FOUR DIMENSIONAL SUPERSTRINGS

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ABSTRACT

We solve completely the constraints of factorization and multiloop modular invariance, for closed string theories in which all internal quantum numbers of the string are carried by free periodic and antiperiodic world-sheet fermions. We derive a simple set of necessary and sufficient rules, and illustrate how they can be used to find the spectrum, one-loop amplitudes and low-energy Lagrangian of many realistic four dimensional chiral models. We prove that modular invariance and factorization ensure the presence of a massless graviton and the correct connection between spin and statistics. We also prove that the existence of a massless spin $3/2$ state ensures the absence of tachyons and the vanishing of the one-loop cosmological constant.

1. INTRODUCTION

It is presently believed that in order to realize the program of string unification [1] of all particle interactions, one must eventually arrive at a theory in four flat space-time dimensions, with $N = 1$ supersymmetry and chiral matter fields. This would presumably be the first step in any effort to affront the real world, and see whether string theories may provide the answers to such long-standing questions in particle physics as the vanishing of the cosmological constant, the gauge hierarchy problem, the explanation of the observed spectrum of fermion masses etc.

A first approach to carrying out this program, was to compactify the known ten-dimensional superstrings [1,2] on a Calabi-Yau manifold [3] or an orbifold [4]. A much simpler proposal [5,6] is to construct string theories directly in four dimensions with nothing fancier than the tools used for constructing the consistent ten-dimensional superstrings [2,7,8,9,10]: all of the string's internal quantum numbers are carried, either by extra flat dimensions compactified on tori, or by free periodic or antiperiodic fermions among which world-sheet supersymmetry is non-linearly realized [11]. Restricting ourselves to this latter case, we solve explicitly in this paper the constraints of factorization and modular invariance of multiloop string amplitudes [8,12] which ensure the existence of a consistent perturbation theory. Using the simple set of rules that generate all solutions, we deduce several general results, namely that factorization and modular invariance imply the presence of a massless graviton and the correct connection between spin and statistics [13], and that the presence of a massless spin-3/2 particle in the spectrum ensures the absence of a tachyon and the vanishing of the one-loop corrections to the cosmological constant. We also explain the basic steps that should be followed for constructing realistic four dimensional models with $N = 1$ supersymmetry and chiral matter fields in complex representations of the gauge group.

The question that naturally arises is whether these theories are particular compactifications of the ten-dimensional superstrings, possibly in the presence of non-trivial background gauge fields [14]. Although this might well be the case, they still stand out by virtue of their simplicity: as will be clear in this

paper, their complete spectrum, vertex operators, one-loop amplitudes, effective low-energy Lagrangian etc. are readily calculable since one uses little more than the old Neveu-Schwartz-Ramond formalism . If nothing else, they thus provide a playground in which the low-energy properties of string theories can be tested. Besides, both these four-dimensional and the ten-dimensional models presumably correspond to perturbatively stable vacua of a second-quantized string theory, and should in this sense be treated on an equal footing; which of those vacua will be dynamically preferred, is a non-perturbative and highly non-trivial question for which we do not have at present any hint.

This paper is organized as follows: in section 2 we introduce our notation and review the constraints of invariance under all superreparametrizations of the world-sheet. In section 3 we solve completely these constraints, and find a simple set of rules that are both necessary and sufficient for the consistency of string perturbation theory to all orders. In section 4 we show how these rules can be interpreted in terms of generalized GSO [17] projections in the Hilbert space of the string theory, and prove that they imply a "sensible" particle interpretation of the one-loop amplitude, including the correct spin-statistics connection; this was the starting point in the work of Kawai, Lewellen and Tye [5,10]. We also prove by means of elementary set algebra that a massless spin $3/2$ state ensures the absence of a tachyon, the existence of equal numbers of bosons and fermions at all mass levels of the string and the vanishing of the one-loop cosmological constant. Finally in section 5 we outline the basic steps that one should follow to construct realistic models: we show how the gauge group, number of supersymmetries and number of chiral families can be modified, and explain a simple trick for deriving the tree-level Lagrangian in the low energy limit.

2. STRINGY CONSTRAINTS

Our notation is as follows: the non-supersymmetric sector of right-moving string excitations contains in addition to the four space-time coordinates $\partial_z X^\mu$ an extra 44 real free fermions $\bar{\phi}^A$ ($A = 1, \dots, 44$). The supersymmetric left-moving sector has in addition to $\partial_z X^\mu$ and their fermionic partners ψ^μ , another 18 real free fermions χ^I, y^I and ω^I ($I = 1, \dots, 6$) among which supersymmetry is non-linearly realized. For the world-sheet supercurrent we take

$$G(z) = \sum_{\mu} \psi^\mu \partial_z X_\mu + \sum_I \chi^I y^I \omega^I \quad (2.1)$$

since it can be shown that all other possibilities will lead to a string theory with only massive fermions and broken space-time supersymmetry [11]. It can be easily verified that with the above choices the (super)conformal anomaly cancels in each sector separately, so that the theory is invariant under infinitesimal (super)reparametrizations of the world-sheet. For topologically non-trivial world-sheets we must still specify the spin-structures, i.e. the properties of spinor fields under parallel transport around non-contractible loops of the surface, and verify that our choice is invariant under large diffeomorphisms not continuously connected to the identity [15], also called modular transformations. Different choices of spin-structures correspond to different string-theories.

To be more precise we consider first a world-sheet with the topology of a torus. After a reparametrization and a Weyl transformation the torus can be represented by a flat parallelogram in the complex plane with sides 1 and τ corresponding to its two non-contractible loops. The one-loop vacuum to vacuum string amplitude can be written as:

$$Z = \int \left[\frac{dr d\bar{r}}{(Im\tau)^2} \right] Z_B^2 \sum_{spinstr.} C \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} Z_{long, 3/2} \begin{bmatrix} a_\psi \\ b_\psi \end{bmatrix} \prod_{f=1}^{64} Z_F \begin{bmatrix} a_f \\ b_f \end{bmatrix} \quad (2.2)$$

We now explain this formula: the measure in square brackets is invariant under the transformations

$$\tau \rightarrow \tau + 1 \quad (2.3a)$$

and

$$\tau \rightarrow -\frac{1}{\tau} \quad (2.3b)$$

which generate the modular group of the torus; the integration is over the upper complex plane moded out by the modular group; Z_B is the bosonic contribution (the determinant of the Laplacian):

$$Z_B = |Im\tau|^{-1/2} |\eta(\tau)|^{-2}$$

where $\eta(\tau) = q^{1/24} \prod_n (1 - q^{2n})$ is the Dedekind eta function with $q = e^{i\pi\tau}$. $Z_F \begin{bmatrix} a_f \\ b_f \end{bmatrix}$ is the contribution of the fermion f (the determinant of the chiral Dirac operator) which depends on its spin structure $(a_f, b_f) \in Z_2 \times Z_2$; following standard conventions $a_f = 1$ or 0 denotes periodic or antiperiodic respectively boundary conditions in the direction 1 , and similarly for b_f in the direction τ . Explicitly, considering τ as a complex temperature, and using standard path-integral techniques we have:

$$Z_F \begin{bmatrix} 0 \\ 0 \end{bmatrix} = Tr[e^{i\tau H_{NS}}] = \frac{\Theta_3^{1/2}(\tau)}{\eta^{1/2}(\tau)} \quad (2.4a)$$

$$Z_F \begin{bmatrix} 0 \\ 1 \end{bmatrix} = Tr[(-)^F e^{i\tau H_{NS}}] = \frac{\Theta_4^{1/2}(\tau)}{\eta^{1/2}(\tau)} \quad (2.4b)$$

$$Z_F \begin{bmatrix} 1 \\ 0 \end{bmatrix} = Tr[e^{i\tau H_R}] = \frac{\Theta_2^{1/2}(\tau)}{\eta^{1/2}(\tau)} \quad (2.4c)$$

$$Z_F \begin{bmatrix} 1 \\ 1 \end{bmatrix} = Tr[(-)^F e^{i\tau H_R}] = \frac{\Theta_1^{1/2}(\tau)}{\eta^{1/2}(\tau)} \quad (2.4d)$$

where H_{NS} and H_R stand for the Hamiltonians in the Neveu-Schwartz and Ramond sectors respectively, $(-)^F$ is the fermion number operator *, and Θ_i are the well known theta functions. These formulae hold for the left-movers and should be complex conjugated for the right-movers.

