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Physics of high energies and cosmic rays.

B.S.Fradkin and M.A.Vasiliev

CANDIDATE TO THE ROLE OF HIGHER - SPIN SYMMETRY

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E.S. Pradkin and M.A. Vasiliev

ABSTRACT

Infinite-dimensional non-abelian superalgebras, related to interacting massless higher-spin fields of all spins $s \geq \frac{3}{2}$, are proposed. The conditions are formulated providing uniqueness for these higher-spin superalgebras denoted as $shs_\rho(1)$ ($\rho=0$ or 1). Among these conditions, the major role is played by the requirement that linearized curvatures of $shs_\rho(1)$ should coincide with the linearized higher-spin curvatures obtained previously. Subalgebras and contractions of $shs_\rho(1)$ are considered. A higher-spin interaction is discussed briefly, which is introduced in terms of the $shs_\rho(1)$ curvatures. It is found that a gravitational interaction of massless higher spins requires non-analyticity in the cosmological constant.

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1. Introduction.

The present paper represents by itself the direct continuation of the investigation of the higher-spin field dynamics initiated in refs. /1,2/ where the quite uniform description for all free massless higher-spin fields on the de Sitter and flat backgrounds has been developed. This form of description is based on the use of non-symmetric tensors and tensor-spinors being abelian gauge fields similar to the $osp(1,4)$ gauge fields of $N=1$ supergravity. In ref./2/, consistent curvatures have been found linearized on the de Sitter background. The actions for free massless higher-spin fields take extremely simple form in terms of these curvatures.

Using the results of ref./2/ as somewhat "initial data", we construct in this paper non-abelian superalgebras generating at the linearized level the curvatures of ref./2/. In what follows, these superalgebras will be denoted as $shs_\rho(1)$ (super higher spins, the parameter ρ takes on values 0 or 1). They are infinite-dimensional, and generate the systems of fields of all spins (each spin $s \geq \frac{3}{2}$ being present only once). The maximum finite-dimensional subalgebras of $shs_\rho(1)$ coincide with $osp(1,4)$, $N=1$ de Sitter superalgebra.*

We hope that if consistent theories, describing massless higher-spin fields interacting with gravity, do exist, the super-

* The algebras $shs_\rho(1)$, considered in the present paper, represent by themselves generalizations of $N=1$ de Sitter superalgebra $osp(1,4)$. Corresponding generalizations of extended de Sitter superalgebras $osp(N,4)$ will be considered in the separate publication by one of us /3/. For these, the designation $shs_\rho(N)$ is reserved.

algebras $shs_p(1)$ may play for these theories the same role as $osp(1,4)$ does for $N=1$ supergravity /4/. Since $shs_p(1)$ contains $osp(1,4)$ as subalgebra, one can also hope that hypothetical higher-spin theories should include supergravity similarly to that as supergravity itself includes Einstein gravity.

Let us stress that as is presently known from the development of supergravity, the question on the structure of local symmetries has two related aspects in theories including gravity. The first one is that of the rigid symmetry G being the symmetry of the vacuum. This symmetry generates geometrical entities, connections and curvatures, which are used for description of a theory. The second aspect, being of most importance, is determination of a true local symmetry L leaving invariant an action of the theory. The relation between G and L has simple nature for internal symmetries only, i.e. for Yang-Mills gauge fields when localization occurs in the standard fashion. In the more general case of theories like gravity and supergravity, the local transformations, corresponding to L , differ from the usual (Yang-Mills) gauge transformations generated by G by some terms proportional to the curvatures of G . Moreover, an action is determined in such theories simultaneously with the determination of L . So, if the symmetry G provides main geometrical quantities convenient for the description of the theory, the true local symmetry L has to be found in the procedure of constructing a consistent dynamics (note that the matters, concerned above, were considered by many authors, see e.g. refs. /4-8/).

In the present paper, we hope to make the next step towards

solution of the higher-spin problem, the problem of introducing a consistent gravitational interaction of massless higher-spin fields. Namely, we propose infinite-dimensional superalgebras $shs_p(1)$ as candidates to the role of higher-spin rigid symmetry G . The problem of constructing a consistent interacting higher-spin theory based on $shs_p(1)$, which includes the determination a true higher-spin local symmetry L , is not solved now, and it will be discussed in this paper only briefly. However, it seems to us that the very fact is sufficiently promising that starting from free higher-spin fields [1,2], we succeeded in constructing the nontrivial higher-spin symmetry $shs_p(1)$. A priori, a more probable result might been the absence of any adequate non-abelian higher-spin symmetry.

In order to simplify the understanding of the main results of the paper for the reader who is not interested in the specific details of their derivation, we present from the very beginning the final expressions for the curvatures originating from the superalgebras $shs_p(1)$. These curvatures are of the form

$$\begin{aligned}
 R_{\nu\mu, \alpha(m), \beta(m)} = & \partial_\nu (\omega_{\mu, \alpha(m), \beta(m)}) - \partial_\mu (\omega_{\nu, \alpha(m), \beta(m)}) \\
 & + \sum_{p+q+s+t=n} \frac{n! m!}{p! q! s! k! t!} \delta(n-p-q) \delta(m-k-l) \times \\
 & \times \delta((p+k)(q+l) + (p+k)(s+t) + (q+l)(s+t) + 1/2) \times \\
 & \times (\omega_{\nu, \alpha(p)} \gamma(s) \beta(k) \delta(t) (\omega_{\mu, \alpha(q)} \gamma(s) \beta(l) \delta(t)) \quad (1.1)
 \end{aligned}$$

where $(\omega_{\nu, \alpha(m), \beta(m)})$ are the gauge fields being commuting (anti-commuting) if the number of spinorial indices², $n+m$, is

² In this paper, we use conventions and notations introduced in

even (odd). In eq.(1.1), summation over the numbers of spinorial indices $\rho, \eta, \beta, \kappa, \ell, t$ is carried out over all non-negative values 0, 1, ... We use also the following notations:

$$\delta(n) = \begin{cases} 1 & \text{at } n=0 \\ 0 & \text{at } n \neq 0 \end{cases}, \quad (1.2)$$

$$|n|_2 = n - 2 \left[\frac{n}{2} \right] \quad (1.3)$$

where $\left[\frac{n}{2} \right]$ denotes the integer part of $\frac{n}{2}$.

The fields $(\mathcal{W})_{\nu, \alpha(n), \beta(m)}$ are assumed to satisfy one of the following two types of hermiticity conditions parametrized by the parameter $\rho = 0$ or 1:

$$(\mathcal{W})_{\nu, \alpha(n), \beta(m)}^\dagger = (-1)^{\frac{1}{2}\rho(n+m-2)(n+m-1)} (\mathcal{W})_{\nu, \beta(m), \alpha(n)}. \quad (1.4)$$

These hermiticity conditions correspond to two different real superalgebras $\text{shs}_\rho(1)$.

The curvatures (1.1) contain all the fields $(\mathcal{W})_{\nu, \alpha(n), \beta(m)}$ with arbitrary $n \geq 0, m \geq 0$. According to the results of ref./2/, we relate all the fields $(\mathcal{W})_{\nu, \alpha(n), \beta(m)}$ with $n+m=2(s-1)$ to the massless particle of the spin s . Thus the superalgebras $\text{shs}_\rho(1)$

ref./2/. Namely, upper and lower indices, denoted by the same letter, are assumed to be (separately) symmetrized. It is also assumed that after symmetrization, the maximum possible number of lower and upper indices, denoted by the same letter, has to be contracted. The number of indices is indicated in brackets. Spinorial indices $\alpha, \beta, \dots, \dot{\alpha}, \dot{\beta}, \dots$ are from the beginning of the Greek alphabet and take on values 1, 2. The space-time indices $\mu, \nu, \rho, \dots$ take on values 0-3.

generate all spins $J \geq 1$ (any spin appears only once). In fact, the superalgebras $shs_{\rho}(1)$ have the structure $shs_{\rho}(1) \oplus o(2)$ where the subalgebra $o(2)$ originates from the generator related to the spin 1 field $(\mathcal{W})_{\nu}, \alpha(0), \beta(0)$, while the superalgebras $shs'_{\rho}(1)$ are simple and generate all spins $J \geq \frac{3}{2}$.

The authors opinion is that the fact, that higher-spin superalgebras are infinite-dimensional and generate spins increasing to infinity, is inevitable in any consistent theory of massless higher spins in the framework of gravity. Specifically, one can hope that the superalgebras $shs'_{\rho}(1)$ provide the bases for consistent higher-spin theories containing all spins $J \geq \frac{3}{2}$. However, the superalgebras $shs'_{\rho}(1)$ do not exhaust all possible candidates to the role of $(N=1)$ higher-spin superalgebras.

Indeed, the superalgebras $shs_{\rho}(1)$ contain subalgebras which themselves pretend to this role. These subalgebras, denoted as $shs(1|\alpha)$ ($\alpha=0$ or 1), can be defined by the same expressions (1.1), (1.4) where it is now assumed that the fields $(\mathcal{W})_{\nu}, \alpha(r), \beta(m)$ are nonvanishing if $|n+m|_4 = 2$ or $|n+m|_4 = 1+2\alpha$, while all other fields have to be set equal to zero (the parameter ρ is irrelevant for $shs(1|\alpha)$). In other words, superalgebras $shs(1|\alpha)$ generate integer spins $2, 4, 6, \dots$ and half-integer spins $\frac{3}{2} + \alpha, \frac{5}{2} + \alpha, \frac{7}{2} + \alpha, \dots$. Let us stress that the maximum finite-dimensional subalgebra of $shs(1|0)$ (and $shs'_{\rho}(1)$) is $osp(1,4)$, while the maximum finite-dimensional subalgebra of $shs(1|1)$ is $sp(4)$. Thus, theories based on $shs(1|0)$ or $shs'_{\rho}(1)$ may contain $N=1$ supergravity as their parts while a theory based on $shs(1|1)$ cannot.

The rest part of the paper is organized as follows. In

Section 2, we formulate initial assumptions used for the determination of the higher-spin superalgebras. In Section 3, we derive equations which follow from the Jacobi identities, and present their general solution (under conditions following from the assumptions of Section 2) corresponding to the complex superalgebra $\mathfrak{shs}(1, \mathbb{C})$. Reality (hermiticity) conditions, extracting real forms $\mathfrak{shs}_\rho(1)$ of $\mathfrak{shs}(1, \mathbb{C})$, are constructed in Section 4. Automorphisms of $\mathcal{P}$ -reflection are also found here. In Section 5, various subalgebras of $\mathfrak{shs}(1, \mathbb{C})$ and $\mathfrak{shs}_\rho(1)$ are described. One part of these subalgebras can also be considered as candidates to the role of higher-spin superalgebras while another part may serve as higher-spin counterparts of Lorentz subalgebra of $\mathfrak{sp}(4)$ in gravity and $\mathfrak{osp}(1, 4)$ in supergravity. In Section 6, we consider a number of contractions of $\mathfrak{shs}_\rho(1)$ and $\mathfrak{shs}(1, \mathbb{C})$. Some peculiar features of flat contractions for higher-spin superalgebras are discussed here. In Section 7, we concern briefly some aspects of higher-spin interaction introduced via the curvatures of $\mathfrak{shs}_\rho(1)$. Finally, we formulate the main results of the paper in Section 8. Some points, being of more technical character, are considered in Appendices. Appendix A contains combinatorial derivation of the form of Jacobi identities used in the main text. In Appendix B, the general solution of Jacobi identities is found which satisfies conditions formulated in Section 3.

2. Initial Assumptions On Higher-Spin Superalgebra.

In this Section, we discuss initial assumptions on the structure of higher-spin superalgebras, used in the constructing

of superalgebras $\text{shs}_p(1)$. We formulate these assumptions in the form of hypotheses. One part of them has clear physical content, while another part receives its justification mainly due to the fact that it provides solution of the task to be unique.

Hypothesis I. A higher-spin superalgebra g contains Lorentz subalgebra $O_4(3,1)$, and splits into a direct sum of finite-dimensional representations of $O_4(3,1)$. In other words, generators (basis) T_A of the higher-spin superalgebra g can be realized as a set of multispinors $T_{\alpha(n),\beta(m)}^i$ (index i enumerates multispinors of the same type) which satisfy the usual commutation relations with Lorentz generators $T_{\alpha(2)} = T_{\alpha(2),\beta(0)}^{i_4}$ and $T_{\beta(2)} = T_{\alpha(0),\beta(2)}^{i_4}$, being of the form

$$[T_{\gamma(2)}, T_{\alpha(n),\beta(m)}^i] = n \epsilon_{\alpha\gamma} T_{\gamma\alpha(n-1),\beta(m)}^i, \quad (2.1)$$

$$[T_{\delta(2)}, T_{\alpha(n),\beta(m)}^i] = m \epsilon_{\beta\delta} T_{\alpha(n),\delta\beta(m-1)}^i. \quad (2.2)$$

Hypothesis II. Any representation of $O_4(3,1)$ occurs in the $(N-1)$ higher-spin superalgebra g once and only once i.e., the set of generators T_A is reduced to $T_{\alpha(n),\beta(m)}$ with arbitrary $n \geq 0, m \geq 0$.

Hypothesis III. Superalgebra g contains de Sitter Lie algebra $\mathfrak{sp}_{dS}(4)$ as subalgebra which, in its turn, contains $O_4(3,1)$. The linearization of the curvatures corresponding to g with respect to $\mathfrak{sp}_{dS}(4)$ gives the linearized higher-spin curvatures of ref./2/.

Hypothesis IV. In the relations

$$\{T_{\alpha\gamma(n),\beta(m)}, T_{\alpha}^{\gamma(n),\beta(m)}\} = C(n,m) T_{\alpha(2),\beta(0)}, \quad (2.3)$$

$$[T_{a(m), \dot{\beta}(m)}, T^{a(m), \dot{\beta}(m)}] = d(n, m) T_{a(0), \dot{\beta}(2)} \quad , \quad (2.4)$$

the coefficients $c(n, m)$ and $d(n, m)$ are different from zero for all $n \geq 0, m \geq 0$.

Hypothesis V. There exists an infinite sequence of Lie superalgebras $\mathcal{G}_K$ ($K=1, 2, \dots, \infty$) for which spinorial indices take on values $1 \dots 2K$, structure coefficients of $\mathcal{G}_K$ do not depend on K explicitly in the multispinorial basis, and $\mathcal{G} = \mathcal{G}_1$ satisfies Hypotheses I.-IV. In other words, the two-componentness of spinors is irrelevant.

The main result of the paper consists in discovering the superalgebras $g = \text{shs}_p(1)$ which turn out to be the only solutions satisfying Hypotheses I.-V. So, let us consider these hypotheses in more detail.

Among Hypotheses I.-V, the most clear ones are Hypotheses I. and III.

Hypothesis I. is almost trivial. In fact, it is equivalent to the requirement that a higher-spin theory, related to the superalgebra g , should only contain the gauge fields $\omega_{\nu, a(n), \dot{\beta}(m)}^i$ realizing finite-dimensional representations of the Lorentz group.

Hypothesis III. contains two points. The first one reduces to the requirement that a hypothetical higher-spin theory, related to the superalgebra g , has to contain gravity described by the fields $\omega_{\nu, \alpha, \dot{\beta}}, \omega_{\nu, a(2), \dot{\beta}(0)}, \omega_{\nu, a(0), \dot{\beta}(2)}$ corresponding to the de Sitter subalgebra $\text{sp}_{4S}(4)$. The second one fixes the relation of g to the linearized results on the

description of free higher-spin fields obtained in ref./2/, which provide somewhat "initial data" for the determination of g . To be concrete, we assume the following linearization /2/:

$$\omega_{\nu, \alpha, \beta} = \bar{h}_{\nu, \alpha, \beta} + \omega_{\nu, \alpha, \beta}^{\pm}, \quad (2.5)$$

$$\omega_{\nu, \alpha(2), \beta(2)} = W_{\nu, \alpha(2)} + \omega_{\nu, \alpha(2), \beta(2)}^{\pm},$$

$$\omega_{\nu, \alpha(0), \beta(2)} = \bar{W}_{\nu, \beta(2)} + \omega_{\nu, \alpha(0), \beta(2)}^{\pm},$$

where the de Sitter background fields $\bar{h}$, W , $\bar{W}$ are of zero order in an expansion parameter while the fields $\omega_{\nu, \alpha, \beta}^{\pm}$, $\omega_{\nu, \alpha(2), \beta(2)}^{\pm}$, $\omega_{\nu, \alpha(0), \beta(2)}^{\pm}$ and all other fields generated by g are of the first order.

The Hypotheses II., IV., and V. are less grounded. It seems that Hypotheses IV., V. have, in fact, technical character, and we shall discuss them in the passages where they will be actually used (for Hypothesis IV. see Appendix B and for Hypothesis V. see Section 3). Here we only note that rejection of Hypothesis IV. leads to appearance of a number of inequivalent solutions. However, all such solutions, which are known to us, correspond to some contractions of $shs_p(1)$ (for more detail see Section 6). It seems to be interesting enough to know whether all superalgebras, satisfying Hypotheses I.-III., V. (or even I.-III.), are exhausted by the superalgebras $shs_p(1)$ and their contractions or not.

