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DIRAC AND WEYL FERMIONS COUPLED TO TWO-DIMENSIONAL SURFACES: DETERMINANTS

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ABSTRACT

The Dirac and Weyl fermion determinants are evaluated following the method indicated by A. Polyakov, P. Wiegman and E. Witten, in the case of fermions coupled to two-dimensional surfaces embedded in R^d .

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1. - INTRODUCTION

This letter generalizes to arbitrary dimensions a previous article¹⁾ which describes the evaluation of the determinants associated with Dirac and Weyl fermions coupled to the extrinsic geometry of a two-dimensional surface embedded in a three-dimensional Euclidean space. This follows the line of thought indicated by Polyakov and Wiegman, as well as Witten²⁾, according to which two-dimensional fermion determinants are computable in closed form when there are as many anomalies as function parameters in the Lagrangian. In all the examples of this type which have been treated so far, the determinants turn out to be expressible in terms of Wess-Zumino actions. The same phenomenon is met in the present example, where the answer for the Weyl fermions is given in terms of a Liouville action, together with a Wess-Zumino term whose integrand is the Chern-Simons density for the group $SO(2) \times SO(d-2)$ evaluated at a connection in $so(2) \oplus so(d-2)$. Let us point out that in the case $d = 3$ this Chern-Simons density is interpretable in terms of a Hopf-invariant density for the surface under consideration¹⁾.

Since the action considered here is the quadratic fermionic part of the Green-Schwarz superstring⁴⁾, the results presented here may be interesting from the point of view of an eventual covariant formulation of superstring theories in which left- and right-handed components are treated asymmetrically⁵⁾. The non-Abelian bosonization carried through here may be useful also for the investigation of the relationship between old (NSR) and new (GS) superstring formalisms⁶⁾.

2. - NOTATIONS

We consider the Dirac action induced on a two-dimensional surface Σ without a boundary embedded in a d -dimensional Euclidean space:

$$\begin{aligned} \int_{\Sigma} L d^2\xi &= \int_{\Sigma} d^2\xi \sqrt{g} \frac{i}{2} g^{\alpha\beta}(\xi) (\gamma_{\alpha}(\xi) \vec{\partial}_{\beta} - \vec{\partial}_{\beta} \gamma_{\alpha}(\xi)) \psi(\xi) \\ &= \int_{\Sigma} d^2\xi i [\sqrt{g} g^{\alpha\beta} \bar{\psi} \gamma_{\alpha} (\partial_{\beta} + R_{\beta}) \psi](\xi) \end{aligned} \quad (1)$$

The notations in Eq. (1) are as follows: $\xi = \{\xi^{\alpha}; \alpha = 1, 2\}$ parametrizes Σ ,

$$g_{\alpha\beta} = \partial_{\alpha} \vec{X} \cdot \partial_{\beta} \vec{X} \quad (2)$$

is the metric induced on Σ by the standard Euclidean metric $\{\delta_{mn}\}$ in R^d ,

$$\gamma_\alpha = \partial_\alpha \vec{X} \cdot \vec{\sigma} \quad (3)$$

are the Dirac matrices induced on Σ by the flat Dirac matrices $\vec{\sigma}$ in R^d , and ϕ lies in the spinor representation of $\text{Spin}(d)$.

R_α is defined by

$$R_\alpha = \frac{1}{2} \gamma^\beta \nabla_\beta \gamma_\alpha \quad (4)$$

where ∇ is the covariant derivative with respect to g . R is equivalently defined by

$$\frac{1}{2} \nabla_\alpha \gamma^\alpha = \gamma^\alpha R_\alpha \quad (5)$$

by virtue of the Gauss-Weingarten³⁾ equations

$$\begin{aligned} \nabla_\alpha \gamma_\beta &= h_{\alpha\beta}^i n_i \\ \partial_\alpha n^i &= -h_{\alpha\beta}^i \gamma^\beta + H_\alpha^{ij} n_j \end{aligned} \quad (6)$$

where the Latin indices are raised and lowered with the flat Euclidean metric in R^d ,

$$n^i = \vec{n}^i \cdot \vec{\sigma} \quad i = 1, \dots, d-2 \quad (7)$$

denotes an orthonormal set orthogonal to Σ ,

$$\begin{aligned} h_{\alpha\beta}^i &= \partial_\alpha \vec{X} \cdot \partial_\beta \vec{n}^i = h_{\beta\alpha}^i \\ H_\alpha^{ij} &= \vec{n}^j \cdot \partial_\alpha \vec{n}^i = -H_\alpha^{ji} \end{aligned} \quad (8)$$

