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ESTABLISHMENT AND CALCULATION OF THE MASS GAP
IN THREE-DIMENSIONAL SU(2) YANG-MILLS THEORY

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A B S T R A C T

We calculate the lowest-lying "glueball" mass in three-dimensional Yang-Mills theory based on the SU(2) gauge group by using the source method. Good scaling was observed in the range $4.5 \leq \beta \leq 5.5$. We obtain $m_{g\alpha} = (8.6 \pm 0.8)/\beta$ or, in terms of the string tension, $m_{g\alpha} = (5.8 \pm 0.7)\sqrt{\sigma}$. We discuss its relevance for the gluon plasma and we report the value $\lambda^{-1} = (2.15 \pm 0.20)g^2(T)T$ for the confinement static correlation length for the SU(2) gluon plasma.

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1. Introduction - Motivation

Our main interest in this present work is to establish and calculate through numerical Monte Carlo simulation the mass gap of the three-dimensional pure Yang-Mills theory based on the $SU(2)$ gauge group.

The motivation for this is twofold:

a) Establishing a mass gap i.e. the lowest-lying "glueball" mass in theories based on gauge groups different than $SU(3)$ and for dimensionality different from four (albeit unphysical) it is of theoretical interest since, although it is a common belief that non-Abelian gauge theories in any dimensions less or equal to four are confining and posses a mass gap, (Feynman has argued that gauge invariance of the functional integral implies a finite energy for any excitation above the ground state¹ and thus dynamical generation of a "glueball" mass) a proof on theoretical grounds is generally missing.

Moreover, pure Yang-Mills theory in three-dimensions is an important model to be analyzed for another reason. It is intermediate in complexity between the four-dimensional and the nearly trivial two-dimensional corresponding theories and yet possesses most of the characteristic features of the four-dimensional one.

Both theories have the same internal group structure, they are both asymptotically free, and there is enough theoretical and numerical evidence that they both confine.^{2,3} Therefore, since the three-dimensional theory exhibits most of the anticipated features of the four-dimensional theory, it is an attractive model to be explored with the hope to understand better the by far more complex four-dimensional theory.

It should be mentioned however, that there are important differences between the two theories: Yang-Mills theory in three-dimensions has no true instantons but possesses Wu-Yang type monopoles. Furthermore in three-dimensions, the square of the bare coupling has dimension of mass so that the theory is super-renormalizable in the ultra-violet but divergent in the infra-red.

However failure of the perturbation expansion in the infra-red does not imply that the theory is inconsistent or does not exist, because infra-red divergences would be cured by dynamical mass generation.⁴

This feature of the theory, namely its super-renormalizability, implies that in the absence of infra-red problems, since the only mass scale in the theory is the square of its coupling g_3^2 , any dimensionful quantity should scale with it. This property is particularly useful for numerical study on the lattice. For instance, if σ is the string tension and m any mass of the theory:

$$\sigma\alpha^2 \sim \frac{1}{\beta^2} \quad , \quad m\alpha \sim \frac{1}{\beta} \quad \text{as } \alpha \rightarrow 0 \quad , \quad \beta \equiv \frac{4}{g_3^2 \alpha} \quad (1)$$

where α is the lattice spacing.

b) Nevertheless, the main motivation for studying the three-dimensional pure Yang-Mills theory is its relevance for QCD at finite temperature (Hot QCD) as the high temperature limit of it.

In recent years a great deal of attention has focused on understanding the behaviour of QCD at non-zero temperatures. This is essential for discussing earlier stages of the evolution of the universe in the context of unified gauge theories.

To this respect it is now well established that hadronic matter when subject to sufficiently high temperatures and/or densities much higher than the confinement scale of QCD (such as might occur in astrophysical processes or in relativistic heavy-ion collision experiments) goes into a phase where its color constituents -quarks and gluons- are no longer confined. This phase is commonly named in analogy to QED, QCD plasma phase, and although its existence is generally accepted, the nature and its details are not completely understood. One would expect that, due to asymptotic freedom, as the temperature-dependent effective coupling constant $g(T) \rightarrow 0$ as $T \rightarrow \infty$, the properties of the plasma phase could be understood within the context of perturbation theory. Unfortunately this is not the case. Perturbative attempts to understand the nature of the QCD plasma phase has so far been hampered by the presence of strong, power behaved divergences in the infra-red, arising from the static magnetic sector of the theory,^{5,6} that prevent an orderly summation of diagrams. These infra-red divergences, due to the self-interaction of gluons, make perturbation theory inadequate in contrast to the usual QED plasma.