Hypothesis II. deserves more attention since this determines the concrete structure of (super)generators.

As follows from the results of ref./2/, we expect that a full set of generators of a higher-spin superalgebra g splits into a number of the subsets containing generators $T_{a(n),\beta(m)}^{i,j}$ with $n+m=2(j-1)$. Any such subset corresponds to some spin s and generates the fields $(\psi_{\beta(n),\alpha(m)}^{i,j})$ with $n+m=2(j-1)$ which have been used in ref./2/ for the description of free massless higher-spin fields. Different subsets are enumerated by the index $i_j=1\dots M_j$ where M_j is the number of massless particles of the spin s . So, Hypothesis II. reduces to the specification of the numbers of particles of different spins in a theory related to the higher-spin superalgebra under consideration.

It seems to be quite natural that there should exist a sequence of superalgebras generalizing extended de Sitter superalgebras $osp(N,4)$ to the higher-spin case (note that this sequence has to be different from the sequence mentioned in Hypothesis V.) In the present paper, we confine ourselves to the consideration of the simplest case generalizing $N=1$ supersymmetry. As $N=1$ supersymmetry does not contain internal symmetries, one can suppose that the corresponding higher-spin superalgebra does not contain internal symmetries as well. In other words, we suppose that for the case under investigation, higher-spin fields do not possess internal indices. The choice $M_j = 1$ for all $j \geq 1$ (Hypothesis II.) is the simplest one satisfying this condition automatically. As the spin-1 gauge fields usually correspond to internal symmetries, the choice $M_j = 1$ at $j \geq \frac{3}{2}$, $M_1 = 0$ seems also to be quite natural. In fact, it will be seen that the difference between these two possibilities is not significant, as the general solution for the case $M_j = 1$

for all $s \geq 1$ has the structure $\text{shs}'_p(1) \oplus \mathcal{O}(2)$ where the $\mathcal{O}(2)$ -factor corresponds to the spin-1 generator $T_{\alpha(\sigma), \beta(\sigma)}$. Consequently, we shall consider the first possibility, $M_j = 1$ for all $j \geq 1$.

In fact, the above arguments in favor of the validity of Hypothesis II. are not strict enough. Indeed, there exist other solutions with some $M_j = 0$ for $j \geq \frac{3}{2}$. However, all such solutions, which are known to us, correspond to some subalgebras of $\text{shs}_p(1)$ (see Section 5.). Hypothesis II. receives its justification mainly as the most simple variant providing existence for the corresponding higher-spin superalgebras $\text{shs}_p(1)$. Nevertheless, it contains an element which seems to be inevitable in attempts to construct a nontrivial symmetry of massless higher-spin fields interacting with gravity. This element is infinite dimensionality of a (rigid) higher-spin symmetry.

Obviously, the property, that higher-spin symmetries turn out to be infinite-dimensional, is very important. Indeed, this implies that ^{any} theory of massless higher-spins interacting with gravity contains infinite number of fields of different spins. Let us stress,

however, that presence of infinite number of fields will not contradict to the known experimental data if one ^(will) succeeds in constructing such a theory where after suitable symmetry breaking all fields, except some finite number of lower-spin fields, will really appear at very high energies (presumably Planck's energies). In fact, the quite similar situation takes place in Kaluza-Klein theories (see e.g. ref./9/). The only difference is that these theories contain infinite number of lower-spin fields and do not contain higher-spin fields at all.

The authors opinion is that the picture, when (elementary) particles of all spins exist in nature but for one or another reason only lower-spin fields can be detected at low energies, is most satisfactory from the aesthetic point of view. We do not know any arguments against infinite number of higher-spin fields before their dynamics is not understood in detail.

An indirect argument in favor of the infiniteness of the higher-spin symmetry is contained in the known paper by Haag, Lopuszansky and Sohnius /10/ since the no-go theorem by these authors is not applicable to the case of infinite number of massless fields. The hypothesis, that systems of interacting higher-spin fields may be infinite, has been also mentioned by some other authors (see e.g. ref. /11/).

The experience, accumulated during prolonged investigation of the higher-spin problem, led the authors of the present paper to the conclusion that infiniteness of the number of massless higher-spin fields, interacting with gravity, is most likely inevitable. Some rough arguments in favour of this conclusion can be given e.g. as follows. Any massless (gauge) spin $s \geq 1$ field is related to a gauge symmetry with the symmetry parameters forming the spin $s' = s - 1$ multispinor. By analogy to supersymmetry when $s' = \frac{1}{2}$, one can expect that higher-spin symmetries mix the fields of different spins. It seems natural to suppose that symmetries with increasing s' mix increasing numbers of fields of different spins (roughly speaking, s' - "supermultiplets"). It can turn out that for any $s'_0 > 1$, corresponding "supermultiplets" always contain massless gauge fields of spins $s_1 > s'_0 + 1$. Since all higher-spin fields are assumed to be massless and hence they are gauge fields, this implies that symmetries with the parameters possessing the spins $s'_1 > s'_0 + 1$ should be presented. In their turn, these new symmetries will form new

"supermultiplets" being broader than the initial ones. They will contain a field of spin $\mathcal{J}_2 > \mathcal{J}_2' + 1$, and so on.

3. Higher-Spin Superalgebra $\text{shs}(1, \epsilon)$.

In this Section, we present final results for the complex superalgebra $\text{shs}(1, \epsilon)$ being the complexification of the real higher-spin superalgebras $\text{shsp}(1)$.

We shall work not in terms of generators (basis) of an algebra but in terms of infinitesimal symmetry parameters (coordinates of an algebra). More exactly, we shall use the notion of "Grassmann shell" of superalgebra, introduced by Beresin /12/.

Given Lie superalgebra g , its generators obey the relations

$$[T_A, T_B] = \mathcal{U}_{AB}^C T_C, \quad (3.1)$$

where $\mathcal{U}_{AB}^C$ are the structure coefficients of g . Let us consider arbitrary elements of the "Grassmann shell" of g which, by definition, are of the form /12/

$$T = \mathcal{J}^A T_A, \quad (3.2)$$

where "coordinates" $\mathcal{J}^A$ are assumed to be even (odd) elements of some Grassmann algebra if T_A is an even (odd) element of g . For any two elements T_1 and T_2 with coordinates $\mathcal{J}_1^A$ and $\mathcal{J}_2^A$, respectively, one finds

$$[T_1, T_2] = [\mathcal{J}_1, \mathcal{J}_2]^A T_A \quad (3.3)$$

where

$$[\mathcal{J}_1, \mathcal{J}_2]^A = \mathcal{U}_{BC}^A \mathcal{J}_1^B \mathcal{J}_2^C. \quad (3.4)$$

By its definition (3.3), the composition law (3.4) satisfies ^{the} antisymmetry condition

$$[\mathfrak{F}_1, \mathfrak{F}_2]^A = -[\mathfrak{F}_2, \mathfrak{F}_1]^A \quad (3.5)$$

and Jacobi identities

$$[[\mathfrak{F}_1, \mathfrak{F}_2], \mathfrak{F}_3]^A + [[\mathfrak{F}_2, \mathfrak{F}_3], \mathfrak{F}_1]^A + [[\mathfrak{F}_3, \mathfrak{F}_1], \mathfrak{F}_2]^A = 0. \quad (3.6)$$

Conversely, any composition law $[\mathfrak{F}_i, \mathfrak{F}_j]^A$, satisfying eqs. (3.5),

(3.6) for all $\mathfrak{F}_1^A, \mathfrak{F}_2^A, \mathfrak{F}_3^A$, generates some Lie superalgebra.

Note that the curvatures and infinitesimal gauge transformations, generated by the superalgebra $\mathfrak{g}$, can be written in the form

$$R_{\nu\mu}^A = \partial_\nu \omega_\mu^A - \partial_\mu \omega_\nu^A + [\omega_\nu, \omega_\mu]^A, \quad (3.7)$$

$$\delta \omega_\nu^A = \partial_\nu \mathfrak{F}^A + [\omega_\nu, \mathfrak{F}]^A, \quad (3.8)$$

where the "commutators" for the gauge fields ω and the parameters $\mathfrak{F}$ are understood in the sense of eq. (3.4).

After these preliminary remarks, let us consider the higher-spin complex superalgebra $\text{shs}(1, \mathbb{C})$. As follows from Hypotheses I, II. of Section 2., the parameters of the higher-spin superalgebra under investigation form the set of multispinors $\mathfrak{F}_{\alpha(n), \beta(m)}$ with all non-negative integer n, m . The most general composition law being in agreement to Hypotheses I., II., is of the form

$$[\mathfrak{F}_1, \mathfrak{F}_2]_{\alpha(n), \beta(m)} = \sum_{p, q, s, k, \ell, t} i^{s+t+1} \frac{n!m!}{p!q!s!k!\ell!t!} \delta(n-p-q) \delta(m-k-\ell) \times \quad (3.9)$$

$$\times \alpha(p, q, s, k, \ell, t) \mathfrak{F}_{1\alpha(p)\beta(s)} \mathfrak{F}_{2\alpha(k)\beta(\ell)} \delta(t) \mathfrak{F}_{2\alpha(q)\beta(\ell)}^{\delta(s)}, \beta(\ell) \delta(t),$$

where the factor $i^{s+t+1} \frac{n!m!}{p!q!s!k!\ell!t!}$ is introduced

for convenience, and $\alpha(p, q, s, k, \ell, t)$ is an unknown function of non-negative integer arguments. The summation over integer variables p, q, s, k, ℓ, t in eq. (3.9) is carried out from 0 to ∞ .

The antisymmetry property (3.5) imposes the following conditions on $\alpha(p, q, s, k, \ell, t)$:

$$\alpha(p, q, s, k, \ell, t) = (-1)^{s+t(p+s+k+t)(q+s+\ell+t)} \alpha(q, p, s, \ell, k, t). \quad (3.10)$$

The sign factor in eq.(3.11) arises, first, due to the lowering of $s+t$ spinorial indices and, second, due to the fact that

$$\tilde{\mathcal{I}}_1 \alpha(n, \beta(m)) \tilde{\mathcal{I}}_2 \gamma(u, \delta(v)) = (-1)^{(n+m)(u+v)} \tilde{\mathcal{I}}_2 \gamma(u, \delta(v)) \tilde{\mathcal{I}}_1 \alpha(n, \beta(m)). \quad (3.11)$$

After some combinatorics explained in Appendix A, one can verify that the application twice of eq.(3.9) leads to the following result

$$\begin{aligned} [[\tilde{\mathcal{I}}_1, \tilde{\mathcal{I}}_2], \tilde{\mathcal{I}}_3] = & - \sum_{\substack{u, v, p, q, s, k, \ell, t, r, u, v, f, g, h}} i^{(s+u+t+f+v+g)} \frac{n!m!}{p!q!s!k!\ell!t!r!u!v!f!g!h!} \times \\ & \times \delta(n-p-q-r) \delta(m-k-\ell-h) \alpha(p+s, q+v, u, k+t, \ell+g, f) \times \\ & \times \alpha(p+q, r, s+v, k+\ell, h, t+g) \tilde{\mathcal{I}}_1 \alpha(p) \gamma(s) \rho(u), \beta(k) \delta(t) \dot{\alpha}(f) \times \\ & \times \tilde{\mathcal{I}}_2 \alpha(q) \gamma(v) \rho(u), \beta(\ell) \delta(g) \dot{\alpha}(f) \tilde{\mathcal{I}}_3 \alpha(r) \gamma(s+v), \beta(h) \delta(t+g). \end{aligned} \quad (3.12)$$

Using eqs. (3.11), (3.12) and making some redefinitions of the summation variables, one can rewrite Jacobi identities (3.6)

in the form

$$\begin{aligned} & \sum_{\substack{u, v, p, q, s, k, \ell, t, r, u, v, f, g, h}} i^{(s+u+t+f+v+g)} \frac{n!m!}{p!q!s!k!\ell!t!r!u!v!f!g!h!} \delta(n-p-q-r) \delta(m-k-\ell-h) \times \\ & \times A(p, q, s, k, \ell, t, u, v, r, f, g, h) \tilde{\mathcal{I}}_1 \alpha(p) \gamma(s) \rho(u), \beta(k) \delta(t) \dot{\alpha}(f) \times \\ & \times \tilde{\mathcal{I}}_2 \alpha(q) \gamma(v) \rho(u), \beta(\ell) \delta(g) \dot{\alpha}(f) \tilde{\mathcal{I}}_3 \alpha(r) \gamma(s+v), \beta(h) \delta(t+g) = 0 \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} A(p, q, s, k, \ell, t, u, v, r, f, g, h) = & \\ = & \alpha(p+s, q+v, u, k+t, \ell+g, f) \alpha(p+q, r, s+v, k+\ell, h, t+g) \\ & + (-1)^{s+u+t+f+(p+s+u+k+t+f)(q+u+\ell+f+r+s+h+t)} \times \\ & \times \alpha(q+u, r+s, v, \ell+f, h+t, g) \alpha(q+r, p, u+s, \ell+h, k, t+f) \end{aligned} \quad (3.14)$$

$$+(-1)^{s+v+t+g+(r+s+v+h+t+g)(p+s+k+t+g+v+l+g)} \times \\ \times \alpha(r+v, p+u, s, h+g, k+f, t) \alpha(r+p, g, u+v, h+k, l, g+f).$$

Thus, in order to find the higher-spin superalgebra, one must solve eqs. (3.10), (3.13) for the function $\alpha(p, g, s, k, l, t)$ satisfying additional restrictions following from Hypotheses I, III, IV. (it is desirable to use only Hypotheses I. and III. but we have not succeeded in solving this task).

Let us stress that, generally, it does not follow from eq. (3.13) that

$$A(p, g, s, k, l, t, u, v, r, f, g, h) = 0. \quad (3.15)$$

The reason is that the quantities

$$\begin{aligned} & \tilde{J}_{1a}(p) \gamma(s) \rho(u), \tilde{\beta}(k) \delta(t) \tilde{\epsilon}(f) \quad \tilde{J}_{2a}(g) \gamma(v) \rho(u), \tilde{\beta}(l) \delta(g) \tilde{\epsilon}(f) \quad (3.16) \\ & \times \tilde{J}_{3a}(r) \gamma(s+v), \tilde{\beta}(h) \delta(t+g) \end{aligned}$$

are not independent at different values of the parameters $p, g, l..$ due to relations analogous to well-known Fierz identities for bispinors. In fact, these relations are consequences of the simple fact that for any set of indices taking on values from 1 to d , the complete antisymmetrization over $d+1$ indices gives zero. In the our case, spinorial indices take on values 1, 2. So, the antisymmetrization over any three spinorial indices gives zero. For example,

$$\epsilon^{\eta_1} (\tilde{J}_{1a} \tilde{J}_{2\eta} \tilde{J}_{3\eta} + \tilde{J}_{1\eta} \tilde{J}_{2\eta} \tilde{J}_{3a} + \tilde{J}_{1\eta} \tilde{J}_{2a} \tilde{J}_{3\eta}) \equiv 0, \quad (3.17)$$

i.e. one of the "Fierz identities" is of the form

$$\bar{\mathfrak{z}}_{1\alpha} \bar{\mathfrak{z}}_{2\beta} \bar{\mathfrak{z}}_3^\delta + \bar{\mathfrak{z}}_{1\beta} \bar{\mathfrak{z}}_2^\delta \bar{\mathfrak{z}}_{3\alpha} - \bar{\mathfrak{z}}_{1\delta} \bar{\mathfrak{z}}_{2\alpha} \bar{\mathfrak{z}}_3^\delta = 0. \quad (3.18)$$

It is not difficult to write down all such relations for the quantities (3.16) (the related topics are analysed in detail e.g. in ref./13/). Then, one must exclude all dependent quantities (3.16) from eq.(3.14), and put equal to zero coefficients against the quantities (3.16) remaining independent. This will give true equations for the function $\alpha(p, q, s, k, \ell, t)$. In practice, it is more suitable to consider the relations of the type (3.18) as some constraints and to introduce all these constraints into right hand side of eq.(3.13) with arbitrary coefficients. However, for the both approaches, the equations arising turn out to be very cumbersome. So, we attempted to solve eq.(3.15) instead of eq. (3.13). Although, eq.(3.15) seems to be much stronger than eq. (3.13), it have more compact and symmetric form that gives the hope to solve it explicitly.

In fact, the possibility of the transition from eq.(3.13) to eq.(3.15) is the content of Hypothesis V. Indeed, we have not used the two-componentness of spinors in the derivation of eq.(3.13). Therefore, any solution of eq.(3.15) generates a sequence of superalgebras with dotted and undotted spinorial indices taking on arbitrary even numbers of values $2k$ and $2\bar{k}$, respectively. On the other hand, if there exists any superalgebra with $k \geq 2, \bar{k} \geq 2$ satisfying eq.(3.13), it does not possesses "Fierz identities" of the type (3.18) and hence eq.(3.15) will (be also) satisfied for this superalgebra. As a result, the whole sequence with arbitrary $k \geq 1, \bar{k} \geq 1$ including the case $k=1, \bar{k}=1$, which is of interest for us, will also exist.