* The origin of this operator in Eqs. (2.4b,d) can be easily understood if one notes that a path integral over periodic fermions in the τ direction can be recast into an integral over antiperiodic fermions by a change of variables: $\psi = e^{\frac{i\pi x}{\tau}} \hat{\psi}$

To finish with the explanations concerning Eq. (2.2), note that in this formula the summation runs over all possible spin-structure assignments that are consistent with two-dimensional supersymmetry (i.e. such that $G(z)$ has well defined parallel transport properties), with coefficients $C \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = C \begin{bmatrix} a_1, \dots, a_{64} \\ b_1, \dots, b_{64} \end{bmatrix}$ that are yet to be specified. Note also that two of the components of X^μ were used to cancel the contribution of the graviton ghosts. Likewise $Z_{long, 3/2} \begin{bmatrix} \mathbf{a}_\psi \\ \mathbf{b}_\psi \end{bmatrix}$ is the contribution of two of the components of ψ^μ and of the world-sheet gravitino ghosts; it equals 1 in the case of the torus and can thus be dropped, but we keep it for later reference. Finally note that the fermions ψ^μ , the supercurrent $G(z)$ and the two-dimensional gravitino must all have the same spin structure, that we denoted by $\begin{bmatrix} \mathbf{a}_\psi \\ \mathbf{b}_\psi \end{bmatrix}$.

The modular transformations generated by (2.3a,b) correspond to cutting the torus along its non-contractible loops, then gluing it back together after an integral number of relative ("Dehn") twists by 2π . This has the effect of mixing the spin structures $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, which are called even because they have an even (in fact zero) number of Dirac zero-eigenmodes. Explicitly under $\tau \rightarrow \tau + 1$

$$\eta \rightarrow e^{i\frac{\pi}{2}} \eta; \Theta_1 \rightarrow e^{i\frac{\pi}{4}} \Theta_1; \Theta_2 \rightarrow e^{i\frac{\pi}{4}} \Theta_2; \Theta_3 \leftrightarrow \Theta_4 \quad (2.5a)$$

while under $\tau \rightarrow -\frac{1}{\tau}$

$$\eta \rightarrow (-i\tau)^{1/2} \eta; \frac{\Theta_1}{\eta} \rightarrow e^{-i\frac{\pi}{2}} \frac{\Theta_1}{\eta}; \frac{\Theta_2}{\eta} \leftrightarrow \frac{\Theta_4}{\eta}; \frac{\Theta_3}{\eta} \rightarrow \frac{\Theta_3}{\eta} \quad (2.5b)$$

To ensure modular invariance we must therefore impose the following conditions on the coefficients C :

$$C \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = -e^{i\frac{\pi}{8} \sum a_f} C \begin{bmatrix} \mathbf{a} \\ \mathbf{a} + \mathbf{b} + \mathbf{1} \end{bmatrix} \quad (2.6a)$$

$$C \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} = e^{i\frac{\pi}{4} \sum a_f b_f} C \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} \quad (2.6b)$$

where $\sum$ here stands for $\sum_{leftmovers} - \sum_{rightmovers}$, and $\mathbf{a}$ and $\mathbf{b}$ are of course 64-component vectors. In order to establish these conditions we have actually:

a) kept formally track of the transformation properties of the completely periodic, odd spin structure $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ even though its contribution to (2.2) vanishes due to the existence of a Dirac zero eigenmode (the constant field), and

b) treated the contributions of different fermions as distinct, even when they had the same spin-structure.

Strictly speaking this is not necessary for establishing the modular invariance of the one-loop vacuum amplitude, Eq.(2.2), but it is necessary for one-loop amplitudes with external momentum-dependent legs, and also as we will now see for multiloop vacuum amplitudes.

A multiloop vacuum diagram is a linear chain of donuts. A spin structure is specified by giving the spin structure of every fermion on each donut separately. The coefficient with which it contributes must be a product of one-loop coefficients :

$$C \begin{bmatrix} \mathbf{a}^{(1)} \mathbf{a}^{(2)} \dots \mathbf{a}^{(l)} \\ \mathbf{b}^{(1)} \mathbf{b}^{(2)} \dots \mathbf{b}^{(l)} \end{bmatrix} = C \begin{bmatrix} \mathbf{a}^{(1)} \\ \mathbf{b}^{(1)} \end{bmatrix} C \begin{bmatrix} \mathbf{a}^{(2)} \\ \mathbf{b}^{(2)} \end{bmatrix} \dots C \begin{bmatrix} \mathbf{a}^{(l)} \\ \mathbf{b}^{(l)} \end{bmatrix}$$

since by cluster decomposition the amplitude should factorize [8,12] when the donuts are pulled infinitely far apart (put differently, the vacuum should dominate in the sum over intermediate states that propagate infinitely far).

Now the modular transformations of the multiloop surface can be generated by [12]:

a) Elementary Dehn twists of each donut separately; these leave the amplitude invariant thanks to factorization and one-loop modular invariance.*

b) An additional Dehn twist for each pair of neighbouring donuts that mixes their spin-structures; thanks to factorization, we need only consider such a twist in the case of the double torus. After conformal gauge fixing the double torus is described by a 2×2 symmetric period matrix Ω ; a modular transformation sends Ω to $(A\Omega + B)(C\Omega + D)^{-1}$ where $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a 4×4 integer matrix

* It is here that the full conditions (2.6a,b) become necessary, since fermions with identical or with odd spin structures on one of the l donuts, need not make an identical or a vanishing, respectively, contribution to the l -loop amplitude

with unit determinant. The choice $A = D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ corresponds to a modular transformation that mixes the spin-structures on the two donuts :

$$\begin{bmatrix} \mathbf{a} & \mathbf{a}' \\ \mathbf{b} & \mathbf{b}' \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{a} & \mathbf{a}' \\ \mathbf{b} + \mathbf{a}' & \mathbf{b}' + \mathbf{a} \end{bmatrix}$$

and thus leads to the required additional constraint among the one-loop coefficients that reads :

$$(-)^{a_\psi + a'_\psi} e^{\frac{i\pi}{4} \sum a_f a'_f} C \begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix} C \begin{bmatrix} \mathbf{a}' \\ \mathbf{b}' \end{bmatrix} = C \begin{bmatrix} \mathbf{a} \\ \mathbf{b} + \mathbf{a}' \end{bmatrix} C \begin{bmatrix} \mathbf{a}' \\ \mathbf{b}' + \mathbf{a} \end{bmatrix} \quad (2.7)$$

The second phase on the left-hand-side of the above equation comes from the known transformation properties of Θ -functions on multiloop surfaces [16]. The first phase comes from the transformation properties of

$$Z_{long, 3/2} \begin{bmatrix} a_\psi^{(1)} a_\psi^{(2)} \dots a_\psi^{(l)} \\ b_\psi^{(1)} b_\psi^{(2)} \dots b_\psi^{(l)} \end{bmatrix}$$

Since the world-sheet gravitino determinant on a multiloop surface is not known explicitly, the modular transformation properties of this factor are only known up to a phase. We can nevertheless fix this unknown phase by demanding that the consistent ten-dimensional superstrings [15] * satisfy the condition (2.7). Note that $(-)^{a_\psi}$ is -1 or +1 according to whether the fermions ψ^μ , and hence also the supercurrent $G(z)$ and the world-sheet gravitino, are periodic or antiperiodic respectively in the direction 1 of the torus.