It seems likely that this non-perturbativity implies the dynamical generation of a mass,⁵ and only a non-perturbative treatment like semiclassical methods or lattice approach would account for such dynamical phenomena.

2. Dynamical confinement and QCD₃ as the static high-temperature limit of QCD at finite temperature.

It is now well established, mainly from extensive Monte Carlo simulation studies⁷, that at short distance scales of order $1/T$ the forces between quarks are Debye-screened, so that the QCD plasma can be described as a free or weakly

interacting gas of quarks and gluons. However for scales larger than $1/g^2(T)T$ the theory is very complicated⁸.

Recently it was conjectured by DeTar⁸ that for these scales the QCD plasma exhibits "confining" features similar to those of the low-temperature hadronic phase. "Confinement" here means dynamical confinement, in the sense that excitations generated by local colored sources will be indistinguishable from excitations generated by local color-singlet sources. The confinement scale is expected to be of order $1/g^2(T)T$.

To understand this conjecture it is necessary to briefly review the three-dimensional pure Yang-Mills theory, its confining feature, and its relevance to QCD at finite temperature, with the help of the so called "dimensional reduction" phenomenon which can be viewed in the context of the decoupling theorem⁹.

As it has been argued by Gross, Pisarski and Yaffe⁵ and Appelquist and Pisarski⁴, different excitations of the plasma, due to thermal fluctuations develop temperature-dependent masses m and, since at finite temperatures space-time is Euclidean m is related to an inverse propagator at zero-momentum. These masses act to dampen long-range correlations as:

$$\langle \phi(t, x), \phi(0, 0) \rangle \sim e^{-m|x|} \quad \text{as } |x| \rightarrow \infty \quad (2)$$

By working out from short distances (high-temperatures), the theory simplifies enormously as various massive excitations decouple from the theory. Fermionic modes have frequencies $\omega_n = (2n+1)\pi T$ they decouple classically since their energy can never vanish. From the bosonic modes $\omega_n = 2n\pi T$, only those with zero energy propagate for spatial distances $\xi \gg \beta = 1/T$. What remains from the four-dimensional theory is an effective ($T=0$) three-dimensional theory of Yang-Mills fields coupled adjointly to a Higgs field ϕ .

The Yang-Mills fields A_i are the space-like components of the vector potential of the four-dimensional theory, and the Higgs field is the time component of the original vector potential A_0 , which at high temperatures develops a perturbative mass:

$$m_\phi(T) = m_e(T) \approx \left(\frac{N}{3}\right)^{1/2} g(T)T \quad \text{for } SU(N) \quad (3)$$

through electrostatic Debye-screening. This has been explicitly shown by Nadkarni¹⁰ by using perturbation theory at the one-loop level. At distances ξ beyond the

electrostatic screening length $m_e^{-1}(T)$, the effective theory simplifies further to pure three-dimensional Yang-Mills theory with (dimensionful) coupling $g_3^2 = g^2(T)T$. The same result has been shown to occur in the strongly-coupled lattice theory by DeTar⁸.

The three-dimensional Yang-Mills theory has been shown to confine³; therefore its spectrum must consist of color-singlet "glueballs" with masses $m_{\sim} g_3^2 = g^2(T)T$, as we discussed at the beginning of this section. Now since this three-dimensional theory is the high-temperature limit of the four-dimensional one, it gives some information about the modes of excitation of the four-dimensional theory. That is that the long-range (distances of order $1/g^2(T)T$) modes of the plasma must be "confined" color-singlet modes in order to coincide with the spectrum of the confined three-dimensional theory at the high-temperature static limit $\omega=0, k \rightarrow 0$.

But what does the mass gap of the three-dimensional theory corresponds to? (Assuming that it is different from zero)

It should correspond to the lightest "glueball" of the theory, presumably with valence composition AA. The mass of this state should necessarily be of order $m_{AA}(T) \sim g^2(T)T$. Since this mass is the lightest "glueball" state of this order, it should determine the inverse static correlation length of this order i.e. The confinement static correlation length of the gluon plasma:

$$a^{-1} = m_{AA}(T) \sim g^2(T)T \quad (4)$$

3. The Source Method

As we mentioned in the introduction, the main goal of this work is to establish and calculate through numerical Monte Carlo simulation the mass gap of the three-dimensional pure Yang-Mills theory based on the SU(2) group.