Before writing "initial data" which follow from Hypotheses I, III, IV, we note that in order to solve eqs.(3.10), (3.14), (3.15) one needs not concrete "initial" values for $\alpha(p, q, s, k, \ell, \epsilon)$ but only the information whether the function $\alpha(p, q, s, k, \ell, \epsilon)$ is equal to zero or different from it at "initial" values of the arguments $p, q, s, \dots$. This is not surprising since e.g. "initial data", related to Hypothesis III., were obtained in ref./2/ from the consistency requirement for the linearized higher-spin curvatures, that is, in fact, from the requirement that some Jacobi identities should be satisfied.

The "initial data", originating from Hypothesis I, follow from eqs.(2.1), (2.2) and are of the form

$$\alpha(1, q, 1, 0, \ell, 0) \neq 0, \quad \alpha(0, q, 0, 1, \ell, 1) \neq 0; \quad (3.19)$$

$$\alpha(2, q, 0, 0, \ell, 0) = \alpha(0, q, 2, 0, \ell, 0) = \alpha(0, q, 0, 2, \ell, 0) = \alpha(0, q, 0, 0, \ell, 2) = 0. \quad (3.20)$$

Taking into account the (anti)symmetry condition (3.10), one also finds from eqs.(3.19), (3.20) that

$$\alpha(q, 1, 1, \ell, 0, 0) \neq 0, \quad \alpha(q, 0, 0, \ell, 1, 1) \neq 0; \quad (3.21)$$

$$\alpha(q, 2, 0, \ell, 0, 0) = \alpha(q, 0, 2, \ell, 0, 0) = \alpha(q, 0, 0, \ell, 2, 0) = \alpha(q, 0, 0, 0, \ell, 2) = 0. \quad (3.22)$$

(eqs. (3.19)-(3.22) and all "initial data" given below are assumed to be satisfied for arbitrary non-negative integer q and ℓ).

Hypothesis III. leads to the relations which can be readily obtained from the comparison of eqs.(3.7), (3.9) with the expres-

sion for the linearized higher-spin curvatures of ref./ 2/ (see eq.(3.28) of ref./2/ where the background field $\hat{h}_{\alpha\beta}$ has to be replaced by $(\omega)_{\alpha\beta}$). These are of the form

$$\alpha(1, g, 0, 0, \ell, 1) \neq 0, \quad \alpha(0, g, 1, 1, \ell, 0) \neq 0; \quad (3.23)$$

$$\alpha(1, g, 0, 1, \ell, 0) = \alpha(0, g, 1, 0, \ell, 1) = 0; \quad (3.24)$$

$$\alpha(g, 1, 0, \ell, 0, 1) \neq 0, \quad \alpha(g, 0, 1, \ell, 1, 0) \neq 0; \quad (3.25)$$

$$\alpha(g, 1, 0, \ell, 1, 0) = \alpha(g, 0, 1, \ell, 0, 1) = 0. \quad (3.26)$$

Finally, as follows from eqs.(2.3), (2.4), Hypothesis IV. gives the conditions

$$\alpha(1, 1, g, 0, 0, \ell) \neq 0, \quad \alpha(0, 0, g, 1, 1, \ell) \neq 0. \quad (3.27)$$

The determination of the (general) solution of eqs.(3.10), (3.14), (3.15) at the conditions (3.19)-(3.27) is a rather cumbersome procedure which is described in detail in Appendix B.

In this Section, we only discuss its main idea, write the final expression for the function $\alpha(p, g, s, k, \ell, t)$ and verify that this expression actually provides the solution of the task.

First of all, we note that eqs.(3.10), (3.14), (3.15), (3.19)-(3.27) transform into an equivalent system of equations under the following "gauge transformations" of the unknown function

$$\alpha(p, g, s, k, \ell, t) \rightarrow \tilde{\alpha}(p, g, s, k, \ell, t) = \frac{\psi(p+g, k+\ell)}{\psi(p+s, k+t)\psi(g+s, \ell+t)} \alpha(p, g, s, k, \ell, t) \quad (3.28)$$

where $\psi(n, m)$ is an arbitrary function of non-negative integer arguments satisfying the condition

$$\psi(n, m) \neq 0 \quad \text{at all } n \geq 0, m \geq 0. \quad (3.29)$$

In other words, if some function α provides a solution of eqs. (3.10), (3.14), (3.15), (3.19)-(3.27) then the function $\tilde{\alpha}$ (3.28) is a solution of these equations too. The "gauge invariance" (3.28) is of simple nature. It expresses the freedom in choosing different Lorentz covariant bases of the superalgebra. The function $\psi(n, m)$ corresponds to the transition from the coordinates $\mathcal{Z}_{\alpha(n), \beta(m)}$ to the new coordinates

$$\tilde{\mathcal{Z}}_{\alpha(n), \beta(m)} = \psi(n, m) \mathcal{Z}_{\alpha(n), \beta(m)}. \quad (3.30)$$

Obviously, all solutions, related by the transformations (3.28), are equivalent. Consequently, we often call the transformations (3.28) "gauge transformations" (to avoid misunderstandings, we stress that the "gauge transformations" (3.28) have nothing to do with the true higher-spin gauge transformations).

Eqs. (3.14), (3.15) are non-linear. One can solve these equations because the number of equations strongly exceeds the number of unknown quantities. Indeed, the unknown function $\alpha(p, q, s, k, l, t)$ has six integer arguments, while eq. (3.15) contains twelve arbitrary integer parameters. On the one hand, this fact greatly simplifies constructing the solution of the task but, on the other hand, the existence of the solution, satisfying physical "initial conditions", is nontrivial in this situation and can be treated as some miracle.^{*}

^{*} Note that the fact, that the number of equations exceeds the number of unknown quantities, is only partially related to disregarding "Fierz identities" in the framework of Hypothesis V. Indeed, if one uses the "Fierz identities", the unknown

The main idea, which enables one to solve the task, is quite simple. Using transformations (3.28), one can achieve that at least some part of the values of the function α , which are different from zero due to eqs. (3.19), (3.21), (3.23), (3.25), (3.27), becomes equal to 1. Then one can find such values of the twelve integer parameters in eqs. (3.14), (3.15) that, use by of "zero" initial data (3.20), (3.22), (3.24), (3.25), eqs. (3.14), (3.15) become linear equations for the function $\alpha(p, q, s, k, l, t)$ whose arguments are related in some way one to another. Usually, it is not very difficult to solve these linear equations. As a result, the function $\alpha(p, q, s, k, l, t)$ can be determined in some broader region of the parameters p, q, s, k, l, t as compared to that of initial conditions (3.19) - (3.27). Then the same procedure can be applied to the "extended initial conditions" and so on until the function $\alpha(p, q, s, k, l, t)$ will be completely determined. Besides this linear procedure, the only non-linear element, used for the determination of the solution, is taking into account the simple facts that e.g. if $a \cdot b \neq 0$, then $a \neq 0$ and $b \neq 0$. Let us stress once again that in order to solve eqs. (3.14) by the method described above, it is very important to know that for some values of arguments the function $\alpha(p, q, s, k, l, t)$ is equal to zero and for some others it is non-vanishing.

The main result of this Section and of the whole paper is that, up to the freedom (3.28) in the choice of the basis, the system of equations (3.10), (3.14), (3.15), (3.19)-(3.27) has the following unique solution: $\alpha(p, q, s, k, l, t) = 1$ if there are at least two odd numbers among the numbers $(p+k)$, $(q+l)$, $(s+t)$, and $\alpha(p, q, s, k, l, t) = 0$ otherwise. This solution can be also written in the following two equivalent forms:

function will contain six independent arguments as before while Jacobi equations will contain ten independent parameters (this fact follows e.g. from the results of ref. /13/).

$$d(p, q, s, k, \ell, t) = \frac{1}{2} (1 - (-1)^{[(p+k)(q+\ell) + (p+k)(s+t) + (q+\ell)(s+t)]}) \quad (3.31)$$

$$d(p, q, s, k, \ell, t) = \delta((p+k)(q+\ell) + (p+k)(s+t) + (q+\ell)(s+t) + 1/2). \quad (3.32)$$

Let us verify that eq.(3.31) (equivalently, eq.(3.32)) provides the solution of the task. The antisymmetry condition (3.10) is satisfied trivially since, first, it can be seen from eq.(3.33) that $d(p, q, s, k, \ell, t) = -d(q, p, s, \ell, k, t)$ and, second, the sign factor in eq.(3.10) is canceled due to the delta-function in right hand side of eq.(3.32). It is also readily verified that the function d (3.32) satisfies "initial data" (3.19) - (3.27). Finally, direct insertion of eq.(3.31) into eq.(3.14) gives

$$\begin{aligned} A(p, q, s, k, \ell, t, u, v, z, f, g, h) = \\ = \frac{1}{4} \{ 1 - (-1)^{[(p+s+k+t)(q+v+\ell+g) + (p+q+k+\ell+g+v+s+t)(u+f)]} \\ - (-1)^{[(p+q+s+v+k+\ell+t+g)(z+h) + (p+q+k+\ell)(s+t+v+g)]} \\ + (-1)^{[(p+s+k+t)(q+v+\ell+g) + (p+s+q+v+k+t+\ell+g)(u+f+z+h) + (p+q+k+\ell)(s+v+t+g)]} \\ + (-1)^{[(p+s+u+k+t+f)(q+u+\ell+f+z+s+h+t) + s+u+t+f] } \times \\ \times [1 - (-1)^{[(q+u+z+s+\ell+f+h+t)(v+g) + (q+u+\ell+f)(z+s+h+t)]} \\ - (-1)^{[(q+z+u+s+\ell+h+t+f)(p+k) + (q+z+\ell+h)(u+s+t+f)]} \\ + (-1)^{[(q+u+z+s+\ell+f+h+t)(v+g+p+k) + (q+u+\ell+f)(z+s+h+t) + (q+z+\ell+h)(u+s+t+f)]} \\ + (-1)^{[(z+s+v+h+t+g)(p+s+k+t+g+v+\ell+g) + s+v+t+g] } \times \\ \times [1 - (-1)^{[(z+v+p+u+h+g+k+f)(s+t) + (z+g+h+v)(u+k+p+f)]} \\ - (-1)^{[(z+p+u+v+h+k+g+f)(q+\ell) + (z+p+h+k)(u+v+g+f)]} \\ + (-1)^{[(z+p+u+v+h+k+g+f)(q+\ell+s+t) + (z+p+h+k)(u+v+g+f) + (z+v+h+g)(p+u+k+f)]} \} \end{aligned} \quad (3.33)$$

It is straightforward to verify that the twelve terms in right hand side of eq.(3.33) cancel in pairs. Namely, the following pairs of terms cancel:

first and seventh, second and twelfth, third and ninth, fourth and sixth, fifth and eleventh, eighth and tenth.

According to eq.(3.32), the composition law (3.9) for the superalgebra $\text{shs}(1; \mathbb{C})$ reads

$$\begin{aligned} [\tilde{\mathcal{I}}_1, \tilde{\mathcal{I}}_2]_{a(n), \beta(m)} &= \sum_{p+q+s+k+t} i^{s+t-1} \frac{n!m!}{p!q!s!k!t!} \delta(n-p-q) \delta(m-k-t) \times \\ &\times \delta((p+k)(q+t) + (p+k)(s+t) + (q+t)(s+t) + 1/2) \times \\ &\times \tilde{\mathcal{I}}_{1a(p)} \gamma(s), \beta(k) \dot{\delta}(t) \tilde{\mathcal{I}}_{2a(q)} \gamma(s), \beta(t) \dot{\delta}(t). \end{aligned} \quad (3.34)$$

The gauge transformations (3.8) can be written in the form

$$\begin{aligned} \delta \omega_{\gamma, a(n), \beta(m)} &= D_{\gamma} \tilde{\mathcal{I}}_{a(n), \beta(m)} = \partial_{\gamma} \tilde{\mathcal{I}}_{a(n), \beta(m)} + \\ &+ \sum_{p+q+s+k+t} i^{s+t-1} \frac{n!m!}{p!q!s!k!t!} \delta(n-p-q) \delta(m-k-t) \times \\ &\times \delta((p+k)(q+t) + (p+k)(s+t) + (q+t)(s+t) + 1/2) \times \\ &\times \omega_{\gamma, a(p)} \gamma(s), \beta(k) \dot{\delta}(t) \tilde{\mathcal{I}}_{a(q)} \gamma(s), \beta(t) \dot{\delta}(t). \end{aligned} \quad (3.35)$$

The curvatures of $\text{shs}(1; \mathbb{C})$ are given by eq.(1.1) and satisfy the usual

Blanchi identities

$$\epsilon^{\nu\mu\rho\beta} D_{\rho} R_{\nu\mu, a(n), \beta(m)} = 0 \quad (3.36)$$

where the covariant derivative D_{γ} is defined by eq.(3.35). The gauge transformation (3.35) for the curvatures (1.1) reads

$$\begin{aligned} \delta R_{\nu\mu, a(n), \beta(m)} &= \sum_{p+q+s+k+t} i^{s+t-1} \frac{n!m!}{p!q!s!k!t!} \delta(n-p-q) \delta(m-k-t) \times \\ &\times \delta((p+k)(q+t) + (p+k)(s+t) + (q+t)(s+t) + 1/2) \times \\ &\times R_{\nu\mu, a(p)} \gamma(s), \beta(k) \dot{\delta}(t) \tilde{\mathcal{I}}_{a(q)} \gamma(s), \beta(t) \dot{\delta}(t). \end{aligned} \quad (3.37)$$

To conclude this Section, we stress once again that the superalgebra $\text{shs}(1; \mathbb{C})$ (3.34) provides the unique solution for complexifications of higher-spin superalgebras satisfying Hypotheses I.-V. The very fact of existence the consistent curvatures (1.1), which reduce at the linearized level to the

linearized curvatures proposed in ref./2/ for the description of free massless higher-spin fields, seems to be very promising for overcoming the problem of introducing a gravitational interaction of massless higher-spin fields. The fact, that the higher-spin superalgebra $shs(1;\mathbb{C})$ has simple and elegant structure, seems to be favourable too.

4. Hermiticity Conditions And P-Transformations

In order to speak about hermiticity of an action which can be constructed in terms of the gauge fields and curvatures originating from the higher-spin superalgebra $shs(1,\mathbb{C})$, one must impose some hermiticity conditions on the gauge fields. These hermiticity conditions should be in accordance with the structure of the curvatures in the sense that the conjugate curvatures have to be expressed in terms of the initial ones. In fact, this requirement is equivalent to the choice of some real form of the initial complex superalgebra (in the our case, $shs(1;\mathbb{C})$).

Let gauge fields ω_ν^A , related to some complex superalgebra $\mathfrak{g}^{\mathbb{C}}$, obey hermiticity conditions of the form

$$(\omega_\nu^A)^\dagger = C^A_B \omega_\nu^B, \quad (4.1)$$

where C^A_B is some numerical matrix. In addition, we require the following conditions to be satisfied:

$$(R_{\nu\mu}^A)^\dagger = C^A_B R_{\nu\mu}^B. \quad (4.2)$$

This leads to the following conditions on the matrix C^A_B :

$$([\omega_\nu, \omega_\mu]^A)^\dagger = C^A_B [\omega_\nu, \omega_\mu]^B \quad (4.3)$$

or equivalently,

$$\bar{U}_{BC}^A C^B_C C^C_F = -C^A_B U_{FC}^B. \quad (4.4)$$

Besides, the requirement that $((\omega_\nu^A)^\dagger)^\dagger = \omega_\nu^A$ gives

$$\bar{C}^A_B C^B_C = \delta^A_C. \quad (4.5)$$

Eqs.(4.4), (4.5) imply that the matrix $-C^A_B$ provides some involution μ of $\mathfrak{g}^{\mathbb{C}}$. In fact, the hermiticity condition (4.2) corresponds to consideration of some real form $\mathfrak{g}^R$ of $\mathfrak{g}^{\mathbb{C}}$ extracted by the conjugation

operation β related to the involution μ (the questions concerning real forms of complex Lie superalgebras, their relations to conjugations and involutions, as well as hermiticity conditions for the gauge fields are discussed in more detail e.g. in ref./14/).