Eqs. (2.6) and (2.7) suffice to ensure the absence of global reparametrization anomalies on an arbitrary genus surface. It is another one of the miracles of string theory that they also suffice, as we will soon see, to ensure the presence of a massless graviton, and the correct spin-statistics of string excitations. We first turn our attention, however, to deriving the general solution of these consistency conditions.

* We may restrict ourselves to the 8 left-movers of the 10d superstrings, because of holomorphic factorization. Then the condition (2.7) must be satisfied by the coefficients $C \begin{bmatrix} 0 \\ 0 \end{bmatrix} = C \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -C \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -C \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

3. THE GENERAL SOLUTION

In order to solve the consistency conditions, we will need to rewrite them in a more convenient form by introducing some new notation. First we will denote by F the set of all 64 fermionic coordinates, and by 2^F the set of all subsets of F . Now we can specify a general spin structure assignement on the torus in either of the following two ways:

i) as an ordered tetraplet $[A_1; A_2; A_3; A_4]$ where A_1, A_2, A_3 , and A_4 are the sets of fermions with spin structure $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ respectively; clearly A_1, A_2, A_3 , and A_4 form a partition of F .

ii) as an ordered pair $(\alpha|\beta)$ where α and β are the sets of fermions that are periodic in the directions 1 and τ , respectively, of the torus; α and β are arbitrary subsets of F .

The bilingual dictionnary allowing us to pass from one notation to the other is:

$$[A_1; A_2; A_3; A_4] = (A_1 \cup A_2 | A_1 \cup A_4) \quad (3.1a)$$

$$(\alpha|\beta) = [(\alpha \cap \beta); (\alpha - \alpha \cap \beta); (F - (\alpha \cup \beta)); (\beta - \alpha \cap \beta)] \quad (3.1b)$$

Notation (ii) will prove more convenient for our purposes, but some equations may be more transparent when translated back to language (i).

It will be useful to associate to every set of fermions X the phases:

$$\varepsilon_X = e^{\frac{i\pi}{8}n(X)}$$

and

$$\delta_X = \begin{cases} -1, & \text{if } \psi^\mu \in X \\ +1, & \text{otherwise} \end{cases}$$

as well as a parity operator that counts the fermions in X modulo two :

$$(-)^X f = \begin{cases} -f(-)^X & \text{if } f \in X \\ f(-)^X & \text{otherwise} \end{cases}$$

We also need to define $n_L(X)$ and $n_R(X)$ which are the numbers of left-moving and right-moving, respectively, fermions in the group X , as well as the net

number of fermions in the group X : $n(X) = n_L(X) - n_R(X)$. Finally, we define the product of two fermion sets to be their symmetric difference:

$$\alpha\beta = \alpha \cup \beta - \alpha \cap \beta$$

With this product rule, 2^F becomes a commutative group with the empty set $\emptyset$ as the identity, and with all elements equal to their own inverse; note also that $F\alpha$ is the complement of α .

Consider now an arbitrary spin-structure assignment $(\alpha|\beta)$. In order that it be consistent with world-sheet supersymmetry, which is required for eliminating negative metric states, we should make sure that the supercurrent $G(z)$, Eq. (2.1), is either periodic or antiperiodic under parallel transport around the non-contractible loops of the torus. This means that the sets $F - \alpha$ and $F - \beta$ of fermions that are antiperiodic in the directions 1 and τ , respectively, of the torus should either

a) contain an odd, 1 or 3, number of fermions from every additive term of the supercurrent (and hence also contain the ψ^μ), or

b) contain an even, 0 or 2, number of these fermions (and hence not the ψ^μ).

A little thought will convince the reader that these conditions are conveniently summarized by demanding that

$$[G(z), (-)^\alpha]_{-\delta_\alpha} = [G(z), (-)^\beta]_{-\delta_\beta} = 0$$

where $[\ , \]_+$ here stands for the anticommutator and $[\ , \]_-$ for the commutator. Let $\Upsilon \subset 2^F$ be the collection of all fermion sets that satisfy the relation $(-)^X G = \delta_X G (-)^X$; since $(-)^{XY} = (-)^X (-)^Y$ and $\delta_{XY} = \delta_X \delta_Y$, it follows easily that Υ is closed under symmetric differences, and is thus a subgroup of 2^F . Spin structure assignments that are consistent with world-sheet supersymmetry correspond to elements in $\Upsilon \times \Upsilon$.

Let us return now to the one-loop amplitude. We will use the same symbol that specifies a spin structure assignment, to also denote its contribution to

this amplitude; thus

$$[A_1; A_2; A_3; A_4] = \int_{\tau} \Theta_1^{A_1} \Theta_2^{A_2} \Theta_3^{A_3} \Theta_4^{A_4} \quad (3.2)$$

where we have here omitted the integration measure and the bosonic contribution, and we used the shorthand Θ_i^X for $(\frac{\Theta_i}{\eta})^{\frac{n_L(X)}{2}} (\frac{\Theta_i}{\eta})^{\frac{n_R(X)}{2}}$. The full one-loop amplitude is of course a linear combination

$$Z = \sum_{\alpha, \beta \in \Upsilon} C_{(\alpha|\beta)}(\alpha|\beta)$$

The coefficients $C_{(\alpha|\beta)}$ are constrained by the string consistency conditions, Eqs. (2.6a,b) and (2.7), which ensured the absence of anomalies under global reparametrizations of the string world-sheet (for any number of loops). Translated in our new language these conditions take the elegant form:

$$C_{(\alpha|\beta)} = -\varepsilon_{\alpha} C_{(\alpha|F\alpha\beta)} \quad (3.3a)$$

$$C_{(\alpha|\beta)} = -\varepsilon_{\alpha\cap\beta}^2 C_{(\beta|\alpha)} \quad (3.3b)$$

and

$$C_{(\alpha_1|\beta_1)} C_{(\alpha_2|\beta_2)} = \delta_{\alpha_1} \delta_{\alpha_2} \varepsilon_{\alpha_1 \cap \alpha_2}^2 C_{(\alpha_1|\alpha_2\beta_1)} C_{(\alpha_2|\alpha_1\beta_2)} \quad (3.4)$$

These are the basic equations which we will now solve completely. Our method will be to first derive some properties that must necessarily characterize any solution, then in the end show that they are also sufficient.