Although we have one less dimension, this simplified problem is not free of computational difficulties therefore, in order to pick the most suitable method for this particular problem, the following considerations should be taken into account.

- a) From the string tension measurements for SU(2)₃ of Ambjorn, Hey and Otto¹¹ and Ambjorn, Olesen and Peterson¹² the continuum sets in at $\beta \geq 5$, where the values for the various small loops are quite close to unity, therefore, straightforward correlation methods do not seem to be the appropriate ones for the mass gap.
- b) Another consequence is that, for such high values of β , the 120-element icosahedral subgroup of SU(2) does not provide a good approximation for the full

SU(2), so that the latter has to be used, which of course is more time consuming.

c) Perhaps the most important obstacle is that the SU(2)₃ mass gap turns out to be larger¹² compared to the SU(2)₄ which implies small "signal/noise" ratios.

Fortunately, there is a variety of new techniques which have developed, forming the "second generation" which have superseded the old straightforward correlation methods. To mention the most popular ones, Variational methods¹³, Source methods¹⁴, the Langevin equation¹⁵, fixed boundary conditions^{16,17} and adjoint Polyakov loops¹⁸.

In this particular problem we believe that the most appropriate choice will be the the source method and for the updating algorithm Creutz's heat bath with the full SU(2).

Although there are excellent descriptions of the method in the literature^{16,19}, we will briefly review it and compare it with the common method of "measuring" correlations between fluctuations in the vacuum, in order to point out the advantages and disadvantages of the source method.

The common method of "measuring" correlations between fluctuations in the vacuum "measures" the correlation at a distance t of a gauge invariant, translationally invariant operator $\{\phi\}$:

$$\bar{\phi}(t) = \frac{\langle 0 | \phi(t) \phi(0) | 0 \rangle}{\langle 0 | 0 \rangle} \equiv \frac{\int [dU] \phi(t) \phi(0) \exp(-\beta S[U])}{\int [dU] \exp(-\beta S[U])} \quad (5)$$

where ϕ is a gauge invariant function of link variables $U_\mu(x)$.

Inserting a complete set of intermediate eigenstates:

$$\bar{\phi}(t) = \sum_n \frac{|\langle 0 | \phi | n \rangle|^2}{\langle 0 | 0 \rangle} e^{-(E_n - E_0)t} \quad (6)$$

where the sum runs over all the excitations of the system to which the operator $\{\phi\}$ couples.

The idea is to extract the lowest-lying mass excitation by going to large distances where only the lowest mass state of mass m_g contributes. Then truncating the sum to the first two terms, we fit for a mass m'_g using:

$$\bar{\phi}(t) \underset{t \rightarrow \infty}{\approx} \frac{|\langle 0 | \phi | 0 \rangle|^2}{\langle 0 | 0 \rangle} + \frac{|\langle 0 | \phi | g \rangle|^2}{\langle 0 | 0 \rangle} e^{-m'_g t} \quad (7)$$

Since $\langle \phi \rangle$ is assumed translationally invariant $E_1 - E_0 = m_g$ (the glueball mass).

Note that the coefficients in the expansion (7) are positive definite, so that $m_g' \geq m_g$ i.e. m_g' is always an upper bound to the true glueball mass m_g .

The most important disadvantage of this method is that, as it actually measures correlated fluctuations, the signal usually disappears into the noise after a few lattice spacings. The maximum separation at which glueball measurements have been reported in the literature is three lattice spacings. At such short distances the reliability of the truncation to a single state eq. (7) puts some doubts on the method.

