$ [$i, a \in \underline{3}$ of $SU(3) \subset SU(6)$] to form $P \times P' \times \text{Wilson-line singlets}$. The $P' = \alpha$ sectors give us the antiparticles. Thus as a whole we obtain from the untwisted sector (apart from the supergravity and 'dilaton' particles) three chiral superfields transforming as $(\underline{3}, \underline{15}, 0)$ of $SU(3) \times SU(6) \times U(1) \times SO(14)' \times U(1)'$.

Let us now consider the twisted sectors. They are associated with string modes which expand around any of the 27 fixed points of the Z orbifold. Since there are Wilson lines in one of three complex dimensions of the orbifold, the 27 fixed points are split into three groups of nine fixed points according to the string boundary conditions. The first nine fixed points, which we denote by TT (trivial twisted), are the ones with coordinates $(0, m, n)d$, $(m, n = 0, \pm 1)$. The bosonic string coordinates have the usual closed-string boundary conditions: $x^I(\pi) = x^I(0) + 2\pi v^I$; $x^I(\pi) = \theta x^I(0) + (\text{lattice vectors without Wilson lines})$. The states associated with these nine fixed points do not feel the presence of the Wilson line; thus the zero modes verify the usual Z orbifold condition

$$\frac{1}{2} (p^I + v^I)^2 + \tilde{N} - \frac{2}{3} = 0 \quad (6)$$

The states of these sectors are also the usual ones of the Z -orbifold twisted sector. For $\tilde{N} = 0$ one has $(p^I + v^I)^2 = 4/3$, and the nine states transforming as $[(1, \underline{15}) + 2(1, \underline{\bar{6}})]$ under $SU(3) \times SU(6) \times U(1)$ will be massless. For $\tilde{N} = 1/3$ one has $(p^I + v^I)^2 = 2/3$, and the $9 \times 3 = 27$ states transforming as $(\underline{\bar{3}}, 1)$ are massless. All these states survive under the relevant projection in this case, which is on $P \times P'$ invariant states.

The second set of nine fixed points, which we denote by NTT-1 (non-trivial twisting 1), are those with coordinates $(+1, m, n)d$, $(m, n = 0, \pm 1)$. The bosonic string coordinates have the closed-string boundary conditions: $x^I(\pi) = x^I(0) + 2\pi(v^I + a^I)$; $x^i(\pi) = \theta x^i(0) + 2\pi e_1^i +$ (shifts without Wilson lines), since there is a Wilson line a^I associated with the lattice vector e_1^I , and all these nine fixed points require a shift by e_1^I to be really fixed points. The associated twisted sectors do feel the presence of the Wilson lines, and the massless modes have to satisfy

$$\frac{1}{2} (p^I + v^I + a^I)^2 + \tilde{N} - \frac{2}{3} = 0 \quad (7)$$

There are only solutions to this with $\tilde{N} = 0$, and they correspond to elements of the $(p^I + v^I + a^I)$ shifted lattice, with p^I being $(0, 0, \dots, 0)$; $(-1, 0, -1, 0, \dots)$; $(0, -1, -1, 0, \dots)$; $(0, 0, -1, \dots, -1, 0)$; $(0, 0, -1, \dots, 0, -1)$; $(-1/2, -1/2, -1/2, \pm 1/2, \pm 1/2, \pm 1/2, -1/2, -1/2)$. Taking into account the chirality of the twisted right-handed vacuum, the quantum numbers of these states correspond to a $[(\underline{\bar{3}}, 1) + (1, \underline{\bar{6}})]$ under $SU(3) \times SU(6) \times U(1)$. No massless states with $SO(14)' \times U(1)'$ quantum numbers appear. All these states survive the GSO-like projection, which in this case projects onto states with $\exp 2\pi i [(p^I + v^I + a^I)(v^I + a^I)] = \alpha$, $\alpha = e^{2\pi i/3}$. Finally, the third set of nine fixed points, which we denote by NTT-2 (non-trivial twisting 2), are those with coordinates $(-1, m, n)d$, $(m, n = 0, \pm 1)$. The bosonic string coordinates have the closed-string boundary conditions: $X^I(\pi) = X^I(0) + 2\pi(v^I - a^I)$, $x^i(\pi) = \theta x^i(0) - 2\pi e_1^i +$ (shifts without Wilson lines). The massless modes have to satisfy

$$\frac{1}{2} (p^I + v^I - a^I)^2 + \tilde{N} - \frac{2}{3} = 0 \quad (8)$$

One can easily check that they correspond to a $[(\underline{\bar{3}}, 1) + (1, \underline{\bar{6}})]$, although in this case the $\underline{\bar{6}}$ has a $U(1)$ charge opposite to the one in the NTT-1 sector. Again all these states survive projection. In summary, the matter content of this model with respect to $SU(3) \times SU(6) \times U(1)$ is

$$U \rightarrow 3 (3, 15, 0)$$

$$TT \rightarrow 9 [(1, 15, 0) + (1, \bar{6}, a) + (1, \bar{6}, -a)] + 27 (\bar{3}, 1, 0)$$

