A Geometrical Approach for Open and Closed Strings

K. Bardakci*

Physics Department, University of Geneva and CERN
CH-1211 Geneva (Switzerland)

UGVA-DPT 1986/12-525

* On sabbatical leave from and address after May 30, 1987 : Department of Physics and Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720, USA.

CERN-Pre 87-012

- 1 -
C1

ABSTRACT

We show that, a candidate for the open string lagrangian proposed earlier has a large tangent space invariance group, in addition to being invariant under extended reparametrization transformations. We also simplify the underlying differential geometry and extend it to the closed string. In the latter case, the well known constraint relating the number of the right and left movers emerges naturally.

CERN LIBRARIES, GENEVA

CM-P00062968

1. Introduction

To find a field theoretical description of string theories is an old problem that has attracted renewed interest recently. In the non-covariant light cone gauge, the problem was solved long ago by Kaku and Kikkawa (1). The modern problem is to discover a gauge invariant formulation which permits, among others, covariant choices for gauge fixing. The importance of the reparametrization invariance in this context was already recognized in one of the earliest attempts in this direction (2). The present author has proposed to extend the global reparametrization group into a gauge group by making the group parameters dependant on the string coordinates (3,4). It was shown in (4) that this gauge principle leads to a non-linear lagrangian, whose quadratic part reproduces the correct free string dynamics at the classical level.

The purpose of the present paper is twofold. First, we would like to modify the treatment of the open string given in (4) slightly to make it simpler and clearer. We will also establish the existence of certain symmetries in addition to reparametrization invariance, which are important in understanding the absence of certain fields in the lagrangian. The second aim of the paper is to extend the formalism to include the closed string. The generalization is straightforward, and one nice feature of our approach is that, the constraint

$$(L_0 - \bar{L}_0) |s\rangle = 0, \quad (1.1)$$

to be imposed on the physical states, emerges naturally as a gauge condition. This should be contrasted with other approaches, where this condition had to be imposed by hand (5).

At this point, we would like to briefly compare our approach with the well developed BRST approach (6,7) to string field theory (8-14). The main advantage of the treatment presented here is that it follows from a geometrical gauge principle and the interaction is included from the beginning, since the non-linear gauge invariance necessarily results in an interacting theory. On the other hand, it is incomplete in one important respect; the fermionic coordinates which appear naturally in the BRST formalism are missing. Presumably, these will be introduced in the process of covariant gauge fixing. However, it is clear that a better way is to introduce the fermionic coordinates along with the bosonic ones from the beginning, and then extend the gauge group so that it acts on the new coordinates as well as the old ones. Starting with the same fields and the same gauge principle we have here, but making the fields dependant on the additional fermionic coordinates, it appears possible to make contact with the BRST formalism along the lines suggested by Baulieu et. al. (15). We hope to return to this problem in the future.

The paper is organized as follows. In section 2, we review the differential geometry in loop space needed in the case of the open string and we then extend it to the closed string. Some of this material is already covered in (4); we include it here to make the paper self contained. Also, the treatment given here is more streamlined and the material on the closed string is new. In section 3, we construct the lagrangian for the open string, and we show that there is a symmetry group in the tangent space, which enables one to gauge away a large number of field components. Although the final lagrangian is essentially the same, the construction given here differs somewhat from the one given in (4). The difference is that in the present treatment the tangent space is decoupled from the coordinate space, so that there are two independent commuting transformations available; the

diffeomorphisms in the coordinate space and in addition the tangent space invariance group. This puts our approach more in line with the usual treatment of gravity. Section 3 contains the construction of the closed string lagrangian, and the problem of gauge fixing for the free part of the lagrangian is discussed. We show that all the constraints satisfied by the physical states result from gauge fixing. Section 4 summarizes our conclusions and lists future problems.