The source method consists of locating a translationally invariant source J on the lattice (usually at the origin) which disturbs the vacuum and "measures" the fall-off of the correlation with distance¹⁶.

$$\bar{\phi}_J(t) = \frac{\langle 0 | \phi(t) J(0) | 0 \rangle}{\langle 0 | J(0) | 0 \rangle} \equiv \frac{\int [dU] J(0) \phi(t) \exp(-\beta S[U])}{\int [dU] J(0) \exp(-\beta S[U])} \quad (8)$$

Inserting again a complete set of intermediate eigenstates:

$$\bar{\phi}_J(t) = \sum_n \frac{\langle 0 | \phi | n \rangle \langle n | J(0) | 0 \rangle}{\langle 0 | J(0) | 0 \rangle} e^{-(E_n - E_0)t} \quad (9)$$

Using the translational invariant property of J and ϕ , for t sufficiently large, we again truncate the sum:

$$\bar{\phi}_J(t) \underset{t \rightarrow \infty}{\approx} \langle 0 | \phi | 0 \rangle + \frac{\langle 0 | \phi | g \rangle \langle g | J(0) | 0 \rangle}{\langle 0 | J(0) | 0 \rangle} e^{-m_g' t} \quad (10)$$

In the present case only the $\phi(t)$ fluctuates, so the method should have a much better "signal/noise" ratio, and thus allows us to go to larger distances away from the source, where the $\phi_J(t)$ is truly dominated by its lowest mode and the asymptotic expansion eq. (10) is indeed reliable.

The important disadvantage of the method, however, is that the coefficients in the expansion (10) are not positive definite, so that at non-asymptotic times m_g' is no longer an upper bound of the true mass m_g . Therefore, the source method will in general give no information of m_g at non-asymptotic distances, and can only be used if one can measure signals at large enough distances, to the point where we can accurately identify the asymptotic exponential fall-off.

4. Calculations and results.

We chose to work mainly on a symmetric periodic lattice $N_s^2 \times N_t$, $N_s = N_t = 10$ with $\beta = 3.0, 3.5, 4.0, 4.5, 5.0, 5.5$.

We used the standard Wilson action for SU(2) and placed a "white" source at the $t=0$ plane, i.e. $U_{ij}=1$, $i,j=1,2$, at $t=0$.

We "measure" expectation values of 1×1 , 1×2 and 2×2 space-like plaquettes as a function of the (time) distance from the source.

The statistical uncertainty of the data was chosen to be the standard deviation of the mean, corrected for correlation of successive measurements:

$$\Delta x^2 = \sigma^2 = \frac{1}{N-1} \left(\frac{1}{N} \sum x_i^2 - \bar{x}^2 \right) \left(1 + 2 \frac{\langle x_i x_{i+1} \rangle - \bar{x}^2}{\langle x^2 \rangle - \bar{x}^2} \right) \quad (11)$$

where x_i are the individual measurements.

For a periodic lattice we expect an asymptotic behaviour:

$$\bar{\phi}_J(t) = C \cosh \left(\left(\hat{t} - \frac{1}{2} N_t \right) m_g \alpha \right) + \langle \phi \rangle \quad (12)$$

where $\hat{t}\alpha = t$. Therefore, we best-fit the data to the three-parameter $(C, m_g \alpha, \langle \phi \rangle)$ form eq. (12) by minimizing the χ^2 -function:

$$\chi^2 = \sum_n \frac{[\phi_n - \phi(t_n)]^2}{\sigma_n^2} \quad (13)$$

ϕ_n and σ_n are the expectation values and errors of the operator $\{\phi\}$ at a distance t_n from the source.

Typically we consumed 10^3 sweeps to thermalize the lattice and 2×10^4 sweeps for measurements. In some cases we went up to 4×10^4 as we move further from the source.

In figs. 1a - 1c we present typical fits for 1×2 and 2×2 plaquettes. To see better the exponential fall-off we plot the quantity $\tilde{\phi}_J(t) = \bar{\phi}_J(t) - \langle \phi \rangle$, versus the time distance in lattice units.

Our results are extracted solely from these two operators. The single plaquette was proved to be a bad "antenna" as it "receives" the weakest signal from the source,

resulting in poor statistics.

For the error analysis we adopted the following procedure: First we note that the parameter $\langle\phi\rangle$ in all fits is very little correlated with the other two parameters of the fit. Therefore we fix it to its best-fit value and we check how much can we vary C and $m_g\alpha$, holding the χ^2 -value in a reasonably accepted level of confidence. We define the uncertainty in the parameter $m_g\alpha$ as the distance between its minimum and maximum acceptable values.

We were not able to extract sensible fits for $\beta > 5.5$ presumably because the values for the various loops we measure are very close to 1. Also, we do not present the measurements for $\beta = 3.0$ because the "signal/noise" ratio seems to be very small resulting in poor statistics.

