CERN-TH.4620/86

COMMENTS ABOUT THE GEOMETRY OF NON-LINEAR SIGMA MODELS

Robert Coquereaux^{*)}

CERN - Geneva

and

Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 Bures-sur-Yvette, France

ABSTRACT

Geometrical aspects of several classes of σ models are studied. The geometrical meaning of perturbative quantum field theory is discussed in the content of non-linear σ models. Results on the one-loop effective action are recovered and generalized.

*) On leave of absence from the Centre de Physique Théorique, Section 2, CNRS Luminy, Marseille, France.

INTRODUCTION.

The present paper contains two parts which are largely independent.

In the first we analyse the geometry of several global situations which may occur when one has to study the dynamics of a field described locally by a map from space (time) M to a manifold F (usually a homogeneous space). Indeed, the words "non-linear σ models" have been used frequently in the past to refer to several classes of models which are quite distinct and our first aim (in Part 1) is to establish a clear distinction between them.

In Part 2, we select one particular class of models (that we may call the genuine non-linear σ models !) which involve the dynamics of a field described globally by a map φ between two Riemannian manifolds (M,g) and (F,h) ; we then analyse perturbative quantum field theory in this content and show that, at the one-loop order, it involves the study of a linear differential operator acting on sections of a vector-bundle. By using the heat kernel method, it is then easy to get the structure of the divergences which appear at one loop. The results are given for an arbitrary background gravitational field in M and an arbitrary metric in the target space F (which needs not be a homogeneous space). If we choose flat 4-dimensional space-time and take for F a sphere S^d with constant curvature, our results recover and generalize those of [1] (where the case $d=3$ was worked out and applied to low energy phenomenology). Our main observation here is that there is a vector bundle naturally associated with a non-linear σ -model : this fact both provides a simplification of the concepts and shortens calculation thanks to an important mathematical literature on the subject.

As already stated, Parts 1 and 2 are quite independent and the reader interested only in the perturbative aspect of these models can go directly to Part 2.

1. SEVERAL CLASSES OF NON-LINEAR σ -MODELS.

The "non-linear σ models" refer to situations in which we have (at least locally) a field defined on a manifold (space time M) and valued in another one F (usually a Lie group G or a homogeneous space G/H , but not a vector space - hence the words "non-linear"). We will assume that $F = G/H$ and will distinguish 3 kinds of global situations :

Case a: Let P be a right¹⁾ principal fibre bundle (assume it is non-trivial), with base M and structure group G ; let H be a (closed) subgroup of G , let ρ be a left action of G on the homogeneous space $F = G/H$ (ρ is sometimes called "non-linear representation" in the physical literature); we construct the associated fibre bundle $E = P \times_{\rho} F$ which has base M and typical fibre F . Elements u of E can be written $u = z.f = z.g.(g^{-1})f$, with $z \in P$, $f \in F$, (notation which generalizes the notation $v = e_{\mu}.v^{\mu}$ for vectors); the quantity u has components f in the "frame" z . Matter fields will be described by sections of E : they are maps $x \in M \rightarrow \psi(x) = z(x).f(x) \in E$ which, via the choice of a local moving "frame" $z(x)$, to a space-time point x , associate a point $f(x)$ in $F = G/H$; they can therefore be described locally by the field $f(x)$ valued in F . Notice that the structure group G does not act on the points of E (it acts only on their components); it does not act on

¹⁾ Here and below one can replace consistently the words "right" by "left" and $G/H = \{gH/g \in G\}$ by $H \backslash G = \{Hg/g \in G\}$.

the sections of E either. Actually, the only finite dimensional group acting on the point of E (and inducing the identity on the base M) is the group $K \doteq N/H$ where N is the normalizer of H in G : $N = \{n \in G/nH = Hn\}$; it acts globally from the right on E as follows
 $u = z.f = z.(aH) \rightarrow un = z.fn = z.aHn = z.anH$, with $f = aH$. In general H will be non-trivial but we can study the case where $H = \{e\}$, then we will distinguish the following two subcases :

a1). We let ρ be the action of G on G from the left; then E is obviously isomorphic with P itself and is a right principal bundle (the right action of G on E coming from the fact that in this case $N/H = G$) .