Apart from the hermiticity properties of the gauge fields, we are interested in their properties with respect to spatial reflections. The gauge fields and curvatures are required to transform linearly under the $\mathcal{P}$ -reflections:

$$\mathcal{P}(\omega_{\hat{\nu}}) = \rho^A_B \omega_{\hat{\nu}}^B, \quad (4.6)$$

$$\mathcal{P}(R_{\hat{\nu}\hat{\mu}}) = \rho^A_B R_{\hat{\nu}\hat{\mu}}^B, \quad (4.7)$$

where ρ^A_B is some invertible matrix and, by definition, if an arbitrary vector (one-form) $A_{\hat{\nu}}$ has components (A_0, A_1, A_2, A_3) , then $A_{\hat{\nu}}$ has the components $(A_0, A_1, -A_2, A_3)$, i.e. the reflection under consideration corresponds to the change of the direction of the x_2 -axis. The relations (4.6), (4.7) are consistent if

$$[\mathcal{P}(\omega_{\hat{\nu}}), \mathcal{P}(\omega_{\hat{\mu}})]^A = \rho^A_B [\omega_{\hat{\nu}}, \omega_{\hat{\mu}}]^B, \quad (4.8)$$

or equivalently,

$$\rho^B_F \rho^C_G \omega_{\hat{\nu}}^A = \rho^A_B \omega_{\hat{\nu}}^B. \quad (4.9)$$

In other words, the matrix ρ^A_B provides some automorphism of the superalgebra $\mathfrak{g}^{\epsilon}$.

In the framework of $\text{shs}(1; \mathbb{C})$, we shall assume that the hermitian conjugation $\dagger$ and $\mathcal{X}_2$ -reflection $\mathcal{P}$ transform dotted spinorial indices into undotted ones and vice versa, i.e.

$$\mathcal{P}(\omega_{\hat{\nu}, \dot{\alpha}(n), \dot{\beta}(m)}) = \rho(n, m) \omega_{\hat{\nu}, \beta(m), \dot{\alpha}(n)}, \quad (4.10)$$

$$(\omega_{\hat{\nu}, \dot{\alpha}(n), \dot{\beta}(m)})^{\dagger} = C(n, m) (\omega_{\hat{\nu}, \alpha(m), \dot{\alpha}(n)})^{\dagger}, \quad (4.11)$$

where the nonvanishing coefficients $\rho(n, m)$ and $c(n, m)$ correspond to the matrices ρ^A_B and C^A_B and obey the conditions (4.9) and (4.4), (4.5), respectively. The structure of the relations (4.10) and (4.11) is familiar for

the formalism of two-component spinors. It originates from the invariance of the flat vierbein $\mathcal{H}_{\alpha\beta}^{\mu\nu} = (\mathbb{I}_{\alpha\beta}, \mathcal{S}_{\alpha\beta})$ ($\mathbb{I}$ is unit 2×2 matrix, $\mathcal{S}_{\alpha\beta}$ are Pauli matrices) under (i) hermitian conjugation and (ii) transposition and the change of the $\mathcal{S}_{\alpha\beta}$ sign.

Using eq. (3.34), one can readily see that condition (4.8) imposes the following restrictions on the coefficients $\rho(n, m)$:

$$\begin{aligned} & \delta(|(p+k)(q+\ell) + (p+k)(s+t) + (q+\ell)(s+t) + 1|_2) \times \\ & \times (\rho(p+q, k+\ell) - \rho(p+s, k+t) \rho(q+s, \ell+t)) = 0. \end{aligned} \quad (4.12)$$

Similarly, comparing eqs. (4.3) and (3.34), one can see that the function $c(n, m)$ should obey the same equation

$$\begin{aligned} & \delta(|(p+k)(q+\ell) + (p+k)(s+t) + (q+\ell)(s+t) + 1|_2) \times \\ & \times (c(p+q, k+\ell) - c(p+s, k+t) c(q+s, \ell+t)) = 0. \end{aligned} \quad (4.13)$$

(the additional sign factors due to the complex conjugation and reversing the order of product factors turn out to be compensated if one takes into account the presence of the δ -function).

Determination of the general solution of eq. (4.12) is not very difficult task since, like for the Jacobi identities in the form (3.14), (3.15), the number of equations greatly exceeds the number of unknown quantities. Let us describe the procedure of the determination of the general solution for eq. (4.12) in detail as this can be regarded as somewhat illustration for the analogous but more complicated procedure of the determination of the general solution for the Jacobi identities.

From eq. (4.12) taken at $q=s$, 1st, $|s+t|_2=1$, one finds

$$\rho(p+s, k+t) (1 - \rho(2s, 2t)) = 0 \quad \text{at } |s+t|_2 = 1. \quad (4.14)$$

Since $\rho(n, m) \neq 0$ for all $n \geq 0$, $m \geq 0$, it follows from eq. (4.14) that

$$\rho(2s, 2t) = 1 \quad \text{at } |s+t|_2 = 1. \quad (4.15)$$

On the other hand, it follows from eq. (4.12) at $p=q$, $k=1$, $|q+1|_2=1$ combined with eq. (4.15) that $[\rho(q+s, \ell+t)]^2 = 1$ at $|q+1|_2=1$ or equivalently,

$$[\rho(n, m)]^2 = 1 \quad \text{at } n \geq 0, m \geq 0, n+m > 0. \quad (4.16)$$

Using eq.(4.16), we set

$$\rho(1, 1) = (-1)^q \quad (q = 0 \text{ or } 1). \quad (4.17)$$

Then, setting in eq.(4.12) $p=t=1$, $s=k=0$, one finds

$$\rho(q+1, \ell) = (-1)^q \rho(q, \ell+1),$$

and it immediately follows that

$$\rho(q, \ell) = (-1)^{q \cdot \ell} \pi(q + \ell). \quad (4.18)$$

From eq.(4.12) taken at $k=l=t=0$, one finds that the function $\pi(n)$ (4.18)

Obeies the equation

$$\delta(p \cdot q + p \cdot s + q \cdot s + 1/2) (\pi(p+q) - \pi(p+s) \pi(q+s)) = 0, \quad (4.19)$$

and it follows from eqs.(4.15), (4.16) that

$$(\pi(n))^2 = 1 \quad \text{at } n > 0, \quad (4.20)$$

$$\pi(n) = 1 \quad \text{at } |n|_4 = 2. \quad (4.21)$$

Using eq.(4.20), one can set

$$\pi(1) = (-1)^1, \quad \pi(4) = (-1)^4. \quad (4.22)$$

Then, from eq.(4.19), taken at $p=3$, $s=1$, it follows that

$$\pi(q+3) = (-1)^q \pi(q+1) \quad \text{at all } q \geq 0. \quad (4.23)$$

In its turn, one finds from eq.(4.23) that $\pi(q+4) = \pi(q)$ for all $q > 0$, i.e.

$$\pi(q) = \tilde{\pi}(|q|_4) \quad \text{at } q > 0. \quad (4.24)$$

Comparison of eqs.(4.21) - (4.24) gives that

$$\pi(n) = (-1)^1 \quad \text{at } |n|_4 = 1, \quad (4.25)$$

$$\pi(n) = (-1)^4 \quad \text{at } |n|_4 = 0, n > 0, \quad (4.26)$$

$$\pi(n) = (-1)^{\lambda+\gamma} \quad \text{at } |n|_4 = 3. \quad (4.27)$$

It is readily seen that eqs. (4.21), (4.25)-(4.27) can be written in the following equivalent form

$$\pi(n) = (-1)^{\lambda \cdot n} i^{\gamma(n+1)n/2-2} \quad \text{at } n > 0. \quad (4.28)$$

Finally, using eq. (4.18), one finds

$$\rho(q, \ell) = (-1)^{q \cdot \lambda + \ell \cdot \lambda} i^{\gamma(q+\ell+1q+\ell)/2-2} \quad (4.29)$$

at $q \geq 0, \ell \geq 0, q+\ell > 0$

where $\lambda = 1q + \lambda/2$. It is straightforward to verify that the function $\rho(n, m)$ (4.29) satisfies the initial equation (4.12) and thus provides its general solution. This solution contains three arbitrary parameters λ , γ and γ taking on values 0 or 1. Note also that the coefficient $\rho(0, 0)$ remains arbitrary as it corresponds to the $\Theta(2)$ - factor in the decomposition $\text{shs}(1; \mathbb{C}) = \text{shs}'(1; \mathbb{C}) \oplus \Theta(2)$.

Since eq. (4.13) for the function $\tilde{c}(n, m)$ coincides with eq. (4.12), we also find that

$$c(q, \ell) = (-1)^{\nu q + \mu \ell} i^{\rho(q+\ell+1q+\ell)/2-2} \quad \text{at } q \geq 0, \ell \geq 0, q+\ell > 0, \quad (4.30)$$

where the arbitrary parameters ν, μ and ρ take on values 0 or 1. However, as follows from eq. (4.11), the condition (4.5) leads in this case to the additional restriction

$$c(q, \ell) \cdot \bar{c}(\ell, q) = 1. \quad (4.31)$$

Taking into account eq. (4.30), we see that eq. (4.31) requires that

$$\nu = \mu. \quad (4.32)$$

For $\nu + \mu = 1$, the solutions (4.30) for the function $c(n, m)$ do not provide involutions of $\text{shs}(1; \mathbb{C})$. Nevertheless, following to the method applied in refs. /15, 16/ to the superalgebra $\text{osp}(1, 4; \mathbb{C})$, one can use these solutions in order to construct hermiticity conditions similar to those of eq. (4.11) but containing the Klein operator in the fermion sector. In fact, these modified

hermiticity conditions will correspond to the transition from ordinary Lie superalgebras to (generalized) $Z_2 \times Z_2$ -graded Lie superalgebras as demonstrated /14,15/ for the Lie superalgebras $osp(N,4;\mathbb{C})$ related to (extended) supergravity. Physically, the case $\nu = \mu$ corresponds to negative cosmological constant (anti-de Sitter case), while the case $\nu + \mu = 1$ corresponds to positive cosmological constant (de Sitter case). Unfortunately, as demonstrated /14,16/ for supergravity, involutions of $Z_2 \times Z_2$ -graded Lie superalgebras corresponding to the case $\nu + \mu = 1$ cannot be realized, as hermitian conjugation in a positive definite Hilbert space and thus lead to ghosts. Therefore, in what follows we confine ourselves to consideration of the anti-de Sitter case $\nu = \mu$ only.

Arbitrariness in the parameters ν, μ and ρ can be restricted even further by noticing that using linear changes of variables which do not modify the structure of the curvatures (i.e. automorphisms), one can always find such fields $\tilde{\omega}(n,m)$ which possess the vanishing parameter $\tilde{\nu}$. Indeed, it is easily seen that the fields $\tilde{\omega}$ are

$$\tilde{\omega}(n,m) = (-1)^{\nu \cdot n} \omega(n,m). \quad (4.33)$$

Thus, taking into account eq.(4.32), we set in what follows $\nu = \mu = 0$.

The resulting function $c(n,m)$ is

$$c(n,m) = i^{\rho(n+m+1)(n+m-2)} \equiv (-1)^{\frac{1}{2}\rho(n+m-1)(n+m-2)}, \quad \rho = 0 \text{ or } 1. \quad (4.34)$$

The remaining arbitrariness in the parameter ρ corresponds to the two inequivalent real forms $shs_{\rho}(1)$ of $shs(1;\mathbb{C})$ (for convenience, we shall assume that eq.(4.34) takes place for all $n+m \geq 0$ i.e., that eq.(1.4) is true not only for $shs'_{\rho}(1)$ but also for $shs_{\rho}(1)$ including the $\mathfrak{o}(2)$ -factor).

If one requires that the $\mathcal{P}$ -transformed fields $\mathcal{P}(\omega_{\nu,d(n),\beta(m)})$ should satisfy the same hermiticity conditions as the untransformed ones (i.e. that the operation $\mathcal{P}$ is an automorphism of $shs_{\rho}(1)$), this leads to the fol-

lowering restriction on the parameters in the $\mathcal{P}$ -transformation law (4.10), (4.29):

$$\mathcal{K} = \lambda. \quad (4.35)$$

Indeed, it is easily seen that the condition (4.35) provides commutativity of the $\mathcal{P}$ -automorphism (4.10), (4.29) with the involution given by eqs. (4.11), (4.34).

Note that in order to construct the $\mathcal{P}$ -operation corresponding to the reflection of all spatial axes, it is sufficient to combine the $\mathcal{P}$ -operation introduced above with the internal automorphism of $\text{shs}'_\rho(1)$, corresponding to the rotation around the $\mathcal{X}_2$ -axis. This leads to the familiar expressions containing symplectic forms $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ explicitly.

5. Subalgebras Of $\text{shs}(1;\mathcal{G})$ And $\text{shs}'_\rho(1)$.

In this Section, we consider subalgebras of $\text{shs}(1;\mathcal{G})$ and $\text{shs}(1)$, confining ourselves to those of them which themselves contain Lorentz algebra as subalgebra. In particular, it will be shown that the maximum finite-dimensional subalgebra of $\text{shs}'_\rho(1)$ coincides with the $N=1$ de Sitter superalgebra $\text{osp}(1,4)$. As for infinite-dimensional subalgebras, our interest to them is twofold. First, those infinite-dimensional subalgebras which contain $\text{sp}(4)$, the algebra generating the gravitational gauge fields, can themselves pretend to the role of higher-spin (super)algebras which do not satisfy Hypothesis II. Second, one can hope that one or another infinite-dimensional subalgebra, which contains Lorentz subalgebra $\mathcal{O}_4(3,1)$ but does not contain de Sitter subalgebra $\text{sp}_{\mathcal{G}}(4)$, may play the role of an infinite-dimensional counterpart of the local Lorentz symmetry in gravity, the symmetry which remains undeformed after transition to local symmetries. It is clear that determination of possible candidates to the role of infinite-

dimensional counterparts of Lorentz symmetry may be very useful in constructing a higher-spin field dynamics.

We begin with infinite-dimensional subalgebras. Let us decompose the full linear space L of superalgebra $shs(1, \epsilon)$ into the direct sum of subspaces L_{AB} ,

$$L = \sum_{AB} \oplus L_{AB} \quad (5.1)$$

Here the indices A, B are defined modulo four ($A, B = 0, 1, 2, 3$) and satisfy the additional restriction

$$|A+B|_2 = 0, \quad (5.2)$$

while the basis elements of the subspaces L_{AB} coincide with the generators $T_d(n), \beta(m)$ with the parameters n and m satisfying conditions

$$|n+m|_4 = A, \quad (5.3)$$

$$|n-m|_4 = B \quad (5.4)$$

(in fact, condition (5.2) is the consequence of eqs. (5.3), (5.4)).

Now let us note that eq. (3.34), defining the superalgebra $shs(1; \epsilon)$, can be rewritten in the following symbolic form

$$\begin{aligned} [\tilde{J}_1, \tilde{J}_2](n, m) = & \sum_{n_1 m_1 n_2 m_2 s t} \beta(n_1, m_1, n_2, m_2, s, t) \delta(n - n_1 - n_2 - 2s) \times \\ & \times \delta(m - m_1 - m_2 - 2t) \delta(|(n_1 + m_1)(n_2 + m_2) + s + t + 1|_2) \times \\ & \times \tilde{J}_1(n_1, m_1) \tilde{J}_2(n_2, m_2), \end{aligned} \quad (5.5)$$

where $\beta(\dots)$ are some coefficients, $\tilde{J}(n, m)$ denote multi-spinors $\tilde{J}_d(n), \beta(m)$, and parameters s and t describe the numbers of contracted dotted and undotted indices, respectively. Using simple relations of the type $|2n|_4 = 2|n|_2$ and $|2K|_2 = 0$ and also restrictions due to the δ -functions in right hand side of eq. (5.5), one can readily see that the following relations are true

$$\{L_{A_1 B_1}, L_{A_2 B_2}\} \subset L_{|A_1 + A_2 + 2A_1 \cdot A_2 + 2|_4 \mid B_1 + B_2 + 2B_1 \cdot B_2 + 2|_4} \quad (5.6)$$

In particular, we find that

$$\{L_{2B_1}, L_{AB_2}\} \subset L_{|A \mid B_1 + B_2 + 2B_1 \cdot B_2 + 2|_4}, \quad (5.7)$$

$$[L_{A_1 B}, L_{A_2 B}] \subset L_{1/A_1 + A_2 + 2A_1 A_2 + 2I_4} B, \quad (5.8)$$

and

$$[L_{AB_1}, L_{AB_2}] \subset L_{2/B_1 + B_2 + 2B_1 B_2 + 2I_4}, \quad (5.9)$$

$$[L_{A_1 B}, L_{A_2 B}] \subset L_{1/A_1 + A_2 + 2A_1 A_2 + 2I_4} 2, \quad (5.10)$$

It follows from eqs. (5.7)-(5.10) that the following subspaces $\mathcal{G}_A$ and $\mathcal{H}_A$ form subalgebras of $\text{shs}(1; \mathbb{C})$:

$$\mathcal{G}_2 = L_{22} \oplus L_{20}, \quad \mathcal{G}_A = \mathcal{G}_2 \oplus L_{AA} \oplus L_{1A + 2I_4} \quad \text{at } A \neq 2; \quad (5.11)$$

$$\mathcal{H}_2 = L_{22} \oplus L_{02}, \quad \mathcal{H}_A = \mathcal{H}_2 \oplus L_{AA} \oplus L_{1A + 2I_4} \quad \text{at } A \neq 2. \quad (5.12)$$

Obviously, the subspaces $f_{AB} = \mathcal{G}_A \cap \mathcal{H}_B$ also form subalgebras of $\text{shs}(1; \mathbb{C})$. However, only the following subalgebras f_{AB} appear to be really new:

$$f_{22} = L_{22}, \quad (5.13)$$

$$f_{AB} = L_{22} \oplus L_{AB} \quad \text{at } 1/A_2 = 1/B_2 = 1. \quad (5.14)$$

All other subalgebras f_{AB} turn out to be isomorphic to the subalgebras (5.11)-(5.14) as follows:

$$f_{21} = f_{23} = f_{12} = f_{32} = f_{22}; \quad (5.15)$$

$$f_{02} = f_{01} = f_{03} = \mathcal{H}_2, \quad f_{20} = f_{10} = f_{30} = \mathcal{G}_2; \quad (5.16)$$

$$f_{00} = \mathcal{G}_0 = \mathcal{H}_0 \quad (5.17)$$

(we also include into eq. (5.17) the isomorphism $\mathcal{G}_0 = \mathcal{H}_0$).