(9)

$$NTT-1 \rightarrow 9 [(\bar{3}, 1, 0) + (1, \bar{6}, a)]$$

$$NTT-2 \rightarrow 9 [(\bar{3}, 1, 0) + (1, \bar{6}, -a)]$$

One gets the antiparticles from the sector twisted by θ^{-1} and $(-v^I)$. Notice how all four sectors are interrelated in order to obtain an anomaly-free result. This model contains nine families from the TT sector which feel only the $SU(6) \times U(1)$ interactions. The other three sectors give rise to nine further families which feel also the $SU(3)$ interactions. The gauge group $SO(14)' \times U(1)'$ is hidden. This example is not particularly interesting from the phenomenological point of view, but we have chosen it here because it has all the features of a general model based on the Z orbifold with a Wilson line.

One can obtain more interesting models by using different combinations of point-group embedding (v^I 's) and Wilson lines (a^I 's). Some examples are shown in table 2. Thus, for instance, one may get rid of the untwisted matter by putting some part of the Wilson line on top of the shift v^I . That is done in example 1 of table 2 which leads to a model with nine E_6 families (all from the TT sector) and a hidden $E_6' \times SU(3)^2$. Very often the addition of a Wilson line has the effect (in the case of the canonical v^I) of eliminating the families which otherwise would have appeared in the NTT-1 and NTT-2 sectors. This happens in examples 1 to 4 in table 2. Thus the addition of Wilson lines is an efficient way of controlling the number of generations of a given orbifold. It is then easy to obtain models with three families of 'quarks' and 'leptons'. This is the case of example 6 in the table. It starts from a 'non-canonical' v^I shift originating an $SU(9)$ which is broken by the Wilson line down to an $SU(5) \times [SU(2) \times U(1)]^2$ model with three $SU(5)$ families all originating in the untwisted sector (the twisted sector does not usually contain interesting matter for non-canonical v^I 's).

More three-family models may be obtained by adding an appropriate second Wilson line on another complex dimension, e.g. $a_3^I = a_4^I \equiv \bar{a}^I$. This should be done taking care that the usual modular invariance conditions are respected. In this case the gauge group is broken down to the subgroup commuting with both Wilson lines. The twenty-seven fixed points split into nine sectors of three fixed points each. If one starts from a canonical v^I , a second Wilson line can easily convert a model with nine generations (from the TT sector) into a model with three generations, since the TT sector has now only three fixed points. This is what happens in the examples shown in table 3. There are no families from the untwisted sector, since some part of the Wilson lines overlap with v^I , and then those families are projected out. All generations in the twisted sectors are also eliminated by the Wilson lines except for those in the TT sector. Since that sector contains three fixed points [e.g. $(0, 0, m)$, $m = 0, \pm 1$], we get three generations. Obviously we can build many models with different gauge

groups by adding two Wilson lines around two complex dimensions and considering different embeddings, and the examples shown in the table are just a few possibilities. It is rather remarkable that such a large variety of models (and generations) can be obtained just by adding Wilson lines to the simplest supersymmetric orbifold, the Z orbifold.

In principle one can obtain models which look more like the Standard Model, such as $SU(3) \times SU(2) \times U(1)^n$, but this does not guarantee that other phenomenological problems, such as neutrino masses and too fast proton decay, are solved. Furthermore, it is not clear that the additional $U(1)$'s present would not cause additional phenomenological problems. In this context it would be useful to study the effect of Wilson lines on non-Abelian orbifolds in which the additional $U(1)$'s could be absent.

There is another interesting point related to $U(1)$'s in this type of orbifold construction. Very often the $U(1)$'s obtained upon compactification are anomalous. This is true not only for models with additional Wilson lines but also for the usual non-canonical embeddings of table 1 without any Wilson lines. This fact is not new and has recently appeared in different contexts [8, 11, 12]. Since the theories we are discussing are modular-invariant, one suspects that some mechanism, probably related to the anomaly cancellation mechanism of Green and Schwarz, must be present to take care of anomalies. In fact the ten-dimensional ‘Wess–Zumino’ counter term [13] $B \wedge \text{Tr} (F \wedge F \wedge F \wedge F)$ yields, upon compactification [11], a one-loop four-dimensional coupling proportional to $B_{\mu\nu} \text{Tr} (F_{\rho\sigma}) \epsilon^{\mu\nu\rho\sigma}$, F being the field strength of the anomalous $U(1)$. After a duality transformation, this term acquires the aspect of the usual Goldstone boson to gauge boson coupling, and the anomalous $U(1)$ gauge boson becomes massive—hence harmless. Since this mechanism seems to involve just one chiral superfield, it seems to generate a SUSY-breaking D-term^{*)}. This is not necessarily a problem as long as the ‘observed’ matter does not have anomalous $U(1)$ quantum numbers.