In table 1. we present the values of the mass gap $m_g\alpha$ for $3.5 \leq \beta \leq 5.5$.

Scaling

In fig.2 we plot the dimensionless quantity $M\alpha\beta$ against β . We observe that for $\beta \geq 4.5$ the data are consistent with $1/\beta$ scaling. The dashed line is the best-fitted straight line $y=mx+a$ with a negligible slope $m=-0.01$ and y-intersection $a=8.6$, so the straight line is practically horizontal and gives:

$$M\alpha = (8.6 \pm 0.8) / \beta \quad (14)$$

Using the string tension values for $SU(2)_3$ of ref. [12] in the range where it scales, we find for the mass gap:

$$M = (5.8 \pm 0.7) \sqrt{\sigma} \quad (15)$$

In fig. 3 we plot the quantity $M\alpha$ versus β . The solid line is the best-fitted a/β curve we found in the previous fit.

Finally we report the value for the confinement static correlation length for the $SU(2)$ gluon plasma.

$$a^{-1} = (2.15 \pm 0.20) g^2(T) T \quad (16)$$

The infrared magnetic cut-off

The infrared magnetic cut-off or more simply the magnetic mass that is expected to cure the remaining infra-red divergences coming from the self interaction of the

space-like gluons, was first measured by Monte-Carlo simulations²⁰ under the assumption that it is of order $g^2(T)T$. All these works gave essentially consistent values for the magnetic mass: $m_{\text{magn}}=0.25g^2(T)T$ for $SU(2)$. DeTar⁸ argues that it is possible that both phenomena, namely dynamical confinement of the long-range high-temperature plasma modes and magnetostatic screening can occur in the same theory. However, the following problem would arise:

As we said before any mass from the spectrum of the three-dimensional theory is necessarily of order g_3^2 and corresponds to a confining mode of the plasma. In addition, if the magnetic mass is of order $g^2(T)T$ it should be included in the spectrum of the three-dimensional theory.

The possible resolution to this contradiction is that the magnetic mass is not of order $g^2(T)T$ and therefore, the three-dimensional analog of it, is absent from the mass spectrum of the three-dimensional theory.

Indeed, Kajantie and Kapusta,²¹ by using linear response theory and self-consistent non-perturbative solutions of the Schwinger-Dyson equations, calculated the magnetic mass: $m_{\text{magn}}=(3/8n)^{1/2}(N/3)^{3/4}g^{3/2}(T)T$ for $SU(2)$.

They also showed that the data of the Monte-Carlo simulations of refs. [20] are also consistent with a $g^{3/2}(T)T$ behaviour for the magnetic mass.

A more precise analysis on bigger lattices and with better statistics that could clearly distinguish between the $g^2(T)T$ and $g^{3/2}(T)T$ behaviour is strongly required for a decisive answer to this problem.

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TABLE 1.

β	$m_g\alpha$
3.5	2.66 ± 0.24
4.0	1.90 ± 0.17
4.5	1.91 ± 0.19
5.0	1.79 ± 0.13
5.5	1.56 ± 0.17

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FIGURE CAPTIONS

Figs. 1a-1c Typical fits of the quantity $\tilde{\phi}_J(t) = \bar{\phi}_J(t) - \langle \phi \rangle$ as a function of $t=t/\alpha$ for:

- (a) the 1x2 loop at $\beta=5.5$
- (b) the 2x2 loop at $\beta=4.5$
- (c) the 2x2 loop at $\beta=4.0$

All measurements were taken on a symmetrical periodic lattice of size 10^3 .

Fig.2

The $SU(2)_3$ mass gap times β ($M\alpha\beta$) versus β . The horizontal line is the straight line $y=-0.01x+8.61$, best-fitted to the data.

Fig.3

The $SU(2)_3$ mass gap $M\alpha$ versus β . The solid line represents the best-fitted a/β curve we found in the previous fit.

fig. 1a

10x10x10 LATTICE 1X2 PLQT.

BETA = 5.5

fig.1b

10x10x10 LATTICE

2X2 PLQT.

BETA = 4.5

fig.1c

10x10x10 LATTICE 2X2 PLQT.

BETA = 4.0

fig.2

fig.3