2. Review of the Loop Space Differential Geometry.

We first discuss the open string, for which the coordinates of the right and left movers, introduced in (4), are the same

$$X_r^\mu(\epsilon) = X_l^\mu(\epsilon) \equiv X^\mu(\epsilon), \quad (2.1)$$

where $0 \leq \epsilon \leq 1$ and the Lorentz index μ will be frequently suppressed. A scalar field $\phi(X)$ is a functional of $X(\epsilon)$, and under the reparametrizations of ϵ , it transforms as follows :

$$\cup \phi(X) \cup^{-1} = \phi(\bar{X}), \quad (2.2a)$$

where,

$$\bar{X}(\tilde{\epsilon}) \equiv X(\epsilon), \quad (2.2b)$$

with

$$\tilde{\epsilon} = g(\epsilon, \bar{X}).$$

By making the function g a functional of X as well, the global reparametrizations have been promoted into a gauge group. We also need the infinitesimal version of eq. (2.2) :

$$\tilde{\epsilon} = \epsilon + f(\epsilon, X), \quad (2.3a)$$

and

$$\cup \phi(X) \cup^{-1} = \phi(X) - \int d\epsilon f(\epsilon, X) L_\epsilon [\phi(X)], \quad (2.3b)$$

where $f(\epsilon, X)$ is infinitesimal periodic function of ϵ , and the conformal operator L_ϵ is defined by

$$L_\epsilon \equiv X_\mu'(\epsilon) \frac{\delta}{\delta X_\mu(\epsilon)} \quad (2.4a)$$

The prime on X denotes differentiation with respect to the argument ϵ . The L 's satisfy the well-known commutation relations given by

$$[L_\epsilon, L_{\epsilon'}] = \delta'(\epsilon - \epsilon') [L_\epsilon + L_{\epsilon'}], \quad (2.4b)$$

where we have neglected the central charge, since we are working at the classical level. We will discuss this point further later on. In addition to scalars, it is natural to define contravariant and covariant vectors in the standard fashion :

$$\cup V^{\mu\epsilon}(X) \cup^{-1} = \int d\epsilon' \frac{\delta X^{\mu\epsilon}}{\delta \bar{X}^{\mu'\epsilon'}} V^{\mu'\epsilon'}(\bar{X}), \quad (2.5)$$

$\cup V_{\mu\epsilon}(X) \cup^{-1} = \int d\epsilon' \frac{\delta \bar{X}_{\mu\epsilon}}{\delta X^{\mu'\epsilon'}} V_{\mu'\epsilon'}(\bar{X}),$
where $X^{\mu'\epsilon'} \equiv X^{\mu'}(\epsilon')$. The transformation matrix is given by

$$\frac{\delta X^{\mu\epsilon}}{\delta \bar{X}^{\mu'\epsilon'}} = \delta_{\mu\mu'} \delta(\epsilon - g(\epsilon, \bar{X})) + \bar{X}^{\nu\epsilon'} \frac{\delta g(\epsilon, \bar{X})}{\delta \bar{X}^{\nu'\epsilon'}} \quad (2.6)$$

Higher order tensors can be defined in a similar way. However, the space of the tensors so defined is too large, and it has to be suitably restricted to be useful for string theories. The reason for this can be read off from eq.

(2.6). The second term on the right hand side depends on the gradient of the function g with respect to the coordinate $\bar{X}$. We wish to replace this gradient by the conformal operator L , so that the transformation laws can be written in terms of L 's only. This is motivated by the observation that the equations that determine string dynamics should be solely expressible in terms of the conformal operators. The conformal operator is the component of the gradient along the tangential direction, which suggests that we project the tangential components of the vectors in eq. (2.5). If, for example, the contravariant vector is of the form

$$V^{\mu\delta}(X) = X'^{\mu\delta} V^{\delta}(X), \quad (2.7a)$$

it is easy to check, using (2.6), that this form is preserved under the transformations given by (2.5). Therefore, (2.7a) is a restriction compatible with the vector transformation law, and one can easily write down the corresponding transformation rule for the reduced vector V^{δ} :

$$U V^{\delta}(X) U^{-1} = [g'(\epsilon, \bar{X})]^{-1} \{ V^{\delta}(\epsilon, \bar{X}) + \int d\epsilon' V^{\delta'}(\bar{X}) L_{\delta'}^{\delta} [g(\epsilon, \bar{X})] \}, \quad (2.7b)$$

where the superscript on L indicates that L is given by (2.4a), with X replaced by $\bar{X}$. Similarly, one can define a reduced covariant vector V_{δ} by

$$V_{\delta} = X'^{\mu\delta} V_{\mu\delta}, \quad (2.8)$$

and show that it transforms under the inverse of the transformation given in eq. (2.7b).