4. CONCLUSIONS

Let us conclude with a few comments. We have shown how the addition of Wilson lines on orbifold tori is an efficient method for controlling the unbroken gauge group as well as the number of generations left after compactification. Thus, for example, orbifolds with a large Euler characteristic may yield realistic models if appropriate Wilson lines are present. In this context it would be interesting to see the connection between orbifold-like compactifications and the four-dimensional string models discussed in refs. [7] and [8]. Pre-existing Wilson lines could also be useful in the left–right asymmetric orbifolds recently considered by Narain, Sarmadi and Vafa [14]. The orbifold-type constructions seem to have the advantage that one has a certain intuitive control on the gauge interactions and number of generations to be expected from a given model. In the present paper we have studied, in some detail, the case of the Z orbifold, and we have shown how this very simple example can lead to interesting three-family models with gauge groups such as E_6 , $SU(3)_L$

^{*)} The existence of this mechanism seems to be known by a number of people. We thank L. Alvarez–Gaumé, J. Distler, W. Lerche and N. Schellekens for discussions about these points.

$\times \text{SU}(3)_R \times \text{SU}(3)_c$, $\text{SU}(6) \times \text{U}(1)$, $\text{SU}(5) \times [\text{SU}(2) \times \text{U}(1)]^2$, etc. These do not exhaust the list of possibilities, and it would be interesting to search for examples closer to the observed Standard Model.

To our knowledge these seem to be the first systematic sources of orbifold models with three generations in the literature^{*)}. It is amusing that, in all the models discussed above, the fact that one model has three generations is directly or indirectly related to the fact that there are three compactified complex extra dimensions.

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^{*)} See also the example of ref. [6].

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Table 1

Models without Wilson lines. [All $U(1)$ quantum numbers have been omitted.]

Example	Shift	Gauge group	Untwisted sector	Twisted sector
1	$(\frac{1}{3}\frac{1}{3}\frac{2}{3} \dots)(0 \dots)$	$E_6 \times SU(3) \times E_8'$	$3[(27, 3), 1']$	$27[(27, 1), 1'] + 81[(1, \bar{3}), 1']$
2	$(\frac{1}{3}\frac{1}{3}\frac{2}{3} \dots)(\frac{1}{3}\frac{1}{3}\frac{2}{3} \dots)$	$[E_6 \times SU(3)]^2$	$3[(27, 3)(1, 1)' + (1, 1)(27, 3)']$	$27[(1, \bar{3}), (1, \bar{3})']$
3	$(\frac{1}{3}\frac{1}{3} \dots)(\frac{2}{3} \dots)$	$[E_7 \times U(1)] \times [SO(14) \times U(1)]'$	$3(56, 1') + 3(1, 1') + 3(1, 14') + 3(1, 64')$	$27[(1, 1') + (1, 14')] + 81(1, 1')$
4	$(\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{2}{3} \dots)(\frac{2}{3} \dots)$	$SU(9) \times SO(14)' \times U(1)'$	$3(84, 1') + 3(1, 14') + 3(1, 64')$	$27(\bar{9}, 1')$

Table 2

Models with one Wilson line. [All $U(1)$ quantum numbers have been omitted.]