a2). We let ρ be the adjoint action of G on G : $\rho(g) k = gkg^{-1}$, then $E = \text{Ad}P$ although locally a product of M and G is not a principal bundle (it is usually called the adjoint bundle) and there is no global G action defined on it. Notice that matter fields entering the Wess-Zumino term are of this type (indeed sections of E are gauge transformations).

Case b. Let Q be a left principal bundle (assume it is non-trivial) with base M and structure group $K = N/H$ where N is the normalizer of the subgroup H of G . Using now the right action of K on G/H we build the associated bundle $\tilde{E} = G/H \times_K Q$ which again has base M and typical fibre $F = G/H$. Elements $\tilde{u}$ of E can be written $\tilde{u} = f.\tilde{z} = fn.n^{-1}\tilde{z}$ with $\tilde{z} \in Q$, $f \in F$, $n \in K$. Matter fields $x \in M \rightarrow \tilde{\psi}(x) = f(x).\tilde{z}(x)$ are sections of E and can again be locally described (via the choice of a local moving frame $\tilde{z}(u)$ in Q) by the field $f(x)$ valued in F . Notice that the structure group

K no longer acts on the points of E (it acts on their components) but the group G acts here globally (from the left $(\tilde{u} = f.\tilde{z} \in E \rightarrow g\tilde{u} = gf.\tilde{z} = (gaH).\tilde{z}$, where $f = aH$). In the particular case where H is trivial ($H = \{e\}$), we are in the principle bundle situation.

Case c. We choose $E = M \times G/H$ a trivial bundle with base M and fibre G/H . Matter fields (sections of E) are of the kind $x \in M \rightarrow \psi(x) = (x, f(x)) \in E$ and can be globally defined by the map $x \in M \rightarrow f(x) \in G/H$. In this case, we have both a global action of G (from the left) and a global action of $N|H$ (from the right).

REMARK 1. Notice that in case a) the gauge group (group of vertical automorphisms of the associated principal bundle) is described locally by maps from M to G , whereas in case b) it is described locally by maps from M to $N|H$. What is meant by "gauge group" depends of course upon the choice we make when we describe in a particular way a given geometric situation. This explains why it is commonly said that "there is a hidden gauge invariance under H "; indeed if we choose to describe maps to G/H by maps to G , there is an arbitrariness in the description which is measured by H (G is a H -principal bundle over G/H).

REMARK 2. In our discussion of cases a,b,c we assumed that the pair of Lie groups (G,H) was somehow given. The discussion would be slightly different if we start with a manifold F which happens to be a homogeneous space for some Lie group ($F = G/H$); indeed, there exist often several Lie groups which are transitive on the same manifold (for instance $S^5 = SO(6)/SO(5) = SU(3)/SU(2)$).

Let us choose the simple example of $F = S^3$, then we can write S^3 as one of the following quotients :

$$G = SU(2) , H = \{e\} \quad \text{then} \quad N|H = SU(2)$$

$$G = SU(2) \times U(1) , H = \text{diag}(U(1)) \quad \text{then} \quad N|H = U(1)$$

$$G = SU(2) \times SU(2) , H = \text{diag}(SU(2)) \quad \text{then} \quad N|H = Z_2 .$$

In all three cases the quotient is S^3 (but only in the first is the quotient a group quotient); now, for each of these three cases, we may have a situation of the type a,b (or c).

REMARK 3. Some material related to this section may be found in Refs. [2] and [3].

2. PERTURBATIVE THEORY.

In all this section, we assume that we are in the trivial case c) of 1, therefore, the field $x \rightarrow \varphi(x)$ is a map from M (typically space-time) to some finite dimensional manifold F (usually with a structure of homogeneous space for a Lie group ($F = G/H$), but we do not need to assume it for the moment).