From eq. (5.6), it follows that subspaces

$$n = \sum_{A=0}^3 \oplus L_{AA}, \quad m = \sum_{A=0}^3 \oplus L_{A1-A1_4}; \quad (5.18)$$

$$K = L_{22} \oplus L_{00} \quad (5.19)$$

($K = n \cap m$) form subalgebras of

of n , m and K with the subalgebras $shs(1, \mathbb{C})$ as well. It can be readily seen that intersections constructed above do not lead to new subalgebras.

In fact, we are only interested in those subalgebras of $shs(1, \mathbb{C})$ which are invariant under involutions transforming dotted spinorial indices into undotted ones and vice versa, and thus, generate nontrivial subalgebras of $shs_p(1)$. It follows from eqs. (5.3), (5.4) that such involutions transform L_{AB} into $L_{A\dot{B}} - B_{\dot{A}}/4$. Using this fact, one can see that among various independent subalgebras of $shs(1, \mathbb{C})$ described above, only the subalgebras $\mathfrak{g}_A$ (at all A), $\mathfrak{h}_2, K$ and $\mathfrak{f}_{22}$ turn out to be invariant under the involutions under consideration. Simultaneously, these subalgebras are invariant under the P-automorphism constructed in previous Section.

Let us consider these subalgebras in more detail. Only two of them are Lie superalgebras possessing nontrivial odd (fermion) sector. These are $\mathfrak{g}_1$ and $\mathfrak{g}_3$.

The superalgebra $\mathfrak{g}_1$ generates nonvanishing gauge fields $\mathcal{W}(n, m)$ with $|n+m|_4 = 1$ or $|n+m|_4 = 2$. In other words, $\mathfrak{g}_1$ generates massless fields with the spins $s = 2k+2$ and $s = 2K + \frac{3}{2}$ ($k=0, 1, 2, \dots, \infty$) (let us remind that $s = \frac{1}{2}(n+m)+1$). Quite similarly, the superalgebra $\mathfrak{g}_3$ generates spins $s = 2k+2$ and $s = 2K + \frac{5}{2}$ ($k=0, 1, 2, \dots$). Both $\mathfrak{g}_1$ and $\mathfrak{g}_3$ provide the examples of nontrivial superalgebras which do not satisfy Hypothesis II. Generally, one cannot exclude the possibility that these subalgebras can be utilized for building nontrivial higher-spin theories. Taking this into account, we denote $\mathfrak{g}_1$ and $\mathfrak{g}_3$ as $shs(1|0)$ and $shs(1|1)$, respectively (in fact, we assume that $shs(1|\alpha)$ at $\alpha=0$ or 1 are real subalgebras of $shs_p(1)$ but, omit the index p as the difference between the cases $p=0$ and $p=1$ becomes irrelevant for these subalgebras).

The crucial difference between $shs(1|0)$ and $shs(1|1)$ consists in the following. The maximum finite-dimensional subalgebra

of $\text{shs}(1|0)$ is the de Sitter superalgebra $\text{osp}_{d|2}(1,4)$, while the maximum finite-dimensional subalgebra of $\text{shs}(1|1)$ coincides with the de Sitter Lie algebra $\text{Sp}_{d|2}(4)$. Therefore, one can speculate that applying some truncation procedure to a hypothetical higher-spin theory based on $\text{shs}(1|0)$, it will be possible to obtain usual $N=1$ supergravity. As for $\text{shs}(1|1)$, analogous truncation can only lead to pure gravity but not, to, e.g., a theory of spins 2 and $5/2$. In fact, this is in agreement with the results of ref. /17/ where inconsistency of "hypergravity" (a theory of massless spins 2 and $5/2$) was shown.

Let us now discuss the remaining subalgebras $\mathfrak{g}_2$, $\mathfrak{g}_0$, $\mathfrak{h}_2$, $\mathfrak{k}$ and $\mathfrak{f}_{22}$ which all turn out to be Lie algebras. First of all, it is readily seen that $\mathfrak{g}_0$ coincides with the even subalgebra of $\text{shs}(1, \mathbb{C})$, while $\mathfrak{g}_2$ is the even subalgebra of any of the superalgebras $\mathfrak{g}_1$ or $\mathfrak{g}_3$. These Lie algebras may correspond to boson sectors of theories based on the superalgebras $\text{shs}_{d|2}(1)$ and $\text{shs}(1|\mathbb{C})$.

The algebras $\mathfrak{h}_2$, $\mathfrak{f}_{22}$, and $\mathfrak{K}$ generate the nonvanishing fields $\mathcal{W}(n, m)$ with $|n-m|_4=2$, $|n-m|_4=|n+m|_4=2$, and $|n|_2=|m|_2=0$, respectively. Obviously, $\mathfrak{f}_{22} \subset \mathfrak{h}_2$ and $\mathfrak{f}_{22} \subset \mathfrak{k}$. For all these subalgebras, the related fields $\mathcal{W}(n, m)$ do not contain the vierbein $\mathcal{W}(1, 1)$, i.e. $\mathfrak{h}_2$, $\mathfrak{k}$ and $\mathfrak{f}_{22}$ do not contain $\text{sp}_{d|2}(4)$. Therefore, these subalgebras themselves cannot be utilized for building theories including gravity. On the other hand, $\mathfrak{h}_2$, $\mathfrak{k}$ and $\mathfrak{f}_{22}$ contain the Lorentz subalgebra $\mathcal{O}_d(3, 1)$ with the gauge fields $\mathcal{W}(2, 0)$ and $\mathcal{W}(0, 2)$. Possibly, one of these subalgebras may play in hypothetical higher-spin theories the role similar to that of ordinary Lorentz subalgebra in gravity and supergravity (see the beginning of this Section). Note that the algebra $\mathfrak{f}_{22}$ belongs to all infinite-dimensional subalgebras of $\text{shs}(1, \mathbb{C})$, considered above. Thus, $\mathfrak{f}_{22}$ seems to be more pro-

bable candidate to the role of an infinite-dimensional counterpart of the Lorentz algebra than h_2 and k . One more candidate to this role belongs to another set of infinite-dimensional subalgebras of $shs(1; \mathbb{C})$ being now in order.

Obviously, the basis elements, possessing only dotted or only undotted spinorial indices, form infinite-dimensional subalgebras of $shs(1; \mathbb{C})$. In what follows, the subalgebra with the basis elements $T_a(n), \dot{A}(0)$ will be denoted as u , while the subalgebra with the basis elements $T_a(n), \dot{B}(m)$ will be denoted as v . Intersections of u and v with the subalgebras $\mathcal{G}_A, \mathcal{H}_A, \mathcal{F}_{AB}$, n, m and k also form subalgebras of $shs(1; \mathbb{C})$. Taking into account the fact that u and v have nontrivial intersections only with subspaces L_{1AA} and L_{1A-1A} , respectively, it is easily verified that among subalgebras arising as a result, only the following ones appear to be really new:

$$\mathcal{S}_A = u \cap \mathcal{G}_A, \quad \mathcal{t}_A = v \cap \mathcal{G}_A. \quad (5.20)$$

Subalgebras $\mathcal{S}_1, \mathcal{S}_3$ and $\mathcal{t}_1, \mathcal{t}_3$ turn out to be Lie superalgebras. Their spin content coincides with that of $\mathcal{G}_1$ and $\mathcal{G}_3$ but corresponding gauge fields are $\mathcal{W}(n, 0)$ or $\mathcal{W}(0, n)$. The subalgebras $\mathcal{S}_0$ and $\mathcal{t}_0$ coincide with the even subalgebras of u and v , respectively, while $\mathcal{S}_2$ and $\mathcal{t}_2$ are the even subalgebras of $\mathcal{S}_1$ and $\mathcal{t}_1$ or equivalently, $\mathcal{S}_3$ and $\mathcal{t}_3$.

Involutions, transforming dotted indices into undotted ones, transform subalgebras u and $\mathcal{S}_A$ into v and $\mathcal{t}_A$, respectively, and vice versa. Using eq. (5.5), one can see, however, that

$$[\mathcal{S}_A, \mathcal{t}_B] = 0 \quad \text{at} \quad 1A \cdot B|_2 = 0, \quad (5.21)$$

i.e. $\mathcal{S}_0$ and $\mathcal{S}_2$ commute with $\mathcal{t}_B$, while $\mathcal{t}_0$ and $\mathcal{t}_2$ commute with $\mathcal{S}_A$. As a result, the subspaces

$$\mathcal{U}_{AB} = \mathcal{S}_A \oplus \mathcal{t}_B \quad \text{at} \quad 1A \cdot B|_2 = 0 \quad (5.22)$$

will also form subalgebras of $shs(1; \mathbb{C})$. It is important that the algebras $\mathcal{U}_{00}$ and $\mathcal{U}_{22}$ are invariant under the involutions under

consideration. Therefore, corresponding real forms of $\mathcal{U}_{00}$ and $\mathcal{U}_{22}$ form subalgebras of $\mathfrak{shs}_p(1)$.

The algebra $\mathcal{U}_{00}$ generates the fields $\mathcal{W}(n,0)$ and $\mathcal{W}(0,n)$ with $|n|_2=0$, while the algebra $\mathcal{U}_{22}$ generates the fields $\mathcal{W}(n,0)$ and $\mathcal{W}(0,n)$ with $|n|_4 = 2$. The algebra $\mathcal{U}_{22}$ seems to be most interesting as it belongs to all infinite-dimensional subalgebras of $\mathfrak{shs}_p(1)$ constructed above. In other words, $\mathcal{U}_{22}$ is the minimum infinite-dimensional subalgebra of $\mathfrak{shs}_p(1)$, containing the Lorentz subalgebra with the fields $\mathcal{W}(2,0)$ and $\mathcal{W}(0,2)$. Thus, if one supposes that a higher-spin theory possesses an infinite-dimensional symmetry analogous to the Lorentz symmetry in (super)gravity, then this symmetry should contain the symmetry originating from r_{22} .

The maximum finite-dimensional subalgebra of $\mathfrak{shs}(1, \mathbb{C})$ is that with the basis elements $T_{\alpha}(n, \beta(m))$ at $0 \leq n+m \leq 2$. Indeed, one can readily see by use of (3.34) that, first, for any given $\mathfrak{F}_1(n, m)$ and $\mathfrak{F}_2(n', m')$ with $n+m \leq 2$, $n'+m' \leq 2$ their commutator $[\mathfrak{F}_1, \mathfrak{F}_2](n'', m'')$ satisfies the same restriction $n''+m'' \leq 2$. and, second, successive products ("commutators") of an arbitrary element with the parameters $\mathfrak{F}(n_0, m_0)$ at $n_0+m_0 > 2$ generate infinite number of new parameters $\mathfrak{F}(n, m)$ with $n+m > 2$ (more precisely, an arbitrary basis element $T_{\alpha}(n, \beta(m))$ taken at $n+m > 2$ together with all Lorentz elements $T_{\alpha}(2, \beta(0))$ and $T_{\alpha}(0, \beta(2))$ generate some infinite-dimensional subalgebra of $\mathfrak{shs}(1; \mathbb{C})$).

The basis element $T_{\alpha}(0, \beta(0))$ corresponds to $\mathfrak{o}(2)$ factor in the decomposition $\mathfrak{shs}(1; \mathbb{C}) = \mathfrak{shs}'(1, \mathbb{C}) \oplus \mathfrak{o}(2)$. For the parameters $\mathfrak{F}(n, m)$ with $0 < n+m \leq 2$, we introduce the following designations:

$$\bar{\mathcal{F}}_{\alpha(2)} = \bar{\mathcal{F}}_{\alpha(2), \dot{\alpha}(0)}, \quad \bar{\mathcal{F}}_{\dot{\alpha}(2)} = \bar{\mathcal{F}}_{\alpha(0), \dot{\alpha}(2)}, \quad \bar{\mathcal{F}}_{\alpha\dot{\alpha}} = \bar{\mathcal{F}}_{\alpha(1), \dot{\alpha}(1)}, \quad (5.23)$$

$$\bar{\mathcal{F}}_{\alpha} = \bar{\mathcal{F}}_{\alpha(1), \dot{\alpha}(0)}, \quad \bar{\mathcal{F}}_{\dot{\alpha}} = \bar{\mathcal{F}}_{\alpha(0), \dot{\alpha}(1)}. \quad (5.24)$$

Then, one finds from eq.(3.34) that

$$\frac{1}{2} [\bar{\mathcal{F}}_1, \bar{\mathcal{F}}_2]_{\alpha(2)} = \bar{\mathcal{F}}_{1\alpha\gamma} \bar{\mathcal{F}}_{2\dot{\alpha}}^{\gamma} + \bar{\mathcal{F}}_{1\alpha\dot{\beta}} \bar{\mathcal{F}}_{2\dot{\alpha}}^{\dot{\beta}} + i \bar{\mathcal{F}}_{1\alpha} \bar{\mathcal{F}}_{2\dot{\alpha}}, \quad (5.25)$$

$$\frac{1}{2} [\bar{\mathcal{F}}_1, \bar{\mathcal{F}}_2]_{\dot{\alpha}(2)} = \bar{\mathcal{F}}_{1\dot{\alpha}\beta} \bar{\mathcal{F}}_{2\dot{\alpha}}^{\beta} + \bar{\mathcal{F}}_{1\dot{\alpha}\beta} \bar{\mathcal{F}}_{2\dot{\alpha}}^{\beta} + i \bar{\mathcal{F}}_{1\dot{\alpha}} \bar{\mathcal{F}}_{2\dot{\alpha}}, \quad (5.26)$$

$$[\bar{\mathcal{F}}_1, \bar{\mathcal{F}}_2]_{\alpha\dot{\alpha}} = (\bar{\mathcal{F}}_{1\alpha\gamma} \bar{\mathcal{F}}_{2\dot{\alpha}}^{\gamma} + \bar{\mathcal{F}}_{1\dot{\alpha}\beta} \bar{\mathcal{F}}_{2\alpha}^{\beta} + i \bar{\mathcal{F}}_{1\alpha} \bar{\mathcal{F}}_{2\dot{\alpha}}) - (1 \leftrightarrow 2), \quad (5.28)$$

$$[\bar{\mathcal{F}}_1, \bar{\mathcal{F}}_2]_{\alpha} = (\bar{\mathcal{F}}_{1\alpha\gamma} \bar{\mathcal{F}}_2^{\gamma} + \bar{\mathcal{F}}_{1\alpha\dot{\beta}} \bar{\mathcal{F}}_2^{\dot{\beta}}) - (1 \leftrightarrow 2), \quad (5.28)$$

$$[\bar{\mathcal{F}}_1, \bar{\mathcal{F}}_2]_{\dot{\alpha}} = (\bar{\mathcal{F}}_{1\dot{\alpha}\beta} \bar{\mathcal{F}}_2^{\beta} + \bar{\mathcal{F}}_{1\dot{\alpha}\beta} \bar{\mathcal{F}}_2^{\beta}) - (1 \leftrightarrow 2). \quad (5.29)$$

The real Lie superalgebra, defined by eqs.(5.25)-(5.29),

coincides with $\text{osp}(1, 4)$, the $N=1$ supersymmetric extension of the anti-de Sitter Lie algebra, $\text{sp}(4)$, described by the parameters

$\bar{\mathcal{F}}_{\alpha(2)}, \bar{\mathcal{F}}_{\dot{\alpha}(2)}, \bar{\mathcal{F}}_{\alpha\dot{\alpha}}$ (the both values of the parameter ρ in the involution (1.4) are equivalent for the subalgebras $\text{osp}(1, 4)$ of $\text{shs}_{\rho}(1)$).