Action (1) is invariant under the gauge transformations:

$$\partial_\alpha X^i(\xi) \rightarrow \Lambda^i_j(\xi) \partial_\alpha X^j(\xi)$$

$$\psi(\xi) \rightarrow S(\Lambda) \psi(\xi), \quad \bar{\psi}(\xi) \rightarrow \bar{\psi}(\xi) S^{-1}(\Lambda) \quad (9)$$

where Λ is a local $\text{Spin}(d)$ operation constrained by the requirement that there exist $X'^j(\xi)$ such that

$$\Lambda^i_j(\xi) \partial_\alpha X^j(\xi) = \partial_\alpha X'^j(\xi) \quad (10)$$

namely

$$\partial_\alpha \Lambda^i_j(\xi) \partial_\beta X^j(\xi) - \partial_\beta \Lambda^i_j(\xi) \partial_\alpha X^j(\xi) = 0 \quad (11)$$

It is convenient to rewrite action (1) in a different way, as follows.

Introduce a zweibein $e^a_\alpha d\xi^\alpha$, $a = 1, 2$ with

$$e^a_\alpha e_{a\beta} = g_{\alpha\beta} \quad (12)$$

$$\gamma_a = e^\alpha_a \gamma_\alpha \quad (13)$$

Consider the $\text{Spin}(d)$ operation S which rotates γ^a, n^i into the standard Dirac matrices $\vec{\sigma}$:

$$\begin{aligned} S^{-1} \sigma_a S &= \gamma_a(\xi) & a &= 1, 2 \\ S^{-1} \sigma_i S &= n_i(\xi) & i &= 1, \dots, d-2. \end{aligned} \quad (14)$$

One has

$$\begin{aligned} S^{-1} \partial_\alpha S &= \frac{1}{4} \left(\gamma^a \partial_\alpha \gamma_a + n^i \partial_\alpha n_i \right) \\ &= R_\alpha + H_\alpha^{ij} \frac{[n_i, n_j]}{8} + \Gamma_\alpha^{ab} \frac{[\gamma_a, \gamma_b]}{8} \end{aligned} \quad (15)$$

where

$$\Gamma_{\alpha}^{ab} = -e_{\beta}^a \nabla_{\alpha} e^{\beta b} \quad (16)$$

is the Levi-Civita spinor connection on Σ for the induced metric. Substituting Eq. (15) into Eq. (1), and recalling Eqs. (4) and (14), we get:

$$\begin{aligned} L = i \sqrt{g} g^{\alpha\beta} \bar{\psi} S^{-1} & \left[e_{\alpha}^a \sigma_a \left(\partial_{\beta} - \Gamma_{\beta}^{bc} \frac{[\sigma_b, \sigma_c]}{8} - \right. \right. \\ & \left. \left. - H_{\beta}^{ij} \frac{[\sigma_i, \sigma_j]}{8} \right) \right] S \psi \end{aligned} \quad (17)$$

Note that the Dirac operator deduced from Eq. (17) anticommutes with

$$\gamma_3(\xi) = i \gamma_1(\xi) \gamma_2(\xi) \quad (18)$$

which allows us to introduce chiral fermions according to

$$\psi_{\mathbf{R}} = \frac{1}{2} (1 \pm \gamma_3) \psi \quad (19)$$

One can use both reparametrization invariance and the ambiguity on S up to $SO(2) \times SO(d-2)$ gauge transformations to choose the conformal gauge, namely

$$e_{\alpha}^a = \rho^{-\frac{1}{2}} \delta_{\alpha}^a \quad (20)$$

and Lorentz-type gauges, namely

$$g^{\alpha\beta} \partial_{\alpha} H_{\beta}^{ij} = g^{\alpha\beta} \partial_{\alpha} \Gamma_{\beta}^{ab} = 0 \quad (21)$$