Let us begin by writing down some special (and actually redundant) cases of Eq. (3.4) which can be easily derived with the help also of the transposition property (3.3b):

$$C_{(\emptyset|\emptyset)} C_{(\alpha|\emptyset)} = \delta_{\alpha} C_{(\alpha|\emptyset)}^2 \quad (3.5a)$$

$$C_{(\alpha|\emptyset)} C_{(\beta|\emptyset)} = \delta_{\alpha} C_{(\alpha|\emptyset)} C_{(\alpha\beta|\emptyset)} \quad (3.5b)$$

$$C_{(\alpha|\beta)}^2 = \delta_{\alpha} \delta_{\beta} C_{(\alpha|\emptyset)} C_{(\beta|\emptyset)} \quad (3.5c)$$

$$C_{(\alpha|\beta)} C_{(\alpha|\gamma)} = \delta_{\alpha} \delta_{\gamma} C_{(\alpha|\beta\gamma)} C_{(\gamma|\emptyset)} \quad (3.5d)$$

The first relation shows that for every fermion set α either $C_{(\alpha|\emptyset)} = 0$, or else $C_{(\alpha|\emptyset)} = \delta_\alpha C_{(\emptyset|\emptyset)}$. We may for convenience set $C_{(\emptyset|\emptyset)} = 1$ in what follows *, since an overall rescaling of the one-loop amplitude amounts to a redefinition of the string theory coupling constant. Let us denote by

$$\Xi = \{\alpha_1, \alpha_2, \dots\}$$

the collection of fermion sets, for which $C_{(\alpha_i|\emptyset)} = \delta_{\alpha_i}$. Clearly $\emptyset \in \Xi$, and setting $\alpha = \beta = \emptyset$ in Eq. (3.3a) we see that also $F \in \Xi$ always. Next, Eq. (3.5b) shows that Ξ is closed under symmetric differences. It is therefore a subgroup of Υ , generated by some basis $\{b_0 = F, b_1, \dots, b_N\}$; its cardinality is 2^{N+1} , and its generic element is of the form $b_0^{m_0} b_1^{m_1} \dots b_N^{m_N}$, with $m_i = 0$ or 1 . Finally Eq. (3.5c) proves that a general spin structure $(\alpha|\beta)$ contributes to the amplitude if and only if both $\alpha \in \Xi$ and $\beta \in \Xi$, and in this case its coefficient is a sign : $C_{(\alpha|\beta)}^2 = 1$. We summarize these conclusions in a

Lemma 1: The one-loop amplitude of a consistent string theory, in which all fermions are either periodic or antiperiodic, is always of the form

$$Z = \sum_{\alpha, \beta \in \Xi} C_{(\alpha|\beta)} (\alpha|\beta) \quad (3.6)$$

with Ξ a subgroup (under symmetric differences) of Υ containing F , and with $C_{(\alpha|\beta)}^2 = 1$ for all α and β in Ξ .

We will refer to Ξ as the group of parities of the string theory, anticipating its natural interpretation in terms of generalized GSO projections (see the following section). Not all subgroups of Υ are acceptable groups of parities however, since an assignment of signs $C_{(\alpha|\beta)} \in \mathbb{Z}_2$, consistent with all constraints (3.3a,b) and (3.4) is not in general possible. Indeed from Eq. (3.3a) it follows

* The case $C_{(\emptyset|\emptyset)} = 0$ can be easily seen to lead to $C_{(\alpha|\beta)} = 0$ for all α and β , and is therefore trivial.

immediately that for all $\alpha \in \Xi$ we must have $\varepsilon_\alpha^2 = 1$, so that the net number of fermions should be a multiple of eight

$$n(\alpha) = 0(\text{mod}8) \quad \forall \alpha \in \Xi$$

It is not sufficient to impose this condition on the basis elements of Ξ , because ε_α is not a group homomorphism. Instead it satisfies the following identity that is easy to derive by induction, keeping in mind also that $\varepsilon_X^{16} = 1$ always :

$$\varepsilon_{\alpha_1 \alpha_2 \dots \alpha_M} = \left(\prod_i \varepsilon_{\alpha_i} \right) \left(\prod_{i>j} \varepsilon_{\alpha_i \cap \alpha_j}^{-2} \right) \left(\prod_{i>j>k} \varepsilon_{\alpha_i \cap \alpha_j \cap \alpha_k}^4 \right) \left(\prod_{i>j>k>l} \varepsilon_{\alpha_i \cap \alpha_j \cap \alpha_k \cap \alpha_l}^{-8} \right)$$

By raising this equation to the square, we can see that the condition $n(\alpha) = 0(\text{mod}8)$ for all $\alpha \in \Xi$ is equivalent to the following conditions among the basis elements b_i :

$$n(b_i) = 0(\text{mod}8) \tag{3.7a}$$

$$n(b_i \cap b_j) = 0(\text{mod}4) \tag{3.7b}$$

$$n(b_i \cap b_j \cap b_k) = 0(\text{mod}2) \tag{3.7c}$$

There is one final consistency condition that requires the net number of fermions in the intersection of any quadruplet of basis elements to be even

$$n(b_i \cap b_j \cap b_k \cap b_l) = 0(\text{mod}2) \tag{3.8}$$

This is actually stronger than (3.7c), since the basis elements include F . In order to derive this condition we must now do some work. To start with, from Eq. (3.5d) we find that for all α, β , and γ in Ξ

$$C_{(\alpha|\beta)} C_{(\alpha|\gamma)} = \delta_\alpha C_{(\alpha|\beta\gamma)} \tag{3.9}$$

which says that for all α in Ξ , the mapping

$$\delta_\alpha C_{(\alpha|x)} : \Xi \ni x \rightarrow Z_2$$

is a group homomorphism. Clearly

$$C_{(\alpha|x_1 x_2 \dots x_K)} = \delta_\alpha^{K-1} \prod_{i=1}^K C_{(\alpha|x_i)}$$

This , together with the transposition property (3.3b) allows us to express any $C_{(\alpha|\beta)}$ in terms of the coefficients $C_{ij} = C_{(b_i|b_j)}$ of pairs of basis elements. Explicitly, if $\alpha = b_{\alpha_1} b_{\alpha_2} \dots b_{\alpha_A}$ and $\beta = b_{\beta_1} b_{\beta_2} \dots b_{\beta_B}$, we have

$$C_{(\alpha|\beta)} = \left(\prod_{i=1}^A \delta_{b_{\alpha_i}}^{B-1} \right) \left(\prod_{j=1}^B \delta_{b_{\beta_j}}^{A-1} \varepsilon_{b_{\beta_j} \cap (b_{\alpha_1} \dots b_{\alpha_A})}^2 \right) \left(\prod_{i=1}^A \prod_{j=1}^B C_{\beta_j \alpha_i} \right) \quad (3.10)$$

The C_{ij} themselves are not all independent; they must satisfy the symmetry relation (3.3b), as well as the condition

$$C_{ii} = -\varepsilon_{b_i} C_{i0}$$

which follows from (3.3a) and the factorization property (3.9). This leaves us with $\frac{N(N+1)}{2} + 1$ independent signs, for instance C_{00} and C_{ij} for all $i > j = 0, 1, \dots, N$, which we can choose at will. The claim now is that to any such choice there corresponds a consistent string theory, if and only if the basis elements of Ξ obey the condition (3.7a,b) and (3.8). We prove this by brute force (the dum way), by showing that the coefficients $C_{(\alpha|\beta)}$ explicitly constructed in (3.10), obey the general stringy constraints, if and only if these conditions hold.

Since the proof is straightforward, we will only sketch it here. Consider for example the transposition constraint (3.3b) which is satisfied by assumption by the coefficients C_{ij} for pairs of basis elements. Simple book-keeping shows that it is also satisfied by the generic coefficients (3.10), if and only if the following holds:

$$\prod_{j=1}^B \varepsilon_{b_{\beta_j} \cap (b_{\alpha_1} \dots b_{\alpha_A})}^2 = \varepsilon_{(b_{\beta_1} \dots b_{\beta_B}) \cap (b_{\alpha_1} \dots b_{\alpha_A})}^2 \left(\prod_{i=1}^A \varepsilon_{b_{\alpha_i} \cap (b_{\beta_1} \dots b_{\beta_B})}^2 \right) \left(\prod_{i=1}^A \prod_{j=1}^B \varepsilon_{b_{\alpha_i} \cap b_{\beta_j}}^2 \right) \quad \blacksquare$$