Note also that the relation to the tensor-spinor formalism can be

established as follows: $\bar{\mathcal{F}}_{\alpha(2)}, \bar{\mathcal{F}}_{\dot{\alpha}(2)} \rightarrow \bar{\mathcal{F}}_{\alpha\beta}, \quad \bar{\mathcal{F}}_{\alpha\dot{\alpha}} \rightarrow \bar{\mathcal{F}}_{\alpha},$

$\bar{\mathcal{F}}_{\alpha}, \bar{\mathcal{F}}_{\dot{\alpha}} \rightarrow \bar{\mathcal{F}}(\alpha) \quad (\bar{\mathcal{F}}(\alpha) \text{ is the Majorana bispinor}).$

The fact, that $\text{shs}_{\rho}(1)$ contains anti-de Sitter algebra $\text{sp}(4)$,

is the consequence of Hypothesis III. However, we have obtained

that $\text{shs}_{\rho}(1)$ contains $\text{sp}(4)$ together with its $N=1$ supersymmetric

extension, $\text{osp}(1, 4)$, which turns out to be the maximum finite-dimen-

sional subalgebra of $\text{shs}_{\rho}(1)$. Let us stress once again that this

fact enables one to hope that supergravity can be naturally inclu-

ded in the (yet hypothetical) higher-spin theory based on $\text{shs}_{\rho}(1)$.

6. Contractions.

Generally, contraction of an algebra g is understood as some limiting process which enables one to construct from g some algebra g' non-isomorphic to g .

The well-known example of a contraction is provided by the limiting process which enables one to obtain the Gelfand-Likhtman superalgebra corresponding to the flat space from the anti-de Sitter superalgebra $osp(1,4)$. This contraction is realized as follows. Instead of the parameters $\mathcal{E}$ in eqs. (5.25)-(5.29), we introduce the following new parameters $\mathcal{E}$:

$$\mathcal{E}_\alpha(x) = \mathcal{E}_\alpha(x), \quad \overline{\mathcal{E}}_{\dot{\beta}}(x) = \overline{\mathcal{E}}_{\dot{\beta}}(x), \quad \mathcal{E}_{\alpha\dot{\beta}} = \lambda^{-1} \mathcal{E}_{\alpha\dot{\beta}}; \quad (6.1)$$

$$\mathcal{E}_\alpha = \lambda^{-\frac{1}{2}} \mathcal{E}_\alpha, \quad \overline{\mathcal{E}}_{\dot{\beta}} = \lambda^{-\frac{1}{2}} \overline{\mathcal{E}}_{\dot{\beta}}. \quad (6.2)$$

Here λ is arbitrary real parameter, the contraction parameter. In fact, λ coincides with the inverse radius of the related anti-de Sitter space. In gravity and N=1 supergravity, the quantity $(-\lambda^2)$ coincides (up to a positive numerical factor) with the cosmological constant.

In terms of the parameters $\mathcal{E}$, eqs. (5.25)-(5.29) read

$$\frac{1}{2} [\mathcal{E}_1, \mathcal{E}_2]_{\alpha(x)} = \mathcal{E}_{1\alpha\dot{\gamma}} \mathcal{E}_{2\dot{\gamma}}^\beta + \lambda^2 \mathcal{E}_{1\alpha\dot{\beta}} \mathcal{E}_{2\dot{\beta}}^\beta - i\lambda \mathcal{E}_{1\alpha} \mathcal{E}_{2\dot{\beta}}, \quad (6.3)$$

$$\frac{1}{2} \overline{[\mathcal{E}_1, \mathcal{E}_2]}_{\dot{\beta}(x)} = \overline{\mathcal{E}}_{1\dot{\beta}\delta} \overline{\mathcal{E}}_{2\delta}^\beta + \lambda^2 \mathcal{E}_{1\gamma\dot{\beta}} \mathcal{E}_{2\dot{\beta}}^\gamma - i\lambda \overline{\mathcal{E}}_{1\dot{\beta}} \overline{\mathcal{E}}_{2\delta}^\beta, \quad (6.4)$$

$$[\mathcal{E}_1, \mathcal{E}_2]_{\alpha\dot{\beta}} = (\mathcal{E}_{1\alpha\dot{\gamma}} \mathcal{E}_{2\dot{\gamma}}^\beta + \overline{\mathcal{E}}_{1\dot{\beta}\delta} \mathcal{E}_{2\delta}^\beta + i\mathcal{E}_{1\alpha} \overline{\mathcal{E}}_{2\dot{\beta}}) - (1 \leftrightarrow 2), \quad (6.5)$$

$$[\mathcal{E}_1, \mathcal{E}_2]_\alpha = (\mathcal{E}_{1\alpha\dot{\gamma}} \mathcal{E}_{2\dot{\gamma}}^\beta + \lambda \mathcal{E}_{1\alpha\dot{\beta}} \overline{\mathcal{E}}_{2\dot{\beta}}^\beta) - (1 \leftrightarrow 2), \quad (6.6)$$

$$\overline{[\mathcal{E}_1, \mathcal{E}_2]}_{\dot{\beta}} = (\overline{\mathcal{E}}_{1\dot{\beta}\delta} \overline{\mathcal{E}}_{2\delta}^\beta + \lambda \mathcal{E}_{1\gamma\dot{\beta}} \mathcal{E}_{2\dot{\beta}}^\gamma) - (1 \leftrightarrow 2). \quad (6.7)$$

For any $\lambda \neq 0$ or ∞ , the superalgebra given by eqs. (6.3)-(6.7)

is isomorphic to that given by eqs. (5.25)-(5.29) as eqs. (6.1), (6.2) are invertible. However in the limit $\lambda = 0$, eqs. (6.3)-(6.7) provide the new superalgebra, the Golfand-Likhtman superalgebra being the $N=1$ supersymmetric extension of the Poincare algebra. Let us stress that the possibility for the transition to the limiting point $\lambda = 0$ is due to the fact that right hand sides of eqs. (6.3)-(6.7) contain only non-negative powers of λ . Generally, this condition can be not satisfied, i.e. not any parameter can serve as a contraction parameter.

Our interest to contractions of $shs_p(1)$ is twofold. First, some contractions of $shs_p(1)$ generate superalgebras which satisfy Hypotheses I.-III., V. but do not satisfy Hypothesis IV. that seems to be of a technical nature, and is less grounded as compared to others. Second, flat contractions of $shs_p(1)$ are of great physical importance. These are contractions in which the anti-de Sitter subalgebra $sp_{d,2}(4)$ becomes the Poincare algebra p_4 . It is clear that the question whether an appropriate flat contraction of $shs_p(1)$ exists is very significant for possible physical applications.

Before consideration of contractions of $shs_p(1)$, we make two general remarks.

The first one is that any subalgebra h of an arbitrary Lie superalgebra g enables one to construct the contraction of g as follows. Let the parameters $\tilde{f}'$ correspond to the subalgebra h while the parameters $\tilde{f}''$ are those of g/h (i.e. the parameters $\tilde{f}''$ correspond to basis elements of g which do not belong to h). Therefore, relations of the type (3.4) for g are of the following structure

$$[\tilde{f}_1, \tilde{f}_2]' = \tilde{f}_1' \tilde{f}_2' + \tilde{f}_1' \tilde{f}_2'' + \tilde{f}_1'' \tilde{f}_2' + \tilde{f}_2'' \tilde{f}_2'', \quad (6.8)$$

$$[\tilde{f}_1, \tilde{f}_2]'' = \tilde{f}_1' \tilde{f}_2'' + \tilde{f}_1'' \tilde{f}_2' + \tilde{f}_1'' \tilde{f}_2''. \quad (6.9)$$

Let us redefine the parameters as follows: $\mathcal{E}' = \mathcal{F}'$, $\mathcal{E}'' = \lambda^{-1} \mathcal{F}''$ where λ is the contraction parameter. It is seen that after transition to the new parameters $\mathcal{E}$ in eqs. (6.8), (6.9), their right hand sides will only contain non-negative powers of λ , and for $\lambda = 0$, one finds

$$[\mathcal{E}_1, \mathcal{E}_2]' = \mathcal{E}_1' \mathcal{E}_2', \quad (6.10)$$

$$[\mathcal{E}_1, \mathcal{E}_2]'' = \mathcal{E}_1' \mathcal{E}_2'' + \mathcal{E}_1'' \mathcal{E}_2' \quad (6.11)$$

where right hand sides of eqs. (6.10), (6.11) have the same structure as the corresponding terms in right hand sides of eqs. (6.8), (6.9). We denote the superalgebra which appears as a result as $\mathcal{G}_h$. Obviously, $\mathcal{G}_h = \mathcal{G}/\hbar + \hbar$ where $\mathcal{G}/\hbar$ has to be understood as the Abelian superalgebra. The important example of the above procedure is provided by the contraction of the anti-de Sitter algebra $g = sp(4)$ to the Poincare algebra $\mathcal{P}_4$. As is readily seen, in this case $\hbar$ is the Lorentz subalgebra of $sp(4)$, i.e. $\mathcal{P}_4 = sp(4) \otimes (3, 1)$.

The second remark is that the example of the contraction of $osp(1, 4)$ to the Gelfand-Likhtman superalgebra, considered at the beginning of this Section, also admits the following natural generalization. Let subalgebra $\hbar \subset \mathcal{G}$ be a Lie algebra. In this case, the whole set of the parameters $\mathcal{F}$ of $\mathcal{G}$ can be splitted into the subsets: $\mathcal{F}_B', \mathcal{F}_B'', \mathcal{F}_F''$ where $\mathcal{F}_B'$ and $\mathcal{F}_B''$ are even (boson) parameters belonging to $\hbar$ and $\mathcal{G}/\hbar$, respectively, while $\mathcal{F}_F''$ is the whole set of odd (fermion) parameters of $\mathcal{G}$ ($\mathcal{F}_F' \equiv 0$ as $\hbar$ is supposed to be a Lie algebra). Let us introduce the contraction parameter λ as follows:

$$\mathcal{E}_B' = \mathcal{F}_B', \quad \mathcal{E}_B'' = \lambda^{-1} \mathcal{F}_B'', \quad \mathcal{E}_F'' = \lambda^{-\frac{1}{2}} \mathcal{F}_F'' \quad (6.12)$$

Then, one can see that in terms of the parameters $\mathcal{E}$, right hand sides of eqs.(6.8), (6.9) contain non-negative powers of λ . Taking the limit $\lambda \rightarrow 0$, one finds the contracted algebra $\hat{\mathfrak{g}}_1$ given by the relations

$$[\mathcal{E}_1, \mathcal{E}_2]_B = \mathcal{E}'_{1B} \mathcal{E}'_{2B}, \quad (6.13)$$

$$[\mathcal{E}_1, \mathcal{E}_2]_B'' = \mathcal{E}'_{1B} \mathcal{E}''_{2B} + \mathcal{E}''_{1B} \mathcal{E}'_{2B} + \mathcal{E}''_{1F} \mathcal{E}_{2F}, \quad (6.14)$$

$$[\mathcal{E}_1, \mathcal{E}_2]_F'' = \mathcal{E}'_{1B} \mathcal{E}''_{2F} + \mathcal{E}''_{1F} \mathcal{E}'_{2B} \quad (6.15)$$

whose right hand sides coincide with the corresponding terms in right hand sides of eqs.(6.8), (6.9). It is clear that the Gelfand-Likhtman superalgebra is $\widehat{\mathfrak{osp}(1,4)} \circ \mathfrak{o}(3,1)$.

The above described general methods for obtaining contractions can be applied to the superalgebras $\mathfrak{shs}_p(1)$ and their arbitrary finite- and infinite-dimensional subalgebras. Specifically, one can identify subalgebra $\mathfrak{h}$ with $\mathfrak{sp}(4)$, the subalgebra of $\mathfrak{shs}_p(1)$ with the parameters $\mathfrak{F}(n,m)$ at $n+m=2$. In this case, application of the procedure of the first type leads to the "linearized" superalgebras $\mathfrak{shs}_p^L(1) = \mathfrak{shs}_p(1) \circ \mathfrak{sp}(1)$. These can be defined by eq.(3.34) where all the terms, containing nonlinearly the parameters $\mathfrak{F}(n,m)$ with $n+m \neq 2$, have to be omitted. The superalgebras $\mathfrak{shs}_p^L(1)$ generate precisely the linearized higher-spin curvatures found in ref./2/. Moreover, these linearised superalgebras have served as the starting point for the determination of $\mathfrak{shs}_p(1)$ which, in fact, have been required to have $\mathfrak{shs}_p^L(1)$ as their contractions (of Hypothesis III.). On the other hand, $\mathfrak{shs}_p^L(1)$ themselves satisfy Hypotheses I.-III., although these linearised superalgebras of course do not provide for nontrivial solutions. Note also that the N=1 supersymmetric "linearisations" $\mathfrak{shs}_p^{L^3}(1)$ or $\mathfrak{shs}_p(1)$ can be constructed as $\mathfrak{shs}_p(1) \circ \mathfrak{sp}(1,1)$.

Here we shall not consider in detail all contractions of $shs_p(1)$ which can be obtained by use of various subalgebras found in previous Section (the reader can easily construct them if desires). Instead, we only make some qualitative remarks.

The structure of such contractions is clear from eqs.(6.10), (6.11) or (6.13)-(6.15). Namely, for contractions of the first type, one must keep in right hand side of eq.(3.34) only the terms containing the parameters $\mathfrak{F}(n,m)$ of $\hbar$ (all other terms should be omitted). For contractions of the second type, one must keep the above terms and also the terms bilinear in odd (fermion) parameters in those relations among eqs.(3.34) whose left hand sides contain parameters related to $\mathfrak{F}\hbar^3\mu/\hbar$.

Therefore for the both types of contractions, if choosen subalgebra contains $sp(4)$ as subalgebra, the related contraction is non-flat. Such contractions appear if $\hbar$ is identified with one of the subalgebras $\mathcal{G}_A$ of previous Section as well as if $\hbar = sp(4)$ or $\hbar = sp(1,4)$. All these contractions satisfy Hypotheses I.-III., V. However, the authors opinion is that they are less natural and elegant as compared to the superalgebras $shs_p(1)$ themselves. In any case, we have found no advantages of these contractions in attempts to use them instead of $shs_p(1)$.

If $\hbar$ contains as its (finite-dimensional) maximum Lie subalgebra only the Lorents subalgebra $\mathcal{O}_4(3,1)$ with the parameters $\mathfrak{F}(2,0)$ and $\mathfrak{F}(0,2)$ (or $\mathcal{O}_4(3,1) \oplus \mathcal{O}(2)$, the latter with the parameter $\mathfrak{F}(0,0)$) then, for the both types of above approaches, the related contractions of $shs_p(1)$ turn out to be flat in the sense that their (finite-dimensional) maximum Lie subalgebras coincide with $\mathcal{O}_4 \oplus \mathcal{O}(2)$ (the trivial factor $\mathcal{O}(2)$ is

absent in the framework of $\mathfrak{shs}_p'(1)$. Amongst the subalgebras considered in previous Section, such flat contractions can be constructed based on $\mathfrak{h}_2, \mathfrak{f}_{22}, \mathfrak{k}, \mathfrak{z}_{00}, \mathfrak{z}_{22}$ and Lorentz subalgebra itself.

As follows from the general structure of contractions under consideration, all these flat contractions possess the property that the terms, containing $\mathfrak{F}(1,1)$ in right hand sides of corresponding counterparts of eq.(3.34), are of the structure $\mathfrak{F}(1,1) \times \mathfrak{F}^h(n,m)$ where $\mathfrak{F}^h(n,m)$ are the parameters of the choosen subalgebra $\mathfrak{h}$. Assuming that the linearization of the flat (super)algebras is carried out with respect to the vierbein $\mathcal{W}(1,1)$ (i.e. with respect to the Poincare subalgebra), one finds that in the linearized curvatures corresponding to the flat contractions under consideration, the terms with $\mathcal{W}(1,1)$ are of the structure $\mathcal{W}(1,1) \times \mathcal{W}^h(n,m)$ where the fields $\mathcal{W}^h(n,m)$ are the gauge fields of $\mathfrak{h}$.

It is very important that none of the linearized flat curvatures, which can be constructed in the framework of such procedure, have the structure of the flat linearized curvatures introduced in ref./2/ (see eq.(2.44) of ref./2/, which contains the transformation laws corresponding to these flat linearized curvatures). Moreover, it seems that no flat contraction of $\mathfrak{shs}_p'(1)$ exists at all which after linearization might reproduce the linearized flat curvatures of ref./2/.