The next step is the differentiation of the reduced tensors. Differentiating scalars is easy; if, in eq. (2.8), $V_{\mu\delta}$ is a pure gradient, given by

$$V_{\mu\delta} = \frac{\delta \phi}{\delta X^{\mu\delta}} \quad (2.9a)$$

then,

$$V_{\delta} = L_{\delta} \phi \quad (2.9b)$$

transforms as a reduced covariant vector. We also know that, the curl of a covariant vector $V_{\mu\delta}$, given by

$$F_{\mu\delta, \mu'\delta'} = \frac{\delta V_{\mu\delta}}{\delta X^{\mu'\delta'}} - \frac{\delta V_{\mu'\delta'}}{\delta X^{\mu\delta}}, \quad (2.10)$$

transforms as a second order covariant tensor. The reduced covariant tensor is constructed by projecting in the tangential direction in each index:

$$\begin{aligned} F_{\delta, \delta'} &= X'^{\mu\delta} X'^{\mu'\delta'} F_{\mu\delta, \mu'\delta'} \\ &= L_{\delta} V_{\delta'} - L_{\delta'} V_{\delta} + \delta'(\delta - \delta')(V_{\delta} + V_{\delta'}). \end{aligned} \quad (2.11)$$

The term proportional to the derivative of the delta function on the right hand side of this equation arises because L 's fail to commute among themselves. The extension of (2.11) to higher order antisymmetric tensors is entirely straightforward, and in this way, a theory of forms based on the conformal operator L_g can be constructed. Such a theory of forms was already considered by Banks and Peskin (10); however, their tensors had transformation laws different from (2.7b). In the next section, we shall see that if (2.7b) is linearized in an appropriate fashion, the Banks-Peskin transformation laws can be recovered. We will not, however, present the details of this general tensor analysis here, since we do not need it in this paper, and also since it fits naturally with the projected generalization of the present formalism to include fermionic coordinates.

Eq.(2.11) for the curl of a reduced covariant vector, and the transformation law (2.7b) for a contravariant vector, and its inverse for a covariant vector, are among the most important results of the section. Having arrived at these results, one can abstract them and forget about the starting point. From now on, we will consider be abstract operators that satisfy Leibnitz's differentiation rule and the commutation relations (2.4b). Since the transformation law for a vector is expressible in terms of the L 's alone, using their commutation relations, it is easy to verify directly that $\bar{F}_{\epsilon, \epsilon'}$ indeed transforms like a tensor. This abstract point of view has the advantage of allowing one to use the advantage of allowing one to use the appropriate representation for the conformal operators in a given context.

In our approach, the basic dynamical field of string field theory will be the vielbein $e_{\epsilon, \tau}^{\epsilon', \tau'}$. We will also need its inverse defined by

$$\int d\tau e_{\epsilon, \tau}^{\epsilon', \tau'} e_{\epsilon', \tau'} = \delta(\epsilon - \epsilon'). \quad (2.12)$$

The index ϵ refers to the coordinate space and τ to the tangent space*. Under reparametrizations, ϵ transforms as in (2.7b) and τ is untouched:

$$\begin{aligned} U e_{\epsilon, \tau}^{\epsilon', \tau'}(X) U^{-1} &= [g'(\epsilon, \bar{X})]^{-1} \{ e_{g(\epsilon, \bar{X}), \tau}^{\epsilon', \tau'}(\bar{X}) \\ &+ \int d\epsilon' e_{\epsilon, \tau}^{\epsilon', \tau'}(\bar{X}) L_{\epsilon'}^{\bar{X}}(g(\epsilon, \bar{X})) \}. \end{aligned} \quad (2.13)$$

* The transformation law for the vielbein differs from the one given in (4)

In the next section, we will show that the theory is also invariant under a set of tangent space transformations which act only on the variable τ . For the time being, we leave the tangent space group unspecified. We are not going to introduce any dynamical field corresponding to the connection, and therefore, the only covariant tensor available is the torsion constructed from the vielbein according to (2.11):

$$\begin{aligned} T_{\epsilon, \epsilon'; \tau''} &= L_{\epsilon}(e_{\epsilon', \tau''}) - L_{\epsilon'}(e_{\epsilon, \tau''}) \\ &+ \delta'(\epsilon - \epsilon') [e_{\epsilon, \tau''} + e_{\epsilon', \tau''}]. \end{aligned} \quad (2.14a)$$