This equation is not as hopeless as it looks. Using the identity

$$\varepsilon_{\alpha \cap \beta \cap \gamma} = \varepsilon_{\alpha \cap \beta} \varepsilon_{\alpha \cap \gamma} \varepsilon_{\alpha \cap \beta \cap \gamma}^{-2}$$

and the fact that $\varepsilon_X^{16} = 1$, we can express both sides as products of phases whose arguments are either double, or triple or quadruple intersections : $\varepsilon_{b_i \cap b_j}^2$, $\varepsilon_{b_i \cap b_j \cap b_k}^4$ and $\varepsilon_{b_i \cap b_j \cap b_k \cap b_l}^8$. The first two types of phases always occur in

pairs, and therefore cancel thanks to the conditions (3.7). This is not the case for the phases of quadruple intesections however, which only disappear if they are explicitly set equal to the identity for all basis elements. This means that unless the condition (3.8) is also imposed, there is an obstruction in trying to extend the assignement of signs from the basis elements to all of Ξ , because the operations of transposition and factorization do not commute. In a similar manner we show that the stringy constraints (3.3a) and (3.4) are automatically satisfied by the coefficients (3.10), provided (3.7) and (3.8) hold. This means that there are no further obstructions, and completes the demonstration of the following theorem that summarizes the results of this section:

Theorem: Choose any subgroup Ξ of fermion sets, generated by a basis $\{b_0 = F, b_1, \dots, b_N\}$ with the following properties:

$$n(b_i) = 2n(b_i \cap b_j) = 4n(b_i \cap b_j \cap b_k \cap b_l) = 0 \quad \text{modulo 8} \quad (C1)$$

and

$$(-)^{b_i} G(z) = \delta_{b_i} G(z) (-)^{b_i} \quad (C2)$$

For any such Ξ there exist $2^{\frac{N(N+1)}{2}+1}$ consistent string theories, corresponding to all possible choices of signs (+1 or -1) for the coefficients $C_{(F|F)}$ and $C_{(b_i|b_j)}$ for $i > j = 0, 1, \dots, N$. Any given choice can be uniquely extended to a mapping $C_{(\alpha|\beta)} : \Xi \times \Xi \rightarrow Z_2$, with the properties:

$$C_{(\alpha|\beta)} = \epsilon_{\alpha\cap\beta}^2 C_{(\beta|\alpha)} \quad (C3)$$

$$C_{(\alpha|\alpha)} = -\epsilon_\alpha C_{(\alpha|F)} \quad (C4)$$

and

$$C_{(\alpha|\beta)} C_{(\alpha|\gamma)} = \delta_\alpha C_{(\alpha|\beta\gamma)} \quad (C5)$$

The full one-loop amplitude is given by

$$Z = \sum_{\alpha, \beta \in \Xi} C_{(\alpha|\beta)} (\alpha|\beta)$$

Furthermore this construction exhausts all consistent theories for which the internal quantum numbers of the string are carried by free periodic or antiperiodic fermions on the world-sheet.

4. THE SPECTRUM AND OTHER PROPERTIES

We will now give a simple interpretation to Ξ and C , in terms of generalized GSO projections [17] in the Hilbert space of the string theory. The notation introduced in the previous section will prove particularly handy, since it will allow us to derive several general results in a few lines. We will in particular prove that all consistent string theories have a massless graviton, that modular invariance ensures the spin-statistics connection, and that the presence of a massless spin-3/2 particle ensures the absence of tachyons as well as the vanishing of the one-loop cosmological constant.

The Hilbert space of a string theory is a direct sum of different sectors of type $R^\alpha N^{F\alpha}$, in which all fermions in the set α have Ramond or periodic boundary conditions, while all remaining fermions have Neveu-Schwartz or antiperiodic boundary conditions. In a self-explanatory, though slightly abusive notation, we may write the contribution of the spin structure $(\alpha|\beta)$ as

$$(\alpha|\beta) = (-)^\beta R^\alpha N^{F\alpha}$$

meaning that one sums the contributions of all states in the sector $R^\alpha N^{F\alpha}$, with a sign given by the value of $(-)^\beta$ on this state; this follows easily from equations (3.2), (3.1a) and (2.4). Changing once more the overall normalization of the full amplitude, that can be reabsorbed in the string theory coupling constant, we may write

$$Z = \frac{1}{2^{N+1}} \sum_{\alpha \in \Xi} \left[\sum_{\beta \in \Xi} C_{(\alpha|\beta)} (-)^\beta \right] R^\alpha N^{F\alpha}$$

Using the factorization property (C5), and the decomposition of a generic element of Ξ as $\beta = b_0^{m_0} b_1^{m_1} \dots b_N^{m_N}$ where $m_i = 0$ or 1 , we can easily deduce that the term inside the square brackets is a product of projectors:

$$Z = \sum_{\alpha \in \Xi} \delta_\alpha \left[\prod_{i=0}^N \frac{1}{2} (1 + \delta_\alpha C_{(\alpha|b_i)} (-)^{b_i}) \right] R^\alpha N^{F\alpha} \quad (4.1a)$$

or equivalently, since the extra projectors are redundant

$$Z = \sum_{\alpha \in \Xi} \delta_\alpha \left[\prod_{\beta \in \Xi} \frac{1}{2} (1 + \delta_\alpha C_{(\alpha|\beta)} (-)^\beta) \right] R^\alpha N^{F\alpha} \quad (4.1b)$$

The meaning of the subgroup Ξ and the coefficients $C_{(\alpha|\beta)}$ should be now clear; to each element $\beta \in \Xi$, there corresponds i) a sector $R^\beta N^{F\beta}$ of the Hilbert space, and ii) a generalized GSO projection that leaves only those states in $R^\alpha N^{F\alpha}$, whose β -parity is $(-)^{\beta} = C_{(\alpha|\beta)} \delta_\alpha$. The factorization property (C5) ensures that these projections are mutually compatible.

Let us recall some standard facts: the sector $R^\alpha N^{F\alpha}$ of the Hilbert space consists of states formed by acting on the vacuum $|0\rangle_\alpha$ with the creation operators of left- and right-moving string excitations, X_m^μ and $\bar{X}_m^\mu$ respectively, as well as with the creation operators of fermionic excitations, f_m . For $f \in F\alpha$ the oscillators have half-integer frequencies m ; all other frequencies are integer. The vacuum $|0\rangle_\alpha$ must represent the Clifford algebra of zero-modes of all fermions in α . There are two cases:

i) if $\psi^\mu \in \alpha$, it is a space-time spinor as well as a spinor of an internal $SO(n_L(\alpha) + n_R(\alpha) - 2)$; furthermore all states in $R_\alpha N_{F\alpha}$ have half-integer spin;

ii) if $\psi^\mu \in F\alpha$, it is a space-time scalar, in a spinor representation of an internal $SO(n_L(\alpha) + n_R(\alpha))$; furthermore all states built on it have integer spin.

From equation (4.1) we see that the contribution of a state to the vacuum amplitude is negative in the former case and positive in the later; this is in accordance with the fact that integer and half-integer spin excitations must be quantized with Bose and Fermi statistics, respectively. Let us point out here that the requirement that the one-loop amplitude has such a sensible particle interpretation, was the starting point in the work of Kawai, Lewellen and Tye [5,10]; the fact that this is equivalent to factorization and multiloop modular invariance was argued for by Seiberg and Witten [8], and has been demonstrated for ten-dimensional superstrings by Parkes [13].

The effect of making a projection $(-)^{\beta} = \rho$ in the sector $R_\alpha N_{F\alpha}$ depends on whether α and β are disjoint or not. If they are we must only keep those states built on all components of the vacuum $|0\rangle_\alpha$ with an even (when $\rho = 1$) or odd (when $\rho = -1$) number of β -oscillators. If on the other hand $\alpha \cap \beta$ is

not empty, then we define the generalized chirality operator

$$\Gamma_{\alpha \cap \beta} = \prod_{f \in \alpha \cap \beta} f_0 \quad (4.2)$$

where if $\psi^\mu \in \alpha \cap \beta$ the product should, by Lorentz invariance, include all Dirac matrices transverse to the momentum of the state. Now admissible states are those built with an even number of β -oscillators on the components of the vacuum that have generalized chirality ρ , as well as those built with an odd number of β -oscillators on the vacuum of opposite chirality.