The space of field configurations is the infinite dimensional manifold

$E = C^\infty(M, F)$ of smooth maps from M to F ; moreover we assume that M and

F are Riemannian manifolds with metrics $g = (g_{\mu\nu})$ on M and $h = (h_{ij})$ on

F . The map φ is "non-linear" (since F is not a vector space); however, its

Jacobian $d\varphi$ at the point $x \in M$ maps the tangent (vector) space $T(M, x)$ to

the tangent (vector) space $T(F, f)$ where $f = \varphi(x)$ - one can write locally

$d\varphi = (\partial_\mu \varphi^i)$ - and using the metrics g and h we can define its norm $\|d\varphi\|^2(x)$ at the point $x \in M$ (in local co-ordinates it is written $g^{\mu\nu}(x) \partial_\mu \varphi^i(x) \partial_\nu \varphi^j(x) h_{ij}(\varphi(x))$). The action associated to a given configuration is usually taken as the real number $S(\varphi) = \int_M \|d\varphi\|^2$. Finding the stationary points of this functional is not easy and only in special cases all critical points are known. In any case, let φ_0 be one chosen critical point (one "vacuum"); it is important to recognize that, in perturbative quantum field theory, one studies "perturbations" around a chosen vacuum (here φ_0): more precisely, the fields one has to integrate over in the functional integral are elements of the tangent space, at φ_0 , to the (infinite dimensional) space E of field configurations. The space $E = C^\infty(M, F)$ of field configurations is a smooth infinite dimensional manifold and carries also a metric structure (more about it below). If φ is near φ_0 , there is a geodesic (in E) from φ to φ_0 and we formally expand $S(\varphi) - S(\varphi_0)$ in powers of σ_0 where σ_0 is the tangent vector to this geodesic at φ_0 and therefore an element of the infinite dimensional vector space $T(C^\infty(M, F), \varphi_0)$ tangent to the space of field configurations at the point φ_0 (here the "point" is a map). Therefore, the quantum fields describing the fluctuations around the vacuum φ_0 are always linear fields (even when φ_0 is of non-linear nature). The conceptual distinction between the space of field configurations E and the space of fluctuations around a critical point of the action may easily be forgotten when E is a vector space or an affine space (like in gauge theory, but it is of crucial importance in our case. The space of quantum fluctuations at φ_0 is therefore the (infinite dimensional) vector space $T(E, \varphi_0) = T(C^\infty(M, F), \varphi_0)$ (it is the fibre, at φ_0 , of the tangent bundle $T(C^\infty(M, F))$). The main observation is now that this space $T(E, \varphi_0)$ is isomorphic with the space $C^\infty(\varphi_0^*(TF))$ of smooth sections of the pull back, under φ_0 , of the tangent bundle to F

(the vector bundle $\varphi_0^*(TF)$ is built by putting, "above" $x \in M$, the tangent space to F at $\varphi_0(x)$)²⁾; in plain terms, this means that we may describe a quantum fluctuation at φ_0 by a map $\sigma(x) (= \sigma_0^k(x))$ valued in the tangent space to F at $\varphi_0(x)$ (a finite dimensional vector space).

The metric at the point φ of the infinite dimensional manifold E is described as follows: let σ_1 and σ_2 be two vectors of the tangent space $T(E, \varphi)$; we have to define the scalar product $\langle \sigma_1, \sigma_2 \rangle_\varphi$. We know that σ_1 and σ_2 may be considered as sections of the vector bundle $\varphi^*(TF)$, this bundle is equipped with a fibre metric ($x \rightarrow h_{ij}(\varphi(x))$) and we define

$$\langle \sigma_1, \sigma_2 \rangle_\varphi = \int_M \sigma_1^i \cdot \sigma_2^j \cdot h_{ij}(\varphi(x)) d\text{vol}(M).$$