We then get for the right Weyl operator

$$\begin{aligned} D_R &= i \gamma^+ S^{-1} \left(\rho^{-\frac{1}{2}} \partial_+ - \rho^{-\frac{1}{2}} \Gamma_+^{ab} \frac{[\sigma_a, \sigma_b]}{8} - \right. \\ &\quad \left. - \rho^{-\frac{1}{2}} H_+^{ij} \frac{[\sigma_i, \sigma_j]}{8} \right) S \\ &= i \gamma^+ S^{-1} g^{-1} \left(\rho^{-\frac{1}{2}} \partial_+ - \rho^{-\frac{1}{2}} \Gamma_+^{ab} \frac{[\sigma_a, \sigma_b]}{8} \right) g S \\ &\stackrel{\text{Def.}}{=} \gamma^+ \bar{D}_R \end{aligned} \quad (22)$$

where we use the parametrization

$$H_+^{ij} \frac{[\sigma_i, \sigma_j]}{g} = -g^{-1} \partial_+ g$$

$$g \in SO(d-2) \quad (23)$$

and the + index refers to

$$\xi^+ = \frac{\xi^1 + i \xi^2}{\sqrt{2}}$$

3. - COMPUTATIONS

The effective action of right Weyl fermions is formally

$$W = \ln \det \mathcal{D}_R = \ln \det \gamma^+ \bar{\mathcal{D}}_R$$

$$= \ln \det \gamma^+ + \ln \det \bar{\mathcal{D}}_R \quad (24)$$

We regularize W by first dropping $\ln \det \gamma^+$ and working at the variational level:

$$\delta W = \text{tr} \delta \bar{\mathcal{D}}_R \bar{\mathcal{D}}_R^{-1} = \text{tr} \delta \bar{\mathcal{D}}_R \bar{\mathcal{D}}_L \bar{\mathcal{D}}_L^{-1} \bar{\mathcal{D}}_R^{-1}$$

$$= \text{tr} \int_{\epsilon}^{\infty} dt \delta \bar{\mathcal{D}}_R \bar{\mathcal{D}}_L e^{-t \bar{\mathcal{D}}_R \bar{\mathcal{D}}_L} \quad (25)$$

$\bar{\mathcal{D}}_{R,L}$ and W are functions of S, g, ρ , and

$$\delta \bar{\mathcal{D}}_R = \delta S^{-1} S \bar{\mathcal{D}}_R + S^{-1} \delta g^{-1} g S \bar{\mathcal{D}}_R +$$

$$+ \delta \ln \rho^{-\frac{1}{2}} \bar{\mathcal{D}}_R + \bar{\mathcal{D}}_R S^{-1} g^{-1} \delta g S$$

$$+ \bar{\mathcal{D}}_R S^{-1} \delta S \quad (26)$$

where we consider variations of δg , preserving the gauge condition [Eq. (21)].

Substituting Eq. (26) into Eq. (25) and integrating over t, we obtain

$$\delta W = \text{tr} \left\{ S^{-1} \delta S \left(e^{-\epsilon \bar{D}_L \bar{D}_R} e^{-\epsilon \bar{D}_R \bar{D}_L} \right) \right. \\ \left. + \delta \ln f^{-\frac{1}{2}} e^{-\epsilon \bar{D}_R \bar{D}_L} \right. \\ \left. + g^{-1} \delta g \left(e^{-\epsilon S^{-1} \bar{D}_L \bar{D}_R S} e^{-\epsilon S^{-1} \bar{D}_R \bar{D}_L S} \right) \right\} \quad (27)$$

where $\lim_{\epsilon \rightarrow 0} e^{-\epsilon \bar{D}_L \bar{D}_R}$ is determined by the relevant Seeley coefficients:

$$\lim_{\epsilon \rightarrow 0} e^{-\epsilon \bar{D}_R \bar{D}_L} = \frac{\rho}{\epsilon} + \frac{1}{24\pi} \partial_+ \partial_- \ln f + \\ + \frac{1}{32\pi} (\partial_- \Gamma_+^{ab} - \partial_+ \Gamma_-^{ab}) [\gamma_a, \gamma_b] + \\ + \frac{1}{32\pi} (\partial_- H_+^{ij} - \partial_+ H_-^{ij} - 2[H_+, H_-]^{ij}) [n_i, n_j] \quad (28)$$