Consider in particular the value of the parity $(-)^{F\alpha} = \delta_\alpha C_{(\alpha|F\alpha)}$ in the sector $R_\alpha N_{F\alpha}$; from (C3) and (C5) it follows easily that this is equal to $-\epsilon_\alpha$ so that states with an even (if $\epsilon_\alpha = +1$) or an odd (if $\epsilon_\alpha = -1$) number of half-integer oscillators are always projected out. This is consistent with the fact that these states could never satisfy the mass-shell conditions (that come from the zeroth-moment Virasoro conditions):

$$\begin{aligned} M^2 &= -\frac{1}{2} + \frac{n_L(\alpha)}{16} + \sum_{\text{leftmovers}} (\text{frequencies}) = \\ &= -1 + \frac{n_R(\alpha)}{16} + \sum_{\text{rightmovers}} (\text{frequencies}) \end{aligned} \quad (4.3)$$

There is one other set of projections, whose values have been completely fixed by the string consistency conditions (C3-5), and which are therefore not free parameters. These are the values of all β -parities in the pure Neveu-Schwartz sector N^F : $(-)^{\beta} = C_{(\emptyset|\beta)} = \delta_\beta$. This is particularly important; it means that the massless state $\psi_{\frac{1}{2}}^\mu \bar{X}_1^\nu |0\rangle_{\emptyset}$, which contains the graviton, the dilaton and a two-index antisymmetric tensor, can never be projected out of the Hilbert space of a consistent string theory. Let us here pause and summarize these conclusions in a

Corollary: The Hilbert space of a consistent string theory is

$$\mathcal{H} = \bigoplus_{\alpha \in \mathbb{E}} \left(\prod_{\beta \in \mathbb{E}} \frac{1}{2} (1 + (-)^{\beta} \delta_\alpha C_{(\alpha|\beta)}) \right) R^\alpha N^{F\alpha}$$

Modular invariance ensures the presence of a massless graviton, and the correct connection between spin and statistics .

Let us consider next under what conditions the spectrum contains a massless spin-3/2 particle . It follows easily from the mass-shell conditions Eqs. (4.3), that this must be a state $\bar{X}_1^\mu |0\rangle_S$ in $R^S N^{FS}$, with S a set of exactly 8 leftmovers, including in particular the fermions ψ^μ . Without loss of generality we may take

$$S = \{\psi^\mu, \chi^1, \dots, \chi^6\} \quad (4.4)$$

Since $\bar{X}_1^\mu |0\rangle_S$ has an internal $SO(6)$ spinor index, it actually contains as many components as 8 Weyl spin-3/2 particles as well as 8 Weyl spin 1/2 particles. Half of them are always eliminated by the projection $(-)^S = \prod_{f \in S} f_0 = \delta_S C_{(S|S)}$. In order to analyze the effect of the remaining, if any, β -parity projections on this state, consider first the mapping

$$\Xi \ni \beta \rightarrow \beta \cap S \in 2^S$$

which is a group homomorphism from Ξ into the group of all subsets of S . Because of the conditions (C1) all elements in the image $Im(\Xi)$ must satisfy: $n(X) = 0(mod 4)$ and $n(X \cap X' \cap X'') = 0(mod 2)$. It can be easily verified that a subgroup of 2^S containing S and satisfying these conditions, can have at most three generators that may, without loss of generality , be taken to be S , $S_1 = \{\psi^\mu, \chi^1, \chi^2\}$ and $S_2 = \{\psi^\mu, \chi^3, \chi^4\}$. The claim now is that , if $\{S\}$, or $\{S, S_1\}$, or $\{S, S_1, S_2\}$ is a basis for $Im(\Xi)$, then we can always choose some corresponding basis $\{F, S, \zeta_1 \dots \zeta_K\}$ or $\{F, S, b_1, \zeta_1 \dots \zeta_K\}$, or $\{F, S, b_1, b_2, \zeta_1 \dots \zeta_K\}$ of Ξ , such that:

$$b_1 \cap S = S_1 = \{\psi^\mu, \chi^1, \chi^2\} \quad (4.5a)$$

$$b_2 \cap S = S_2 = \{\psi^\mu, \chi^3, \chi^4\} \quad (4.5b)$$

and for all $i = 1, \dots, K$

$$\zeta_i \cap S = \emptyset \quad (4.5c)$$

We will refer to such a basis as a canonical basis for Ξ ; the reason that we can always find a canonical basis, is that if $\zeta_i \cap S = S^m S_1^{m_1} S_2^{m_2}$ is not empty for some basis element ζ_i , then it suffices to redefine our basis replacing ζ_i by

$\zeta_i' = \zeta_i S^m b_1^{m_1} b_2^{m_2}$ which satisfies (4.5c). We will also choose by convention the sets b_1 and b_2 to be the sets of minimal cardinality that satisfy (4.5a,b); because of the condition (C2), each of these sets contains at least 8 left-movers.

Let us go back now to the massless spin-3/2 states $\bar{X}_1^\mu |0\rangle_S$. Their ζ_i -parity is +1, and hence in order to survive the $(-)^{\zeta_i} = \delta_S C_{(S|\zeta_i)}$ projections, we must necessarily impose $C_{(S|\zeta_i)} = -1$ for all i . This implies that $C_{(S|\beta)} = -1$ for all $\beta \in \Xi$ disjoint from S as can be easily verified with the help of the consistency conditions (C3-5). We still have at most two other independent projections, that correspond to the basis elements b_1 and b_2 , if they exist, and which set $\Gamma_{S_i} = -C_{(S|b_i)}$ on these states. Since each of these generalized chirality projections truncates half of the components of $\bar{X}_1^\mu |0\rangle_S$, it follows that the cases of one, two or three generators for $Im(\Xi)$, generally correspond to theories with $N = 4$, $N = 2$ and $N = 1$ respectively massless Weyl gravitinos. Actually there is an exception to this rule: if b_1 is a set of exactly 8 left movers and no right movers, then new massless spin-3/2 states $\bar{X}_1^\mu |0\rangle_{b_1}$ may appear doubling the number of gravitinos; similarly if both b_1 and b_2 are sets of precisely 8 left movers, the number of gravitinos can be quadrupled giving us back an $N = 4$ theory. We will return to this comment in the following section.

For the time being let us point out that the necessary and sufficient condition for the existence of at least one massless gravitino, is that there exist at least one set S containing precisely 8 left-moving fermions, including the ψ^μ , and such that for all $\beta \in \Xi$ disjoint from S , $C_{(S|\beta)} = -1$ *. We are now ready to prove a

Lemma 2: The presence of at least one massless spin-3/2 state, ensures the existence of an equal number of fermions and bosons at each mass-level of the spectrum, the vanishing of the one-loop cosmological constant, and the absence of tachyons.

* If S did not contain the ψ^μ , this condition would be inconsistent with the fact that $C_{(S|FS)} = -\delta_S \epsilon_S = +1$.