Let us remember that forms travel under pull back, therefore the vector bundles $\varphi_0^*(TF)$ is automatically equipped with a connection $A_\mu = \partial_\mu \varphi_0^i \Gamma_i$, which is the pull back under φ_0 of the Levi-Civita connection Γ_i on the target space (we could as well choose a linear connection in F , with non-zero torsion). A_μ is therefore a one-form on M valued in the endomorphisms of the fibre: $A_\mu = (A_{\mu l}^k)$ with $A_{\mu l}^k(x) = \partial_\mu \varphi_0^i \cdot \Gamma_{i l}^k(\varphi_0(x))$. The curvature $F_{\mu\nu}$ of this connection (a two-form on M valued in the endomorphisms of the fibre) is, without surprise, the pull back under φ_0 of the curvature two-form $R_{ij} (= R_{ij l}^k)$ of the target space: $F_{\mu\nu} = \partial_\mu \varphi_0^i \partial_\nu \varphi_0^j R_{ij}$ (or $F_{\mu\nu l}^k = \partial_\mu \varphi_0^i \partial_\nu \varphi_0^j R_{ij l}^k$). Besides, we get also an endomorphism $E (= E_i^k)$ of the vector bundle $\varphi_0^*(TF)$ by pulling back the curvature tensor $R_{ij l}^k$ on the form indices j and l and contracting with the metric g of M : $E_i^k = g^{\mu\nu} \partial_\mu \varphi_0^j \partial_\nu \varphi_0^l R_{ij l}^k$. Notice that we could also consider the Ricci tensor $\rho_{jl} = R_{ij l}^i$ of F and pull it back by φ_0 to get

2)

This vector bundle $\varphi_0^*(TF)$ is actually associated with the principal bundle $\varphi_0^*(OF)$ which is the pull back, via φ_0 , of the orthogonal frame bundle OF of F (the space of all orthonormal frames of F for the metric h). Also the connection A is associated with a principal connection in $\varphi_0^*(OF)$ obtained by pulling back a principle connection in OF (the Levi-Civita if there is no torsion).

a covariant tensor $r_{\mu\nu} = \partial_\mu \varphi^i \partial_\nu \varphi^j g_{ij}$ on M ; finally, notice that the trace of the endomorphism E is equal to the trace of $r(h^{ik} E_{ik} = g^{\mu\nu} r_{\mu\nu})$. The equation that one gets (see for instance [4]) for the fluctuation $\sigma(x)$ (from now on we drop the subindex o of σ) is $(-\nabla^2 + E)\sigma = 0$ where³⁾

$$\nabla_\mu \sigma^k = \partial_\mu \sigma^k + A_\mu^k{}_l \cdot \sigma^l.$$

This equation may be obtained from the Lagrangian density

$$-\frac{1}{2} \nabla_\mu \sigma^k(x) \nabla^\mu \sigma^l(x) h_{kl}(\varphi_o(x)) - \frac{1}{2} \sigma^k(x) E_{kl}(x) \sigma^l(x)$$

which is the second variation of the action evaluated at the stationary point φ_o .

The effective action is given formally at the first non-trivial order (one loop) by the logarithm of the determinant of $D \doteq -\nabla^2 + E$; the method is then quite standard [5][6] (we could also compute one-loop Feynman graphs): we write this logarithm as a Mellin's transform $\frac{i}{2} \text{Log det } D = \frac{-i}{2} \int_0^\infty \frac{d\lambda}{\lambda} \text{Tr } e^{-\lambda D}$, and in order to "localize" the determinant, we use the heat kernel method [7] [8][9][10] to make an expansion of $\theta(\lambda) = \text{Tr } e^{-\lambda D}$ when $\lambda \rightarrow 0_+$. One finds

$$\theta(\lambda) \underset{\lambda \rightarrow 0}{\sim} (4\pi\lambda)^{-d/2} (a_0 + a_1 \lambda + a_2 \lambda^2 + \dots), \quad d = \dim M$$

where

$$a_i = \text{Tr} \int_M u_i(x, x) d\text{vol}(x) \quad (\text{the trace being taken in the fibre}).$$