In fact, the flat linearized curvatures suggested in ref./2/ appear as a result of the contraction twice: first, the linearized contraction ℓ leading to $\mathfrak{shs}_p^\ell(1)$ and, second, the flat contraction $\mathfrak{f}$ applied to $\mathfrak{shs}_p^\ell(1)$. The contraction $\mathfrak{f}$ can be realized as follows. Let us introduce new parameters $\mathfrak{E}$ of $\mathfrak{shs}_p^\ell(1)$ by the relation

$$\mathcal{E}(n, m) = \mu^{1n-m-2} \mathcal{F}(n, m), \quad (6.16)$$

μ being the contraction parameter (the parameters $\mathcal{F}$ are those surviving in eq.(3.34) after the linearized contraction $\mathcal{L}$). In terms of $\mathcal{E}$, the superalgebra $\mathcal{SHS}_\rho^{\mathcal{L}}(1)$ reads

$$\begin{aligned} [\mathcal{E}_1, \mathcal{E}_2]_{\alpha(n), \beta(m)} = & (n \mathcal{E}_1^{\gamma, \delta}_{\alpha, \beta(0)} \mathcal{E}_{2\alpha(n-1)\gamma, \beta(m)} + \\ & + m \mathcal{E}_{1\alpha(0), \beta}^{\gamma, \delta} \mathcal{E}_{2\alpha(m), \beta(m-1)\delta} + \\ & + n \mu^{2+1n-m-1n-m-2} \mathcal{E}_{1\alpha, \beta}^{\gamma, \delta} \mathcal{E}_{2\alpha(n-1), \beta(m)\delta} + \\ & + m \mu^{2+1m-n-1m-n-2} \mathcal{E}_{1\alpha, \beta}^{\gamma, \delta} \mathcal{E}_{2\alpha(n)\gamma, \beta(m-1)\delta} \\ & - (1 - \delta(n+m-2)) \times (1 \leftrightarrow 2) \end{aligned} \quad (6.17)$$

As is easily seen, right hand side of eq.(6.17) contains non-negative powers of μ , and in the limit $\mu \rightarrow 0$ one finds the following flat contraction $\mathcal{SHS}_\rho^{f\mathcal{L}}(1)$ of $\mathcal{SHS}_\rho^{\mathcal{L}}(1)$:

$$\begin{aligned} [\mathcal{E}_1, \mathcal{E}_2]_{\alpha(n), \beta(m)} = & (n \mathcal{E}_1^{\gamma, \delta}_{\alpha, \beta(0)} \mathcal{E}_{2\alpha(n-1)\gamma, \beta(m)} + \\ & + m \mathcal{E}_{1\alpha(0), \beta}^{\gamma, \delta} \mathcal{E}_{2\alpha(m), \beta(m-1)\delta} + \\ & + n \theta(m-n) \mathcal{E}_{1\alpha, \beta}^{\gamma, \delta} \mathcal{E}_{2\alpha(n-1), \beta(m)\delta} + \\ & + m \theta(n-m) \mathcal{E}_{1\alpha, \beta}^{\gamma, \delta} \mathcal{E}_{2\alpha(n)\gamma, \beta(m-1)\delta} \\ & - (1 - \delta(n+m-2)) \times (1 \leftrightarrow 2). \end{aligned} \quad (6.18)$$

The superalgebra $\mathcal{SHS}_\rho^{f\mathcal{L}}(1)$ generates the gauge transformations

given by eq.(2.44) of ref. /2/. Note that in fact the transition (6.16) to the "new" parameters ξ is closely related to the transition to the fields $\tilde{\omega}$ and the parameters $\tilde{f}$ used in ref./2/ in investigation of the flat limit (cf.eq.(3.30) of ref./2/, which differs from eq.(6.16) by some irrelevant rescaling of fields with $f \neq 2$ only).

Thus, the superalgebras shs_p^f arise as f - ℓ -contractions of $shs_p(1)$ (first, 1-contraction and, second, f -one). The nontrivial question is whether an ℓ -contraction of $shs_p(1)$ exists leading to the same result. As mentioned above, it seems that the answer is negative. If this is really the case, this fact may indicate the non-existence of a physical flat limit for an interacting higher-spin theory based on $shs_p(1)$.

Indeed, in order to solve the "extra field problem" along the lines proposed in Section 6. of ref./2/ (see also next Section), it is necessary to use the flat linearized curvatures defined by eq. (2.4) of this reference if one works within the expansion of an interaction over the flat space. Since these flat linearized curvatures cannot be obtained as the linearization of any flat non-linear curvatures, this most probably implies that a higher-spin dynamics does not admit a transition to the physical flat limit before linearization. Thus, we expect that a higher-spin interaction should be non-analytic in the cosmological constant. Some additional arguments in favour of this point of view will be outlined in next Section.

As far as we know, the possibility that a higher-spin interaction can be non-analytic in the cosmological constant was never seriously taken into account in all real attempts to solve the higher-spin problem. Clearly, this non-analyticity, as well as infinity of higher-

spin fields, can turn out to be the crucial points for the higher-spin problem. It should be stressed, however, that speaking about the non-analyticity in the cosmological constant, we assume the phase with unbroken higher-spin gauge symmetries. In any case, these gauge symmetries should be spontaneously broken if one hopes to construct a physically sensible higher-spin theory, that is in a full analogy with ordinary supergravity. But in a spontaneously broken phase, the cosmological constant should be redefined. Therefore, the non-analyticity in the cosmological constant before the breaking does not imply that corresponding higher-spin theories are physically unsatisfactory.

To conclude this Section, we describe one more contraction of $\mathfrak{sh}\mathfrak{sp}(1)$ which was found when we attempted to construct general solution of the Jacobi identities without use of Hypothesis IV. The related contraction parameter μ can be introduced as follows:

$$\mathcal{E}_{\alpha(m),\beta(m)} = \mu^{2-(n+m)-|n+m|_2} \mathfrak{F}_{\alpha(m),\beta(m)} . \quad (6.19)$$

In terms of the parameters $\mathcal{E}$, eq.(3.34) reads

$$[\mathcal{E}_1, \mathcal{E}_2]_{\alpha(m),\beta(m)} = \sum_{p,q,s,k,e,t} i^{s+t+1} \frac{n!m!}{p!q!s!k!e!t!} \delta(n-p-q) \times \quad (6.20)$$

$$\delta(m-k-e) \delta((p+k)(q+e) + (p+k)(s+t) + (q+e)(s+t) + 1/2) \times \\ \mu^{2(s+t) + |p+s+k+t|_2 + |q+s+e+t|_2 - |p+q+k+e|_2 - 2} \times \\ \mathfrak{F}_{2\alpha(p)} \delta(s), \beta(k) \delta(t) \mathfrak{F}_{2\alpha(q)} \delta(s), \beta(e)$$

Using the δ -function in right hand side of eq.(6.20), one can see that this relation contains only non-negative powers of μ . Taking the limit $\mu \rightarrow 0$, we find the following contraction $\mathfrak{sh}\mathfrak{sp}^*(1)$ or $\mathfrak{sh}\mathfrak{sp}(1)_0$

$$[E_1, E_2]_{\alpha(m), \beta(m)} = \sum_{p+q+k+l} i^{s+t-1} \frac{n/m!}{p!q!s!k!l!t!} \delta(n-p-q) \delta(m-k-l) \times$$

$$\times \theta(1-s-t) \delta((p+k)(q+l) + (p+k)(s+t) + (q+l)(s+t) + 1/2) \times$$

$$\times \tilde{E}_{1\alpha}(p)(s), \tilde{E}_{1\beta}(k)\delta(t) \tilde{E}_{2\alpha}(q) \delta(s), \tilde{E}_{2\beta}(l)\delta(t) \quad (6.21)$$

(i.e. right hand side of eq.(6.21) contains only those terms of right hand side of eq.(3.34) for which $s+t \leq 1$).

Note that the superalgebras $\mathfrak{shs}_p^*(1)$ cannot be obtained by use of the approaches discussed above (for any subalgebra of $\mathfrak{shs}_p(1)$). Obviously, $\mathfrak{shs}_p^*(1)$ satisfy Hypotheses I.-III., V. These superalgebras are in many respects similar to $\mathfrak{shs}_p(1)$. It may happen that $\mathfrak{shs}_p^*(1)$ can be utilized for building a nontrivial higher-spin theory instead of $\mathfrak{shs}_p(1)$.

7. Higher-Spin Interaction.

As noted in Introduction, determination of the global higher-spin symmetry is although necessary but only the first step towards a solution of the higher-spin problem. If the global higher-spin symmetry is found correctly, then the second step should consist in constructing a higher-spin dynamics in terms of the geometrical objects (connections and curvatures) related to the higher-spin symmetry.

We shall assume that the higher-spin superalgebra coincides with one of the superalgebras $\mathfrak{shs}_p(1)$ suggested in the present paper. Let us stress from the very beginning that the question on introducing a consistent higher-spin interaction is not solved in this paper and, in fact, is now under investigation. Consequently, the matters discussed in this Section are not arranged in a really sys-

tematic way and often play the role of somewhat leading remarks only.

As the simplest probe variant for the action, we take the following:

$$S = \sum_{\substack{n \geq 0, m \geq 0 \\ n+m \geq 0}} [n+m+1] \frac{1}{2} \frac{1}{n!m!} \beta(n+m) \int d^4x \epsilon^{\mu\nu\rho\sigma} R_{\mu, \alpha(m), \beta(n)} R_{\rho\sigma, \alpha(n), \beta(m)} \quad (7.1)$$

where $R(n, m)$ are the shs_p(1) curvatures (1.1), $\beta(n)$ are some real coefficients, and

$$\epsilon(n) = \theta(n) - \theta(-n), \quad \theta(n) = \begin{cases} 1 & \text{at } n \geq 0 \\ 0 & \text{at } n < 0 \end{cases} \quad (7.2)$$

The action (7.1) is the direct generalization of the linearised actions suggested in ref./2/ for free massless higher-spin fields. At the linearized level, it coincides with the sum of all free higher-spin actions with $s \geq \frac{3}{2}$, having an arbitrary normalizing coefficients $\beta(2(s-1))$ where $2(s-1) = n+m$. The action (7.1) is hermitian and $\mathcal{P}$ -invariant with respect to hermitian conjugation (1.4) and $\mathcal{P}$ -transformations (4.10), (4.23). Note also that the action (7.1) is similar in its form to the actions of gravity and N=1 supergravity proposed in ref./4/.

Let us remind the reader that the higher-spin problem in its initial form consists in the fact that the "naive" introducing of the gravitational interaction breaks the higher-spin gauge symmetries /17,18/. Thus, it is the question on the gauge invariance of the higher-spin action being of most importance. According to the curvature transformation law (3.37), the variation of the action (7.1) under the gauge transformations (3.35) is of the form

$$\delta S = \frac{1}{2} \sum_{p,q,s,k,e,t} i^{p+q+s+k+e+t} \frac{1}{p!q!s!k!e!t!} \times \quad (7.3)$$

$$\times \delta((p+k)(q+e) + (p+k)(s+t) + (q+e)(s+t) + 1/2) \times$$

$$\times [\beta(p+q+k+e) \epsilon(p+q-k-e) - \beta(p+q+k+t) \epsilon(p+q-k-t)] \times$$

$$\times \int d^4x \epsilon^{\nu\mu\rho\delta} R_{\nu\mu, \alpha(p+q), \beta(k+e)} R_{\rho\delta, \gamma(s), \delta(t)} \sum_{\gamma(s), \delta(t)} d(g_{\gamma(s)} \beta(e) \delta(t))$$

where we have symmetrized coefficients in such a way that, taking into account the δ -function, expression (7.3) turns out to be symmetric with respect to the transposition of the curvatures. So, the gauge variation (7.3) of the action (7.1) is different from zero.

Note that if one replaces the quantity $\beta(n+m)\epsilon(n-m)$ in eq. (7.1) by some constant, the gauge variation of the corresponding functional T will vanish identically. This implies that superalgebra $\text{shs}_p(1)$ possesses invariant bilinear form,

$$A = \sum_{\substack{n \geq 0, m \geq 0 \\ n+m > 0}} i^{n+m} \frac{1}{n!m!} \eta_{1, \alpha(n), \beta(m)} \eta_{2, \alpha(m), \beta(n)} \quad (7.4)$$

where the parameters $\eta_{1,2}$ realize the adjoint representation of $\text{shs}_p(1)$. However, the invariant form (7.4) does not provide a possibility for constructing a nontrivial invariant action in terms of exterior forms only (i.e., without explicit use of the metrics and, specifically, the Hodge operation *).

Indeed, the invariant functional

$$T = c \sum_{n,m} i^{n+m} \frac{1}{n!m!} \int d^4x \epsilon^{\nu\mu\rho\delta} R_{\nu\mu, \alpha(n), \beta(m)} R_{\rho\delta, \alpha(m), \beta(n)} \quad (7.5)$$

turns out to be the topological invariant (i.e., an integral of full derivative): its equations of motion are identities $0=0$. Note that this fact is quite general and takes place for an arbitrary gauge group G and an invariant form A in the adjoint representation, that is well-known e.g. for Yang-Mills gauge fields. In addition, the functionals like T turn out to be $\mathcal{P}$ -odd. On the other hand, it should be mentioned that the degeneracy of the equations for the

coefficients against different terms in the free higher-spin actions, noted in ref./2/ and, in fact, providing the very existence of the corresponding free actions, arises mainly due to the existence of the invariant form (7.4).

In our attempts to solve the higher-spin problem, we avoid those variants for higher-spin actions which contain the metric tensor explicitly. The reason for this is that the presence of the gravitational metric tensor, related to the massless spin-2 field, destroys uniformity in the description of different higher spins and leads to too broad spectrum of possible actions ^(being) however, non-geometric in their form and much more complicated as compared to those without explicit use of the metrics like in geometric approaches to gravity and supergravity (see e.g. refs. /4,6-8/).

In the framework of the approach suggested in ref./4/, the action is not invariant under the gauge symmetry G^{loc} originating from the ordinary (Yang-Mills) localization of the initial global symmetry G ($G = sp(4)$ for gravity and $G = osp(1,4)$ for $N=1$ supergravity). Instead, the action is explicitly invariant under the gauge transformations from some subgroup $H \subset G^{loc}$ only (for gravity and supergravity, H coincides with the Lorentz subgroup). As for the symmetries from G/H , in gravity and $N=1$ supergravity the action is invariant /4/ under some modified transformations of gauge fields ω^A , which explicitly contain the curvatures $R_{\mu\nu}^A$. In other words, the action turns out to be invariant under some deformation of gauge transformations related to G/H . In supergravity, these deformed transformations are general coordinate transformations and local supertransformations. For the higher-spin case under investi-

gation, it seems also natural to expect that the undeformed gauge variation of the action has to be non-vanishing.

Let us consider the variation (7.3) of the action (7.1) in more detail. First of all note that, as shown in ref./2/, the linearization of the action (7.1), corresponding to the free theory, is invariant under the linearized (more precisely zero order in dynamical fields) gauge transformation (3.35). This trivially follows from the fact that, in the lowest order in the dynamical fields, the variation of the curvatures is proportional to the background curvatures which are required to be equal to zero in the background de Sitter space under consideration (for more detail see ref./2/).

Now consider the part of the full (non-linearized) variation (7.3) which is linear in the following curvatures

$$\gamma_{\nu\mu, \alpha(2)} = R_{\nu\mu, \alpha(2), \dot{\beta}(1)}, \quad \bar{\gamma}_{\nu\mu, \dot{\beta}(2)} = R_{\nu\mu, \alpha(1), \dot{\beta}(2)} \quad (7.6)$$

generalizing the Riemann tensor in gravity. As is readily seen,

this part of the variation (7.3) reads

$$\delta' S = \sum_{s,t} i^{s+t} \frac{1}{3!t!} (\beta(2) - \beta(s+t+1)) \mathcal{E}(1+s-t) \times \\ \times \int d^4x \epsilon^{\nu\mu\rho\delta} \gamma_{\nu\mu, \alpha(2)} R_{\rho\delta, \alpha(2), \dot{\beta}(1)} \mathcal{F}^{\alpha\delta(s), \dot{\beta}(t)} + h.c. \quad (7.7)$$

It follows from eq.(7.7) that the spin content of the curvatures $R(n, m)$ in eq.(7.7) coincides with that of the parameters $\mathcal{F}(n, m)$. Specifically, the part of the variation (7.7) with the diagonal parameters $\mathcal{F}(n, n)$ contains only diagonal curvatures $R(n, n)$. Note that the diagonal parameters $\mathcal{F}(n, n)$ precisely correspond to the higher-spin gauge parameters in the formalism of symmetric tensors and tensor-spinors. On the other hand, the diagonal curvatures $R(n, n)$ are counterparts of the torsion tensor in gravity and supergravity (for

more detail see ref./2/) which in our terms coincides with $R_{\nu\mu,\alpha,\beta}$ (i.e. $R(1,1)$).