Similarly, $\lim_{\epsilon \rightarrow 0} e^{-\epsilon S^{-1} \bar{D}_L \bar{D}_R S}$ is obtained by substituting into Eq. (28) σ_a, σ_i instead of γ_a, n_i respectively. To sum up, we have:

$$\delta W = \text{tr} \left\{ S^{-1} \delta S [\gamma_a, \gamma_b] \frac{1}{16\pi} (\partial_- \Gamma_+^{ab} - \partial_+ \Gamma_-^{ab}) + \right. \\ \left. + S^{-1} \delta S [n_i, n_j] \frac{1}{16\pi} (\partial_- H_+^{ij} - \partial_+ H_-^{ij} - 2[H_+, H_-]^{ij}) + \right. \\ \left. + \delta \ln f^{-\frac{1}{2}} \frac{1}{24\pi} (\partial_+ \partial_- \ln f + \mu^2 \rho) + \right. \\ \left. + g^{-1} \delta g [\sigma_i, \sigma_j] \frac{1}{16\pi} (\partial_- H_+^{ij} - \partial_+ H_-^{ij} - 2[H_+, H_-]^{ij}) \right\} \quad (29)$$

Recalling Eq. (15), a solution of Eq. (29) can be found in the form

$$W(S, g, \rho) = W_1(\rho) + W_2(S) + W_3(g) \quad (30)$$

with

$$\delta W_1(\rho) = \text{tr} \left\{ \delta \ln \rho^{-\frac{1}{2}} \frac{1}{24\pi} (\partial_+ \partial_- \ln \rho + \mu^2 \rho) \right\} \quad (31a)$$

$$\begin{aligned} \delta W_2(S) = \text{tr} \left\{ S^{-1} \delta S [\gamma_a, \gamma_b] \frac{1}{16\pi} (\partial_- T_+^{ab} - \partial_+ T_-^{ab}) \right. \\ \left. + S^{-1} \delta S [n_i, n_j] \frac{1}{16\pi} (\partial_- H_+^{ij} - \partial_+ H_-^{ij} - 2[H_-, H_+]^{ij}) \right\} \end{aligned} \quad (31b)$$

$$\delta W_3(g) = \text{tr} \left\{ g^{-1} \delta g [\sigma_i, \sigma_j] \frac{1}{16\pi} (\partial_- H_+^{ij} - \partial_+ H_-^{ij} - 2[H_-, H_+]^{ij}) \right\} \quad (31c)$$

Integration of Eq. (31a) yields for $W_1(\rho)$ the Liouville action

$$W_1(\rho) = \frac{2^{\frac{d}{2}}}{96\pi} \int d^2 \xi \left[(\partial_+ \ln \rho)(\partial_- \ln \rho) + \mu^2 \rho \right] \quad (32)$$

It is easy to integrate Eq. (31c) for W_3 , with the result

$$\begin{aligned} W_3(g) = \frac{1}{2\pi} \text{tr} \left(\frac{1}{2} \int_{\Sigma} d^2 \xi \partial_a g^{-1} \partial_a g + \frac{i}{3} \int_{\{Z_t\}} (g^{-1} dg)^3 \right) \\ - \frac{2^{\frac{d}{2}}}{32\pi} \text{tr} \int_{\Sigma} d^2 \xi H_+^{ij} H_-^{ij} \end{aligned} \quad (33)$$

In Eq. (33), $\{\Sigma_t\}$ represents a three-dimensional manifold bounded by Σ . Part of the integration goes back to Polyakov and Wiegman, and we have also used the solution of the gauge condition Eq. (21) via the Poincaré lemma.