The replacing of the expansion of $\theta(\lambda)$ in the Mellin transform produces of course (ultra-violet) divergences: $\det D$ has poles at $d = 0, 2, 4, \dots$ one finds :

3) In some cases it is convenient to choose a normal co-ordinate system in F which allows us to get rid of the $\Gamma_i^k{}_l$ and therefore also of the $A_\mu^k{}_l$.

$$\frac{i}{2} \text{Log det } D = - \int_M \text{dvol} \left\{ \frac{1}{d} \text{Tr } u_0 + \frac{1}{4\pi(d-2)} \text{Tr } u_1 + \frac{1}{(4\pi)^2(d-4)} \text{Tr } u_2 + \dots \right\}$$

It is now important to remark that we are precisely in the most general framework studied in [10] (namely the case of a smooth vector bundle over a Riemannian manifold M and of a second-order differential operator D with leading symbol given by the metric tensor). Moreover D is already written as the rough Laplacian plus an endomorphism acting on zero-forms (σ^i) valued in a vector bundle. Let us give the first three coefficients of this expansion (the reader interested in $\text{Tr } u_3$ which contains 43 terms may refer directly to [10, Th. 4.3]). We call $R_{\mu\nu}^{\rho\sigma}$ the Riemann tensor of M , $\rho_{\nu\sigma} = R_{\mu\nu}^{\mu}{}_{\sigma}$ the Ricci tensor of M and $\tau = g^{\mu\sigma} \rho_{\mu\sigma}$ the scalar curvature of M ⁴⁾. Then one gets (we sum over repeated indices) :

$$u_0 = 1$$

$$u_1 = \frac{1}{6}(\tau - 6E)$$

$$u_2 = \frac{1}{360} (+12\tau_{;\mu\mu} + 5\tau^2 - 2\rho^2 + 2R^2 + 60\tau E + 180E^2 - 60\tau E_{;\mu\mu} + 30F^2) .$$

where $\rho^2 = \rho_{\mu\nu}^{\rho\mu\nu}$, $R^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$, $F^2 = F_{\mu\nu}^{\mu\nu}$.

All the above quantities⁵⁾ are endomorphism valued, i.e. for instance

$$\tau = \tau(\text{unit matrix}) : \tau_k^i = \tau \cdot \delta_k^i \quad \text{or} \quad (E^2)_k^i = E_j^i E_k^j .$$

Several of the above quantities (namely $\tau_{;\mu\mu}$ and $E_{;\mu\mu}$) vanish upon integration (they are total derivative) and we discard them. We are left with

4) In phenomenological applications of σ -models defined on a flat Minkowski space-time, these three terms are of course equal to zero.

5) The reader should be warned that, according to one's favourite conventions, the terms linear in the Riemann tensor and/or in the endomorphism E may enter with a different sign (actually our signs for E and R are opposite to those of [10]).

$$\text{Tr} \int_M u_0 = \text{Tr} \int_M 1 \cdot d\text{Vol} = \dim F \cdot \text{Vol}(M)$$

$$\text{Tr} \int_M u_1 = \text{Tr} \int_M \frac{1}{6} (\tau - 6E) d\text{Vol}$$

$$\text{Tr} \int_M u_2 = \text{Tr} \int_M \frac{1}{360} (5\tau^2 - 2\rho^2 + 2R^2 + 60E + 180E^2 + 30F^2) d\text{Vol}.$$

If M is flat (like in particle physics applications) one gets simply

$$\text{Tr} \int_M u_1 = \text{Tr} \int_M (-E) d\text{Vol}$$

$$\text{Tr} \int_M u_2 = \text{Tr} \int_M \left(\frac{1}{2} E^2 + \frac{1}{12} F^2 \right) d\text{Vol}.$$