Proof: The one-loop vacuum to vacuum amplitude, Eq. (3.6), can be rewritten trivially as

$$Z = \sum_{\alpha, \beta \in \Xi} \frac{1}{4} [C_{(\alpha|\beta)}(\alpha|\beta) + C_{(S\alpha|\beta)}(S\alpha|\beta) + C_{(\alpha|S\beta)}(\alpha|S\beta) + C_{(S\alpha|S\beta)}(S\alpha|S\beta)]$$

since when X runs over all elements of Ξ , so does SX . We will now show that the sum of the four terms inside the square brackets, which we will refer to for short as SUM, is identically zero for all $\alpha, \beta \in \Xi$. Indeed, note first that since the Θ_1 -function vanishes identically, $(X|Y) = 0$ unless $X \cap Y = \emptyset$. Hence we need only concern ourselves with the case in which at least one of the four pairs of elements in square brackets are disjoint; without loss of generality we take $\alpha \cap \beta = \emptyset$. The intersection of S with α on the other hand can contain 0, 4 or 8 elements, and similarly with β . We consider these cases separately:

i) If $S \cap \alpha = S \cap \beta = \emptyset$, then $(S\alpha|S\beta) = 0$, and from the condition of existence of a gravitino: $C_{(S|\alpha)} = C_{(S|\beta)} = -1$. Using the factorization and symmetry properties of the coefficients C we easily deduce:

$$\begin{aligned} SUM &= C_{(\alpha|\beta)}[(\alpha|\beta) - (S\alpha|\beta) - (\alpha|S\beta)] \\ &= C_{(\alpha|\beta)} \int_{\tau} \Theta_2^{\alpha} \Theta_3^{FS\alpha\beta} \Theta_4^{\beta} [\Theta_3^S - \Theta_2^S - \Theta_4^S] = 0 \end{aligned}$$

where the last step follows from the fact that S contains exactly 8 left movers, and from the well known identity among Θ -functions;

ii) If $S \subset \alpha$ or $S \subset \beta$, the same argument goes through if one replaces α by $S\alpha$, or β by $S\beta$;

iii) If $S \cap \alpha = \emptyset$ and $S \cap \beta$ contains precisely 4 left movers, then $(S\alpha|\beta) = (S\alpha|S\beta) = 0$, and $C_{(S|\alpha)} = -1$. We therefore find easily

$$SUM = C_{(\alpha|\beta)} \int_{\tau} [\Theta_2^{\alpha} \Theta_3^{F\alpha\beta} \Theta_4^{\beta} - \Theta_2^{\alpha} \Theta_3^{FS\alpha\beta} \Theta_4^{S\beta}] = 0$$

since β and $S\beta$ contain exactly the same number of both left and right movers. The same argument of course goes through if we interchange the role of α and β ; finally

iv) If $S \cap \alpha$ and $S \cap \beta$ contain both precisely 4 left-moving fermions, then we have $(\alpha|S\beta) = (S\alpha|\beta) = 0$, and also $C_{(S|S\alpha\beta)} = -1$ since $S\alpha\beta$ and S are necessarily disjoint (draw a little picture to believe it). But then using repeatedly the symmetry and factorization properties of the coefficients C we find $C_{(S\alpha|S\beta)} = -C_{(\alpha|\beta)}$ and thus

$$SUM = C_{(\alpha|\beta)} \int_{\tau} [\Theta_2^{\alpha} \Theta_3^{F\alpha\beta} \Theta_4^{\beta} - \Theta_2^{S\alpha} \Theta_3^{F\alpha\beta} \Theta_4^{S\beta}] = 0$$

as before.

What we have actually shown here is that the one-loop vacuum diagram vanishes even before one integrates over the modular parameter τ ; this means that the partition functions of space-time bosons and fermions are identical at all temperatures, and hence there are as many bosons as fermions at each mass level of the spectrum. Finally we prove the absence of a tachyon ^{*}: if the tachyon were a fermion it would belong to some sector $R^{\beta} N^{F\beta}$ with $\psi^{\mu} \in \beta$; but then condition (C2) requires β to have at least 8 left-movers, and the mass-shell condition, Eq. (4.3), shows that no states in this sector can have a negative mass. Thus there can be no tachyons that are space-time fermions, and since fermions and bosons occur in pairs at all mass levels of the theory, there can be no tachyons, period.

A last remark: everything we have done until now can be taken over with trivial modifications to any dimension of space-time $D \leq 10$. The set F of fermion fields contains in this case $D - 2$ "transverse" ψ^{μ} , $3(10 - D)$ additional left movers χ^I , y^I and ω^I , and $32 + 2(10 - D)$ right movers $\bar{\phi}^a$, and the supercharge is appropriately modified. The counting of massless gravitinos of course changes also. We may also easily extend our analysis to the non-heterotic superstring: in four dimensions the set F contains the transverse left- and right-moving fermions ψ^{μ} and $\bar{\psi}^{\mu}$, as well as 18 extra fermionic coordinates on each side. The theorem, corollary and two lemmas of sections 3 and 4 all hold if only the phase δ_X is redefined as -1 if either ψ^{μ} or $\bar{\psi}^{\mu}$ but not both are in X and as +1 otherwise. It is amusing to observe that left-right asymmetric theories can be constructed in this case if one chooses projectors appropriately.

^{*} We thank V.Rivasseau for a discussion on this point

5. REALISTIC MODELS

The number of consistent four-dimensional string theories is so huge that classifying them all would be both impractical and not very illuminating. Instead we will limit ourselves here to a few examples of models with realistic low-energy spectra; these will illustrate the simple set of rules given in the theorem of section 3, and the way in which the gauge group and massless matter fields are obtained. Armed with this simple set of rules, and a good knowledge of group theory, the interested model-builder may then try to construct the theory of his choice.

We will restrict ourselves here to theories that are space-time supersymmetric at the tree-level; this both ensures the vanishing of the one-loop corrections to the cosmological constant, and may help explain the gauge-hierarchy problem. Thus the basis of Ξ , which we will denote by ξ for short, contains at least two elements : F and S . Consider first the simplest choice $\xi_1 = \{F, S\}$. As we explained in the previous section this is a theory * with $N = 4$ space-time supersymmetries. The graviton multiplet contains : the states $\psi_{\frac{1}{2}}^{\mu} \bar{X}_1^{\mu} |0\rangle_{\emptyset}$ which include the graviton, a scalar and an antisymmetric tensor field, the states $\chi_{\frac{1}{2}}^I \bar{X}_1^{\mu} |0\rangle_{\emptyset}$ which include four gravitinos and four spin-1/2 particles in the representation (4) of the internal symmetry group $SU(4) \simeq SO(6)$, and finally the vectors $\bar{X}_1^{\mu} |0\rangle_S$ in the real representation (6) of $SU(4)$. Besides these states the massless spectrum also includes a vector multiplet in the adjoint representation of the gauge group $SO(44)$. Explicitly, this multiplet contains, for each adjoint index, a vector, four spinors and six scalars:

$$\psi_{\frac{1}{2}}^{\mu} \bar{\phi}_{\frac{1}{2}}^a \bar{\phi}_{\frac{1}{2}}^b |0\rangle_{\emptyset}; \quad \bar{\phi}_{\frac{1}{2}}^a \bar{\phi}_{\frac{1}{2}}^b |0\rangle_S; \quad \chi_{\frac{1}{2}}^I \bar{\phi}_{\frac{1}{2}}^a \bar{\phi}_{\frac{1}{2}}^b |0\rangle_{\emptyset} \quad (5.1)$$

Suppose we add now to the basis ξ_1 a new element, ζ_1 , that doesn't break any supersymmetries, that is $\zeta_1 \cap S = \emptyset$ and $C_{(S|\zeta_1)} = -1$. It can be easily

* There are actually four theories, corresponding to all possible choices of signs for $C_{(S|S)}$ and $C_{(F|F)}$, but they only differ in the chiralities of the spinor representations of the internal symmetry groups

checked that $(-)^{\zeta_1} = +1$ in both the $R(\emptyset)N(F)$ and the $R(S)N(FS)$ sector, consistent with the fact that the entire $N = 4$ graviton multiplet survives these projections. The vector multiplet (5.1) on the other hand does not, since the $\bar{\phi}^a$ that belong to ζ_1 must now occur in pairs. As a result the gauge group $SO(44)$ is broken down to $SO(n_R(\zeta_1)) \otimes SO(44 - n_R(\zeta_1))$. In fact there is an exception: if ζ_1 is a set of precisely 8 or 16 rightmovers, then new vector multiplets coming from the sector $R(\zeta_1)N(F\zeta_1)$ will appear, enlarging the gauge group to $SO(44)$ or $E_8 \times SO(28)$ respectively. More generally for any supersymmetric theory let $(R_1 R_2 \dots R_{M_\Xi})$ be the partition of the 44 right movers induced by the basis of Ξ in the following sense: every R_i is an intersection of some number of basis elements, and no intersection of basis elements is a proper subset of R_i . Then the gauge group is at least $G = \bigotimes_i SO(n(R_i))$, and may be enlarged if Ξ contains elements with precisely 8 or 16 right movers. Note that in order to reduce the rank of the gauge group below 22 some of the $n(R_i)$ need be odd. In general a large variety of gauge groups can be obtained in four dimensions, but we will limit ourselves here to orthogonal groups, since these suffice to yield realistic models.