In Einstein gravity and Poincare $N=1$ supergravity, the torsion vanishes due to equations of motion. In fact, the zero torsion equation plays the role of the constraint which expresses the auxiliary fields (Lorentz connection) in terms of the dynamical field (vierbein). Similarly, linearized higher-spin diagonal curvatures $R^L(n,n)$ vanish due to the higher-spin equations of motion /2/. These equations express the auxiliary fields $\omega(n\pm 1, n\mp 1)$ in terms of the dynamical higher-spin fields $\omega(n,n)$. In supergravity, the zero-torsion equation enables one to use the "1.5 order formalism" /19, 20, 5/ in order to prove the invariance of the action under some modified (deformed) supertransformations which leave the constraint $R(1,1)=0$ invariant. Similarly, in the higher-spin case, one can see that if

$$R_{\nu\mu,\alpha,\beta}(n) = 0 \quad (7.8)$$

then, at least in the first nontrivial order in the dynamical fields (in which it is proven that eq. (7.8) follows from equations of motion), there exist some modified transformations with the diagonal parameters $\xi(n,n)$, which leave the action (7.1) invariant. Note that in this order, only the transformation laws of auxiliary fields $\omega(n\pm 1, n\mp 1)$ are modified, while the transformations of the dynamical fields $\omega(n,n)$ are in accordance with eq.(3.35), that, in its turn, is also in a full analogy with the situation in gravity and $N=1$ supergravity.

Thus, the approach under consideration enables one to remove (at least in the lowest nontrivial order in the dynamical fields) those terms in the gauge variation of the action, which are proportional to the gravitational curvatures γ or $\bar{\gamma}$ and the diagonal gauge

parameters $\Xi(n, m)$, i.e. precisely the same terms which were considered in refs./17,18/ when the higher-spin problem was treated in terms of symmetric tensor-spinors on the flat background. The more detailed analysis shows that the compensation of these terms for the action (7.1) is in fact due to nonanalyticity of the higher-spin interaction in the cosmological constant. Although we use in this Section dimensionless units with $\lambda = 1$ (λ is the contraction parameter to the flat space, which coincides with the inverse de Sitter radius), if one introduces λ in accordance ^{once with} eq.(3.30) of ref./2/ or eq.(6.16) of ^{with $\mu^2 = \lambda$} this paper then it can be verified that a part of the terms, providing a nontrivial contribution into the part of the variation of the action under consideration, appears with the coefficients $(\frac{\lambda}{\lambda})^2$. Here the denominator originates from the higher-spin interaction while the numerator originates from the commutators of the background Lorentz covariant derivatives, which are proportional to the background de Sitter curvature: that is to λ^2 .

As for invariance of the action under the gauge transformations with non-diagonal parameters $\Xi(n, m)$ at $n \neq m$ as well as the terms in eq.(7.3) which are not of the form (7.7), this question turns out to be not so simple and it seems that it may be solved only after imposing some constraints which do not follow from the action (7.1). This is also seen from the following arguments.

Let us consider the arbitrary variation of the action (7.1). Using the Bianchi identities (3.36) and integrating by parts, one

$$\begin{aligned}
 \text{finds} \\
 \delta S = \int d^4x \epsilon^{\mu\rho\sigma\tau} \sum_{p, q, s, k, \ell, t} (p+q+s+k+\ell+t) \frac{1}{p!q!s!k!\ell!t!} \times \\
 \times \delta((p+k)(q+\ell) + (p+k)(s+t) + (q+\ell)(s+t) + 1/2) \times \\
 \times (\beta(p+q+k)\ell\epsilon(p+q-k-\ell) - \beta(q+s+\ell+t)\epsilon(q+s-\ell-t)) \times \\
 \times R_{\mu, \alpha(p+q), \beta(k+\ell)}(\omega_p, \alpha(p)_{\mu(s)}, \beta(k)_{\ell(t)}) \delta(\omega_s, \alpha(q+s)(\tau), \beta(\ell+t)) \delta(\epsilon)
 \end{aligned} \quad (7.9)$$

It has been shown in ref./2/ that the variation of the linearized action (7.1) over the fields $\mathcal{W}(n,m)$ with $|n-m| > 2$ vanishes identically, i.e. free massless higher-spin fields are described by the fields $\mathcal{W}(n,m)$ with $|n-m| \leq 2$ only. This fact can be readily seen from eq.(7.9) taken at $p+s=k+t=1$. However, it is also seen from eq. (7.9) that the full (non-linear) action (7.1) contains nontrivially all the fields $\mathcal{W}(n,m)$. As a result, equations of motion for the fields $\mathcal{W}(n,m)$ at $|n-m| > 2$ turn out to be essentially non-linear. Therefore, if one regards the fields $\mathcal{W}(n,m)$ with $|n-m| > 2$ as independent dynamical variables then, at least in the framework of the perturbation expansion, the equations of motion for these fields turn out to be inconsistent.

One possible way for resolution of this "extra field problem" may consist in that the "extra fields" $\mathcal{W}(n,m)$ with $|n-m| > 2$ are not independent dynamical variables but instead have to be expressed from the very beginning in terms of the dynamical fields $\mathcal{W}(n,m)$ with $|n-m| \leq 1$ (recall that the fields $\mathcal{W}(n,m)$ with $|n-m| = 2$ are auxiliary /2/ i.e, they can be expressed in terms of the dynamical fields by virtue of equations of motion). In other words, it seems desirable to impose some constraints which express all the fields $\mathcal{W}(n,m)$ in terms of the dynamical fields $\mathcal{W}(n,m)$ with $|n-m| \leq 1$. It is important, however, that unlike to gravity and supergravity where the "extra fields" are absent, these constraints cannot be obtained from the equations of motion following from the action (7.1). In particular, this circumstance prevents from using such simplifying tricks as e.g. the "1.5-order formalism".

The question on the existence of adequate constraints turns out to be nontrivial enough, and is now under investigation. In

ref. /2/, the linearized constraints have been suggested which express all the fields $\omega(n, m)$ in terms of derivatives of the dynamical fields $\omega(n, m)$ with $|n-m| \leq 1$. More precisely, the orders of derivatives of the dynamical fields containing in $\omega(n, m)$ increase as $|n-m|$. As a result, a higher-spin interaction will contain higher derivatives that is in accordance with the results of ref./21, 22/ where non-gravitational higher-spin interactions were considered. It should be stressed, however, that the free part of the action does not contain higher derivatives. Therefore, one can expect that, at least in the framework of the perturbation expansion, the presence of the higher derivatives in an interaction will not lead to new significant problems.

One can hope that there exist appropriate constraints leading to some deformed gauge transformations which leave the action (7.1) invariant for some coefficients $\beta(n)$. At present, we have not succeeded, however, in constructing a consistent non-linear generalization for the linearized constraints of ref./2/. Perhaps, one must choose a different strategy in order to overcome the "extra field problem". One possibility is that further extensions of the higher-spin superalgebras are needed, generating systems of fields which, in common with the "extra fields" $\omega(n, m)$ with $|n-m| > 2$, will form systems of auxiliary fields, i.e. the fields which can be expressed in terms of the dynamical fields by virtue of equations of motion following from some new action. We hope to consider this interesting possibility in a separate publication.

8. Conclusion.

The main result of the present paper consists in the determination of the higher-spin superalgebras $shs_p(1)$ ($p=0$ or 1) which

have been found in the framework of general assumptions formulated as Hypotheses I.-V. in Section 2, and have to be regarded as direct consequences of the $N=1$ supergravity superalgebra $osp(1,4)$. The most important assumption consists in the requirement that at the linearised level higher-spin superalgebras should reproduce the linearised curvatures found in ref./2/ which have been used in this reference for the description of free massless higher-spin fields in the de Sitter space.

Apart from the superalgebras $shs_p(1)$, we have found some other superalgebras satisfying those assumptions on the structure of higher-spin superalgebras, which seem to be of most physical importance. However all these superalgebras turn out to be either some subalgebras or some contractions of $shs_p(1)$. The authors opinion is that the most probable candidates to the role of the higher-spin superalgebra are the superalgebras $shs_p(1)$ themselves, their subalgebras $shs(1/d) \subset shs_p(1)$ ($d = 0$ or 1) considered in Section 5., and also the contractions $shs_p^*(1)$ of $shs_p(1)$ considered in Section 6, as well as analogous contractions of $shs(1/d)$.

All the higher-spin superalgebras, mentioned above, are infinite-dimensional and generate infinity of fields with increasing spins. For example, the superalgebras $shs_p(1)$ and $shs_p^*(1)$ generate all spins $j \geq 1$ (any spin occurs only once), while the superalgebras $shs(1/d)$ generate spins $j=2K+2$ and $j=2K+\frac{3}{2}+d$ ($K=0, 1, 2, \dots, \infty$). Let us stress that the property that massless higher spins ($j \geq 2$) form infinite systems of fields seems to be inevitable when one looks for a consistent interaction of higher spins in the framework of gravity.

It is not completely clear now whether the solution of the higher-

spin problem can be achieved in the framework of the approach, based on the proposed higher-spin superalgebras. This approach is not free from difficulties which are briefly discussed in Section 7. Nevertheless, we regard the obtained results as very promising and indicating the true direction towards the solution of the higher-spin problem.

In particular, the following result provides efficient support for this point of view. It has been shown in Section 7. that at least in the first nontrivial order in the dynamical fields, there can be compensated the terms of the form $\mathcal{L} R \mathcal{J}$ in the gauge variation of the higher-spin action constructed from the curvatures of $\text{shs}_p(1)$ ($\mathcal{L}$ are the Riemann curvatures, R are the higher-spin curvatures, and $\mathcal{J}$ are the diagonal higher-spin gauge parameters $\mathcal{J}_a(n, \hat{a}(n))$). It should be stressed that these are precisely the terms which could not be compensated when the higher-spin problem was analysed in the framework of symmetric tensor-spinors on the flat background /17, 18/. In fact, the compensation of the above terms is closely related to the following new point which can be of extreme importance for constructing a full solution of the higher-spin problem.

Namely, as follows from the results of Sections 6. and 7., the gravitational interaction of massless higher spins turns out to be non-analytic in the cosmological constant. There are two indications that this is really the case. The first one is that a flat contraction of $\text{shs}_p(1)$ does not exist which possesses appropriate properties at the linearized level (see Section 6.). The second one is that the nontrivial cancellations in the gauge variation of the action of Section 7. are provided by the terms in the

higher-spin interaction which are non-analytic in the cosmological constant.

It is quite possible that both the necessity for the infinite number of higher spins and non-analyticity in the cosmological constant are necessary conditions for the solution of the higher-spin problem. In some sense, the latter requirement resembles the situation in extended supergravity and, specifically, in $N=2$ supergravity where minimal electromagnetic interaction for gravitine turns out to be impossible in the flat space and requires for non-zero cosmological constant /23,20/. Possibly enough, the non-analyticity in the cosmological constant is the main subtle point which prevented from the progress in the solution of the higher-spin problem in the early attempts /17,18/ used the flat background only.

It should be stressed, however, that speaking about the non-analyticity in the cosmological constant, we assume the phase with unbroken higher-spin gauge symmetries. Obviously, this does not imply that a corresponding higher-spin theory will be physically unacceptable since, like for ordinary supergravity, higher-spin gauge symmetries should be broken in the physical phase. Simultaneously, the cosmological constant will receive some new physical value in this phase.

Note added

For some reasons, considerable time passed since the present paper and the paper by one of us /2/ had been prepared for publications. Therefore, we would like to make a few additional comments concerning, first, the new interesting works /24-26/ devoted to constructing consistent interactions of massless higher-spin fields, which have appeared during this period and, second, our own further

results on the gravitational higher-spin interaction which give an efficient support for the direction taken in Ref./2/ and in the present paper.

In Ref./24/, some further progress has been achieved in constructing non-gravitational interactions (in the first nontrivial order) for various systems of $d=4$ massless higher-spin fields, extending the results obtained previously in Refs./21,22/. In Ref./25/, some examples of massless higher-spin-like fields were considered in $d=1,2$, which, however, did not lead to any nontrivial dynamics. Nevertheless, there has been yet no progress in constructing a gravitational higher-spin interaction in $d=4$, the problem being of principal importance.

As is clear from the present paper, the main reason for this situation is that the higher-spin problem was analysed previously against the flat background, which is not an adequate approach, since the gravitational higher-spin interaction turns out to be non-analytical in the cosmological constant. In fact, we are able now to show that the action (7.1), supplemented by constraints imposed on the extra fields, as proposed in Ref./2/, describes correctly the gravitational interaction of $d=4$ massless higher-spin fields at least in the first nontrivial order. The detailed analysis will be reported elsewhere. It is worth mentioning that the non-analyticity in the cosmological constant turns out to be crucial for this analysis, as indicated in Section 7 of this paper.

On the other hand, it should be noted that some qualitative observations contained in Refs./24-26/ remain valid for the gravitational higher-spin interaction as well. These are, first, the necessity of introducing infinite systems of massless higher-spin fields and, second, the fact that higher-spin interactions contain higher derivatives (see also Refs./21,22/). These points receive both a quite natural interpretation in terms of the higher-spin superalgebra proposed in the present paper (see Section 7).

Note also that some arguments were given in Ref./25/ indicating that in dimensions 1 and 2, the sets of spins of the type $2,4,6,\dots$ may be relevant to the

higher-spin problem. These resemble the spectra of $d=4$ higher-spin fields generated by the higher-spin superalgebras $shs(1|d)$ proposed in the present paper (see Section 5).

Finally it should also be mentioned that some attempts to guess a higher-spin symmetry were made by the authors of Refs./24,26/. However, these attempts run immediately into serious difficulties which arise mainly due to the handling of the indices of higher-spin fields as the "world" (base) ones. The main technical problem, arising here, is that the higher-spin fields obey the traceless conditions which prevent one from constructing any sensible higher-spin symmetries in terms of world indices because these conditions should depend nontrivially on the gravitational metric tensor if one wishes to introduce the gravitational higher-spin interaction. In fact, this was the reason that led one of us /1,2/, when starting the investigation of the higher-spin problem, to the search of a language similar to the vierbein formalism in gravity which might enable one to avoid these problems. The results mentioned above show that this turns out really to be the true strategy.

REFERENCES

- /1/. M.A.Vasiliev, Sov. Yad.Fiz. 32 (1980), 855 (in Russian).
- /2/. M.A.Vasiliev, Lebedev Institute preprint N 233 (1986);
submitted to Fortschritte der Physik.
- /3/. M.A. Vasiliev, Lebedev Institute preprint N (1986);
to be published.
- /4/. S.W.MacDowell and F.Mansouri, Phys.Rev. Lett. 38 (1977), 739.
- /5/. P.K.Townsend and P. van Nieuwenhuizen, Phys. Lett.67B (1977),
439.
- /6/. F.Mansouri, Phys. Rev D16 (1977), 2456.
- /7/. R.D'Auria, P.Fré, and T.Regge, in "Supergravity'81" (S.Ferrara
and J.G.Taylor, Eds.), Cambridge Univ., Press, London /New York,
1982; and references therein.
- /8/. R.D'Auria, P.Fré, P.K.Townsend, and P. van Nieuwenhuizen, Ann.
Phys. (N.Y.) 155 (1984), 423.
- /9/. A.Salam and J.Strathdee, Ann. Phys, (N.Y.) 141 (1982), 316;
and references therein.
- /10/. R.Neag, J.Iopuzansky, and M.Sohnius, Nucl. Phys. B88 (1975),
257.
- /11/. C.Fronsdal, Phys. Rev. D18 (1978), 3624.
- /12/. F.A.Berezin, Introduction into the algebra and analysis with
anticommuting variables, Moscow State Univ. Press, Moscow,
1983 (in Russian); Sov. Yad. Fiz. 29 (1979), 1670 (in Russian).
- /13/. M.S.Marinov and V.I. Roginsky, Fortschritte der Physik, 19
(1971), 509.
- /14/. M.A.Vasiliev, Lebedev Institute Preprint N 181 (1985); to be
Published in "Quantum field theory and quantum statistics"
(I.A.Batalin, C.J.Isham and G.A. Vilkovisky, Eds.), Adam

Hilger LTD, Bristol.

- /15/. M.A.Vasiliev, Pis'ma Zh. Eksp.Teor. Fiz. 40 (1984), 437 (in Russian).
- /16/. M.A.Vasiliev, Class.Quant. Grav. 2 (1985), 645.
- /17/. G.Aragone and S.Desser. Phys. Lett. 86B (1979), 161.
- /18/. F.A.Berends, J.W. van Holten, P. van Nieuwenhuizen, and B. de Wit, J.Phys. A13 (1980), 1643.
- /19/. A.H.Chamseddine and P.C.West, Nucl. Phys.B129 (1977) 39.
- /20/. E.S. Fradkin and M.A. Vasiliev, Lebedev Institute Preprint N 197 (1976).
- /21/. A.K.H. Bengtsson, I.Bengtsson and L.Brink Nucl. Phys. B227, (1983), 31; Nucl. Phys. B227 (1983), 41.
- /22/. F.A.Berends, G.J.H.Burgers, and H. van Dam, Z.Phys. 924 (1984) 247.
- /23/. D.Z.Freedman and A.Das, Nucl. Phys. B120 (1977), 221.
- /24/. F.A.Berends, G.J.H.Burgers and H. van Dam, Nucl. Phys. B260 (1985);298; Leiden preprint (1985).
- /25/. A.K.H.Bengtsson, I.Bengtsson, Preprint CERN - TH. 4383/86.
- /26/. A.K.H.Bengtsson, I.Bengtsson, Preprint QMC 86-16.

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