These relations have been computed directly, for instance in [1, formula 8.8] or [4, formula 4.38]. From the definition of E and F and using co-ordinates, we find that

$$\text{Tr} E = \partial_\mu \varphi_o^j \partial^\mu \varphi_o^l R_{ij}{}^i{}_l$$

$$\text{Tr} E^2 = \partial_\mu \varphi_o^j \partial^\mu \varphi_o^l R_{ij}{}^k{}_l \cdot \partial_\nu \varphi_o^m \partial^\nu \varphi_o^n R_{km}{}^i{}_n$$

$$\text{Tr} F^2 = F_{\mu\nu}{}^k{}_l F^{\mu\nu l}{}_k = \partial_\mu \varphi_o^i \partial_\nu \varphi_o^j R_{ij}{}^k{}_l \cdot \partial^\mu \varphi_o^{i'} \partial^\nu \varphi_o^{j'} R_{i'j'}{}^l{}_k$$

Since we have in mind applications of these theories with a 4-dimensional space-time M , (and not a two-dimensional one where renormalization works better !), the presence of a pole at $d=4$ in the formal expansion of $\log \det D$ and the fact that its residue contains terms which were not present in the Lagrangian shows that the model is not renormalizable in 4 dimensions; however, there is a standard trick which allows us to make sense of the theory at one loop :

one just adds the term $a \cdot \text{Tr} \mu_2$ to the Lagrangian with a bare coupling constant a whose infinite part will precisely compensate at one loop the infinities produced by the divergent term of $\log \det D$ (of course the same procedure should be followed at the two-loop level etc.) .

Till now, we have not made a particular hypothesis on the target manifold F , besides the fact that it was a finite dimensional Riemannian manifold (without boundary); there are simplifications if we make a further hypothesis (for instance, if the Riemannian manifold (F, h) is Ricci flat ($\rho_{ij} = 0$) then $\text{Tr } E = 0$) . Quantities like $\text{Tr } E^2$ or $\text{Tr } F^2$ involve the tensor R_{ij}^k which should be evaluated at the point $\varphi_0(x) \in F$; it is therefore a big simplification to assume that the target space is endowed with a metric which has isometries; let us then suppose that F is a homogeneous space for a Lie group ($F = G/H$) and that F is endowed with a G -invariant metric. If $F = G$, (i.e. $H=e$) then a G -invariant metric is only specified by the choice of an arbitrary scalar product h_{ij} in the Lie algebra of G and the Riemann tensor can only be expressed in terms of h_{ij} (we do not need the derivatives) and of the structure constants of G (see for instance [3 formula 2.11] for an explicit formula). If $F = G/H$, a G -invariant metric is specified by the choice of an $\text{Ad}.H$ invariant scalar product in the tangent space at the origin of G/H and again there is a simple formula for the curvature tensor (see for instance [3 formula 3.4 (6)] . The formulae simplify further if we choose a normal metric on G/H ; this will be the case for instance, if G/H is isotropy irreducible and in particular if it is irreducible symmetric : there is only one (up to scale) G -invariant metric in this case (the normal metric) with the further simplification that F is now automatically an Einstein space ($\rho_{ij} = c \cdot h_{ij}$) . Finally,

if we take for F a space of constant curvature (for instance a "round" sphere S^d , formulae simplify dramatically and one gets, up to scale (radius k) $R_{ijkl} = \frac{1}{k^2}(h_{ik}h_{jl} - h_{il}h_{jk})$; in this very particular case, let us choose a local orthonormal frame, then

$$k^2 \cdot F_{\mu\nu}^k = \partial_\mu \varphi_o^i \partial_\nu \varphi_o^j (\delta_{ij}^k - \delta_{il} \delta_{jk}^l) = \partial_\mu \varphi_o^k \partial_\nu \varphi_o^l - \partial_\mu \varphi_o^l \partial_\nu \varphi_o^k$$

$$k^2 \cdot E_i^k = \partial_\mu \varphi_o^j \partial^\mu \varphi_{oj} \delta_i^k - \partial_\mu \varphi_o^k \partial^\mu \varphi_{oi}$$

$$K^2 \cdot \rho_{jl} = (d-1)h_{jl} \quad , \quad d = \dim F \quad , \quad \text{Tr } \rho_{jl} = \frac{d(d-1)}{k^2} \quad .$$