A realistic model must of course have chiral matter fields in a complex representation of the gauge group, which could eventually describe the observed low-energy matter particles. A little thought and the mass-shell conditions (4.3) will convince the reader that such states can only be one of the following two kinds:

- i) $|0\rangle_\beta$ with β a set of 8 left-movers that include ψ^μ , and of 16 right-movers, or
- ii) $\bar{\phi}_{\frac{1}{2}}^a |0\rangle_\beta$ with β a set of 8 left movers that include the ψ^μ , and of 8 right movers.

In both these cases the set β_R of right-movers of β should not itself be an element of Ξ , since if it were the above states would have supersymmetric vector partners ($\psi_{\frac{1}{2}}^\mu |0\rangle_{\beta_R}$ or $\psi_{\frac{1}{2}}^\mu \bar{\phi}_{\frac{1}{2}}^a |0\rangle_{\beta_R}$ respectively), and would therefore belong to real representations of the gauge group. As we discussed in section 4, a β with the above properties will necessarily eliminate half of the gravitinos; this is of course due to the well known fact that there are no scalar matter multiplets of $N = 4$ supersymmetry.

Let us further pursue case (i). The basis of our theory must thus include at least $\xi_2 = \{F, S, b_1\}$, where without loss of generality

$$b_1 = \{\psi^\mu, \chi^1, \chi^2, y^3, \dots, y^6, \bar{\phi}^1, \dots, \bar{\phi}^{16}\}$$

The theory described by ξ_2 has $N = 2$ supersymmetry. Its massless spectrum includes the graviton multiplet, a vector multiplet in the adjoint of the gauge group $SO(16) \otimes SO(28)$, and a scalar multiplet in the real (16, 28) representation of the gauge group; these are the states of the $N = 4$ graviton and $SO(44)$ vector multiplets that survived the b_1 -parity projections. In addition there is the new scalar matter multiplet $|0\rangle_{\beta_1}$ in the (*spinor*, 1) representation of the gauge group; this is a real representation, since out of all orthogonal groups only the $SO(2 + 4n)$ have complex representations. In order to obtain a chiral theory we must thus break $SO(16)$ down to factors that include at least one of the groups $SO(6)$, $SO(10)$ or $SO(14)$.

Suppose we opt for $SO(10)$ grand unification; let $h = \prod_{transverse} \psi_0^\mu$ be the helicity operator, and $\Gamma_{10} = \bar{\phi}_0^1 \bar{\phi}_0^2 \dots \bar{\phi}_0^{10}$ the chirality operator that distinguishes the two inequivalent complex conjugate spinor representations of $SO(10)$. For chiral asymmetry we must demand $h\Gamma_{10}$ to have a well defined value*; taking conditions (C1), (C2) and (4.5b) into account, it can be shown that there is an almost unique set that will impose such a projection, namely

$$b_2 = \{\psi^\mu, \chi^3, \chi^4, \omega^3, \dots, \omega^6, \bar{\phi}^1, \dots, \bar{\phi}^{10}, \bar{\phi}^{17}, \dots, \bar{\phi}^k\}$$

with $k = 22, 30$ or 38 . For $k = 30$ for instance the theory described by $\xi_3 = \{F, S, b_1, b_2\}$ has $N = 1$ supersymmetry, an $SO(10) \otimes SO(6) \otimes SO(14) \otimes SO(14)$ gauge symmetry, and four chiral matter multiplets, two in the (16, 4, 1, 1) and two in the (16, $\bar{4}$, 1, 1) representations of the gauge group. This makes a total of 16 chiral families of $SO(10)$.

The number of chiral families can in fact be easily modified. Adding for instance the basis element

$$\zeta_1 = \{y_3, \dots, y_6, \omega_3, \dots, \omega_6, \bar{\phi}_{17}, \bar{\phi}_{18}, \bar{\phi}_{31}, \dots, \bar{\phi}_{44}, \}$$

* Recall that a Weyl spinor contains positive helicity particles in some representation $\mathbf{r}$ of the gauge group, and negative helicity antiparticles in the complex conjugate representation $\mathbf{r}^*$. If $\mathbf{r}$ were real, a Majorana mass would not be forbidden by gauge invariance and will in general be generated by quantum corrections

eliminates half of the above chiral families . Further reductions by factors of two are possible provided one breaks the "horizontal" $SO(6)$ and/or the $SO(10)$ gauge symmetries. We can also obtain a number of families that is not a power of 2; choosing for instance b_2 with $k = 22$, and adding a basis element $\zeta'_1 = \{\bar{\phi}_{11}, \dots, \bar{\phi}_{14}, \bar{\phi}_{23}, \dots, \bar{\phi}_{26}\}$ leads to a theory with 3×16 families of $SO(10)$; these can then be reduced by means of further projections. We stop here since a classification of all interesting models is beyond the scope of this paper.

We would like to conclude with an important observation that illustrates the main advantage of these theories, namely their calculability. Let (Ξ, C) be some arbitrary theory; to every subgroup $\Xi' \subset \Xi$ there corresponds a naturally induced theory obtained by restricting the coefficients C to Ξ' . From Eq. (4.1) it follows easily that part of the Hilbert space of the theory (Ξ, C) is an exact truncation of the Hilbert space of (Ξ', C) . To see how this remark can be useful, let $\xi = \{F, S, b_1, b_2, \zeta_1, \dots, \zeta_K\}$ be a canonical basis for a $N = 1$ supersymmetric theory; then the subbasis $\xi' = \{F, S, \zeta_1, \dots, \zeta_K\}$ corresponds to an induced $N = 4$ theory. Now the tree-level Lagrangian of the massless modes of an $N = 4$ theory is uniquely specified [18], up to the unknown coupling constants of the semi-simple components of the gauge group. Thus we can immediately obtain by truncation part of the low-energy Lagrangian of the original $N = 1$ theory, including in particular the couplings of all vector multiplets; this is not trivial because this Lagrangian is not polynomial. We can even determine the ratios of gauge coupling constants, by noting that the vector bosons of $\bigotimes SO(n(R_i))$ come from the truncation of the $SO(44)$ multiplet (5.1), which has a single arbitrary coupling. The interactions of the additional massless scalar multiplets, other than those left over by the truncation of the $N = 4$ vector multiplets, cannot be similarly determined; they are however considerably restricted [19] by the fact that they can be obtained by truncating an $N = 2$ supersymmetric theory.

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While this work was being completed we received preprints [20,21]; the first contains the details of the work of Kawai, Lewellen and Tye, on which we already commented in the text. The second is based on group lattices of bosonized world-sheet fermions. Finally we should point out that Thirring interactions are studied in [20], and also by J.Bagger, D.Nemeschansky, N.Seiberg and S.Yankielowicz.

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