Example	Shift	Wilson line	Gauge group	Untwisted sector	Twisted sector	Comments
1	$(\frac{1}{3}, \frac{2}{3}, 0 \dots 0)(0 \dots)$	$(\frac{1}{3}, \frac{2}{3}, 0 \dots)(\frac{1}{3}, \frac{2}{3} \dots)$	$[E_6 \times SU(3)]^2$	Nothing	$9(27, 1) + 27(1, \bar{3}) + 27(1', \bar{3}') + 9(27', 1') + 9(1, 3, 1', \bar{3}')$	Nine E_6 families + ...
2	$(\frac{1}{3}, \frac{2}{3}, 0 \dots)(0 \dots)$	$(0 \dots \frac{1}{3}, \frac{2}{3})(\frac{1}{3}, \frac{2}{3} \dots)$	$[SU(3)]^4 \times E_6 \times SU(3)'$	$3[(3, 3, 3, 1)]$	$9[(1, 3, 3, 1) + (1, \bar{3}, 1, 3) + (1, 1, \bar{3}, \bar{3})] + 27(\bar{3}, 1, 1, 1) + 9[(1, \bar{3}, 1, 1)(\bar{3}, 1)'] + 9[(1, 1, \bar{3}, 1)(\bar{3}, 1)']$	Nine $SU(3)^4$ families + ...
3	$(\frac{1}{3}, \frac{2}{3}, 0 \dots)(0 \dots)$	$(0 \dots \frac{1}{3}, \frac{2}{3})(\frac{2}{3} \dots)$	$[SU(3) \times SU(6) \times U(1)] \times [SO(14) \times U(1)]'$	$3[(3, 15)]$	$9[1, 15 + \bar{6} + \bar{6}] + 27(\bar{3}, 1) + 9[(1, \bar{6} + \bar{6}) + (\bar{3} + \bar{3}, 1)]$	Eighteen $SU(6) \times SU(3) \times U(1)$ families
4	$(\frac{1}{3}, \frac{2}{3}, 0 \dots)(0 \dots)$	$(0 \dots \frac{1}{3}, \frac{2}{3}, \frac{1}{3})(\frac{2}{3} \dots)$	$[SU(6) \times SU(3) \times U(1)] \times [SO(14) \times U(1)]'$	$3[(3, \bar{6})]$	$9[1, 15 + \bar{6} + \bar{6}] + 27(\bar{3}, 1) + 9[(3, 1) + (1, \bar{6})] + 9(1, 1) + \text{hidden matter}$	Nine $SU(6) \times SU(3) \times U(1)$ families + ...
5	$(\frac{1}{3}, \frac{2}{3}, 0 \dots)(\frac{2}{3}, 0 \dots)$	$(0 \ 0 \ \frac{2}{3} \dots)(0 \ \frac{1}{3}, \frac{2}{3} \dots)$	$[SO(10) \times SU(2) \times U(1)]^2$	$3[(16', 2') + (10', 1')] + 3[(10, 2) + (1, 1)]$	$9(1, 10') + \text{singlets and doublets}$	Six $SO(10) \times SU(2)' \times U(1)'^2$ families + ...
6	$(\frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3} \dots)(\frac{2}{3} \dots)$	$(0 \dots \frac{2}{3})(0 \ \frac{1}{3}, \frac{2}{3} \dots)$	$[SU(5) \times SU(2) \times U(1)]^2 \times [SO(10)' \times SU(2)'] \times U(1)'^2$	$3[(\bar{10}, 1, 1) + 3(5, 2, 1) + 3(5, 1, 2) + \text{hidden matter}]$	$9[(\bar{3}, 1, 1)] + \text{singlets and doublets}$	Three $SU(5) \times SU(2)^2 \times U(1)^2$ families

Table 3

Models with two Wilson lines. [All $U(1)$ quantum numbers have been omitted.]

Example	Shift	Wilson line	Gauge group	Untwisted sector	Twisted sector	Comments
1	$(\frac{1}{3}, \frac{2}{3}, 0 \dots)(0 \dots)$	$a_1 = (\frac{1}{3}, \frac{2}{3}, 0 \dots)$ $(\frac{1}{3}, \frac{2}{3}, \dots)$ $a_3 = (\frac{1}{3}, \frac{2}{3})$ $(\dots \frac{1}{3}, \frac{2}{3})$	$[E_6 \times SU(3)] \times [SU(3)]^4$	Nothing	$3[(1, 27)] + \text{singlets and triplets}$	Three E_6 families
2	$(\frac{1}{3}, \frac{2}{3}, \dots)(0 \dots)$	$a_1 = (\dots \frac{1}{3}, \frac{2}{3})$ $(\frac{1}{3}, \frac{2}{3}, \dots)$ $a_3 = (\frac{1}{3}, \frac{2}{3}, \dots \frac{1}{3}, \frac{2}{3})$ $(\dots \frac{1}{3}, \frac{2}{3})$	$[SU(3)]^8$	Nothing	$3[(1, 3, 3, 1) + (1, \bar{3}, 1, 3) + (1, 1, \bar{3}, \bar{3})] + \dots$	Three $[SU(3)]^3$ families + ...
3	$(\frac{1}{3}, \frac{2}{3}, \dots)(0 \dots)$	$a_1 = (\dots \frac{1}{3}, \frac{1}{3})$ $(\frac{2}{3}, \dots)$ $a_3 = (\frac{1}{3}, \frac{2}{3}, \dots)$ $(\dots \frac{1}{3}, \frac{2}{3})$	$[SU(6) \times SU(3) \times U(1)] \times [SU(3) \times SU(2)^2 \times U(1)^4]$	Nothing	$3[(15 + \bar{6} + \bar{6}, 1)] + \dots$	Three $SU(6) \times U(1)$ families + ...