Then, calling simply $\partial_\mu \varphi \partial_\nu \varphi \doteq \partial_\mu \varphi^i \partial_\nu \varphi^j \delta_{ij}$ (from now we drop the subscript o), we get :

$$\text{Tr } F^2 = (2(\partial_\mu \varphi \partial_\nu \varphi) \partial^\mu \varphi \partial^\nu \varphi - 2(\partial_\mu \varphi \partial^\mu \varphi)^2)/k^4$$

$$\text{Tr } E = ((d-1) \partial_\mu \varphi \partial^\mu \varphi)/k^2$$

$$\text{Tr } E^2 = ((d-2) (\partial_\mu \varphi \partial^\mu \varphi)^2 + (\partial_\mu \varphi \partial_\nu \varphi) (\partial^\mu \varphi \partial^\nu \varphi))/k^4 \quad .$$

The last three formulae valid for a σ -model valued in a sphere S^d endowed with its maximally symmetric metric coincide with those of [1, formula 9.4] if we set $d=3$ and $k=1$; indeed these authors computed directly the counterterms to the effective action in a non linear σ -model valued in $G/H = SU(2) \times SU(2)/\text{diag}(SU(2))$, which of course turns out to be the sphere S^3 and is a space of constant curvature if we choose on it the normal metric

inherited from G (the isometry group is $SU(2) \times SU(2)$)⁶⁾. Still working with $F = S^d$, and with a flat space-time we see that the divergent term occurring at the one-loop level is proportional to

$\frac{1}{6}(3d-7)(\partial_\mu \varphi \partial^\mu \varphi)^2 + \frac{2}{3}(\partial_\mu \varphi \partial_\nu \varphi)(\partial^\mu \varphi \partial^\nu \varphi)$ ⁷⁾. To make the theory renormalizable at the one-loop level, one has just to add these two terms to the Lagrangian :

$$L = \frac{f^2}{2} \partial_\mu \varphi \partial^\mu \varphi + l_1^B (\partial_\mu \varphi \partial^\mu \varphi)^2 + l_2^B (\partial_\mu \varphi \partial_\nu \varphi)(\partial^\mu \varphi \partial^\nu \varphi)$$

with bare coupling constants⁸⁾ $l_1^B = l_1 + \frac{1}{6}(3d-7)\frac{1}{\epsilon}$, $l_2^B = l_2 + \frac{2}{3}\frac{1}{\epsilon}$, $\frac{1}{\epsilon}$ being a divergent quantity (proportional to $\frac{1}{n-4}$). The theory makes sense at one loop for any value of the renormalized quantities l_1 and l_2 . In low energy phenomena and taking $d=3$, one interprets F as the pion decay constant and, according to [1], one may relate the phenomenological parameters l_1 and l_2 to the D-wave scattering length in $\pi-\pi$ scattering. It could also be possible to make some phenomenology out of the case $d=5$, in view of the fact that $S^5 = SU(3)/SU(2)$ and that the tangent space to S^5 at the origin carries a real irreducible representation which, on the complex domain, corresponds to the decomposition of the adjoint of $SU(3)$ into the adjoint of $SU(2)$ and the representation $[2] + [\bar{2}] + [1]$ which it is usual to associate with the multiplet

6) Actually, the authors of [1] studied the model by choosing a field valued in the linear representation of $SU(2) \times SU(2)$ (locally $SU(4)$) on R^4 and constrained by a condition of norm equal to 1.