The integration of Eq. (31b) for $W_2(S)$ requires using the Gauss-Weingarten formulae

$$\begin{aligned} d\gamma^a &= \Gamma^a_b \gamma^b + h^a_i n^i \\ dn^i &= -h^i_a \gamma^a + H^i_j n^j \end{aligned} \quad (34)$$

where

$$\begin{aligned} \Gamma^{ab} &= \Gamma^{\alpha}_{ab} d\xi^\alpha \\ h^i_a &= h^i_{a\alpha} d\xi^\alpha \end{aligned} \quad (35)$$

with

$$\begin{aligned} h_{a\alpha}^i &= e_a^\beta h_{\beta\alpha}^i \\ H_{j\alpha}^i &= e_\alpha^a H_{ja}^i \end{aligned} \quad (36)$$

The differentials of Γ , h and H are constrained by the compatibility conditions which follow from

$$d^2 = 0 \quad (37)$$

namely

$$\begin{aligned} d\Gamma^{ab} + h^{ai} h_i^b &= 0 \\ dH^{ij} - H^{ik} H_k^j + h^{ai} h_a^j &= 0 \\ dh_i^a - \Gamma^{ab} h_{bi} - h_j^a H^{ji} &= 0 \end{aligned} \quad (38)$$

(the Peterson-Codazzi equations). Similarly, compatible variations of S are parametrized according to

$$\begin{aligned} \delta\gamma^a &= T^a_b \gamma^b + b^a_i n^i \\ \delta n^i &= -b^i_a \gamma^a + K_j^i n^j \end{aligned} \quad (39)$$

From

$$dS - \delta d = 0 \quad (40)$$

there follow the compatibility conditions

$$\begin{aligned} d\Gamma^{ab} - \delta\Gamma^{ab} - b_i^{[a} h^{b]i} &= 0 \\ db_a^i - \delta h_a^i - \Gamma_{ab} b^{bi} + \Gamma_{ab} h^{bi} \\ &\quad - H^{ij} b_{aj} + K^{ij} h_{aj} = 0 \\ \delta H^{ij} - dK^{ij} + b_a^{[i} h^{j]a} + [H, K]^{ij} &= 0 \end{aligned} \quad (41)$$

One can then check that

$$W_2(S) = \frac{i 2^{\frac{d}{2}}}{16\pi} \int_{\{\Sigma_t\}} \left(T_{ab} d\Gamma^{ba} + H^{ij} dH_{ji} - \frac{2}{3} (H^3)^{ij} \right) \quad (42)$$

solves Eq. (31b), using in particular the gauge conditions Eq. (21). $W_2(S)$ emerges as a Chern-Simons term pertaining to $SO(2) \times SO(d-2)$.

The topological meaning of this term is best understood by noticing that $W_2(S)$ may be rewritten as^{*)}

$$W_2(S) = \frac{-i}{6\pi} \int_{\{\Sigma_t\}} \text{tr} (S^{-1} dS)^2 \quad (43)$$

with

$$S^{-1} dS = \frac{1}{4} \left(\gamma^a d\gamma_a + n^i dn_i \right) \quad (44)$$

as in Eq. (15). In order to check the identity of Eqs. (42) and (43), one needs the compatibility conditions Eqs. (38) and (41).

The case of Dirac fermions is easily dealt with: W_1 gets doubled, W_3 turns into the Polyakov-Wiegman answer for the group $SO(d-2)$ and W_2 cancels out.

4. - CONCLUDING REMARKS

The situation simplifies in a gauge where $H_+ = 0$, in which gauge W_3 has to be merely forgotten, and of course, W_2 has to be evaluated in this gauge.

We also conjecture that W_2 is the integral over half a sphere of a Hopf-invariant density for Σ , just the same as in the $d = 3$ case.

*) The occurrence of such an $SO(d)$ Wess-Zumino term as well as the Polyakov-Wiegman $SO(d-2)$ term in Eq. (33) was essentially known by A.M. Polyakov (private communication to A.S.).

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