7) Given this combination for a particular value of d , it is not obvious it comes from the second coefficient of the heat expansion for a differential operator acting in a vector bundle !

8) Notice that the metric $h_{ij}(y)$ of F at the point $y \in F$ is fixed (it is like a external field) and that it is conceptually dangerous to say that it is "modified" by quantum fluctuations (by writing something like

$h_{ij}^{\text{new}}(\varphi_0(x)) = h_{ij}^{\text{old}}(\varphi_0(x)) [1 + (\text{cte}) \partial_\mu \varphi_0 \partial^\mu \varphi_0]$).

$K^+, K^0, \bar{K}^0, K^-, \eta$.

Let us end this section by suggesting a slight generalization of the previous study which could be of some use in particle physics. We start with the observation that the perturbative study of a non-linear sigma model boils down (at the one-loop level) to the study of the differential operator $\Delta^2 + E$ acting on the sections of the vector bundle $\varphi_0^*(TF)$, E being a well-defined endomorphism of E . The quantum fields (quantum fluctuations) are then scalar fields from the space-time point of view. We may have the idea of keeping the same geometrical data $(g_{\mu\nu}, h_{ij}, \varphi_0)$ but now of studying more general differential operators acting on one-forms (if we want to apply the results to vector-like particles), two-forms... or even spinors valued in the vector bundle $\varphi_0^*(TF)$. The simplest possibility is to add the endomorphism E to the Laplace-Beltrami operator Δ acting on P -forms (or to add the endomorphism E to the spinor Laplacian (square of the Dirac operator)) acting on $\varphi_0^*(TF)$ -valued spinors on M . In the simplest case of 1-forms, we have for example the equation $(\Delta + E) \sigma_\mu = 0$, where the Laplace-Beltrami operator is related to the rough Laplacian $-\nabla^\mu \nabla_\mu$ by the Weitzenböck formula :

$$\Delta \sigma_\mu^i = -\nabla^\lambda \nabla_\lambda \sigma_\mu^i + \rho_\mu^\nu \sigma_\nu^i - F_\mu^{\nu i} \sigma_\nu^j$$

and where the covariant derivative is

$$\nabla_\mu \sigma_\nu^i = \partial_\mu \sigma_\nu^i + A_\mu^i{}_j \sigma_\nu^j + \Gamma_{\mu\nu}^\lambda \sigma_\lambda^i .$$

Assuming a flat space-time, we get the simple equation

$$-\nabla^\lambda \nabla_\lambda \sigma_\mu^i - \partial_\mu \varphi^m \partial^\nu \varphi^n R_{mn}{}^i{}_j \sigma_\nu^j + (\partial_\mu \varphi^j \partial^\mu \varphi^l) R_{kj}{}^i{}_l \sigma_\nu^k = 0 .$$

There is a further simplification if F is a space of constant curvature :

$$-\nabla^\lambda \nabla_\lambda \sigma_\mu^i - (\partial_\mu \varphi^i \partial^\nu \varphi_j - \partial^\nu \varphi^i \partial_\mu \varphi_j) \sigma_\nu^j + (\partial_\nu \varphi^l \partial^\nu \varphi_l) \sigma_\mu^i - (\partial_\nu \varphi^i \partial^\nu \varphi_k) \sigma_\mu^k = 0 .$$

This kind of equation describes the interaction of particles of different spins⁹⁾ and could lead to phenomenologically interesting results.

ACKNOWLEDGEMENTS

I would like to thank Prof. E. de Rafael who triggered my interest in this subject, Prof. M. Jacob at CERN for his kind hospitality and IHES where this work was typed.

9) Actually, our vector bundle not only carries a fibre metric and a connection but the base (space-time M) is endowed with a linear connection and we may study arbitrary tensor fields (not only p -forms) valued in $\varphi_0^*(TF)$.

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