It is convenient to define another version of the torsion tensor, where the indices ϵ and ϵ' are converted into tangent space indices τ and τ' :

$$\begin{aligned} T_{\tau''}^{\tau, \tau'} &= \int \int d\epsilon d\epsilon' e_{\epsilon, \tau}^{\epsilon', \tau'} e_{\epsilon', \tau'}^{\epsilon, \tau} T_{\epsilon, \epsilon'; \tau''} \\ &= \int \int d\epsilon d\epsilon' e_{\epsilon, \tau}^{\epsilon', \tau'} e_{\epsilon', \tau'}^{\epsilon, \tau} [L_{\epsilon}(e_{\epsilon', \tau''}) \\ &- L_{\epsilon'}(e_{\epsilon, \tau''})] + \int d\epsilon e_{\epsilon, \tau}^{\epsilon, \tau} [e_{\epsilon', \tau}^{\epsilon', \tau} \partial_{\epsilon}(e_{\epsilon', \tau'}) \\ &- e_{\epsilon', \tau'}^{\epsilon, \tau} \partial_{\epsilon'}(e_{\epsilon, \tau})]. \end{aligned} \quad (2.14b)$$

This expression agrees with the one given in (4), except for a term linear in the vielbein. This difference is due to the different treatment of the tangent space.

The torsion tensor, being linear in the conformal operators, is a good starting point for the construction of the lagrangian. We will give this construction in the next section, and for the remainder of this section, we will describe the extension of the formalism developed so far to the case of the closed string. This extension is quite straight forward; all one has to do is to double everything. For example, instead of (2.1), we should start with two

different coordinates, one for the right movers and one for the left movers. We label them by an index i , with $i=1,2$, then

$$X_i^\mu(\delta_i) \equiv X^{\mu, i \delta_i} \quad (2.15)$$

are the right (left) moving coordinates for $i=1(2)$. The δ_i 's are related to the more customary complex variables z and $\bar{z}$

by

$$z = \exp(i\delta_1), \quad \bar{z} = \exp(-i\delta_2). \quad (2.16)$$

The reparametrizations are now given by

$$U \phi(X) U^{-1} = \phi(\bar{X}), \quad (2.17a)$$

where,

$$\bar{X}_i(\tilde{\delta}_i) \equiv X_i(\delta_i), \quad (2.17b)$$

with $\tilde{\delta}_i = g_i(\delta_i, \bar{X})$.

We note that we now have two different reparametrization functions g_1 and g_2 , one for each δ_i , and each one is a functional of both X_1 and X_2 . An infinitesimal reparametrization is given by

$$\tilde{\delta}_i = \delta_i + f_i(\delta_i, X). \quad (2.18)$$

The transformation law of vectors is an obvious generalization of eq. (2.5); for example

$$U V^{\mu, i \delta_i}(X) U^{-1} = \int d\epsilon' \int \frac{\delta X^{\mu, i \delta_i}}{\delta \bar{X}^{\mu', i' \delta'}} V^{\mu', i' \delta'}(\bar{X}), \quad (2.19)$$

and similarly for the covariant vectors. Again, the tensor

space is too large, and it can be restricted by defining reduced vectors as in (2.7a) and (2.8):

$$\begin{aligned} V^{\mu, i \delta} &= X'^{\mu, i \delta} V^{i \delta}, \\ V_{i \delta} &= X'^{\mu, i \delta} V_{\mu, i \delta}. \end{aligned} \quad (2.20)$$

The contravariant reduced vector transforms as follows:

$$\begin{aligned} U V^{i \delta}(X) U^{-1} &= [g'_i(\epsilon, \bar{X})]^{-1} \{ V^{i g_i(\epsilon, \bar{X})}(\bar{X}) \\ &+ \int d\epsilon' \sum_j V^{i' \delta'}(\bar{X}) L^{i' \delta'}_{i \delta} [g_i(\epsilon, \bar{X})] \}, \end{aligned} \quad (2.21)$$

where,

$$L_{i \delta} \equiv X'^{\mu, i \delta} \frac{\delta}{\delta X^{\mu, i \delta}}, \quad (2.22a)$$

satisfies the commutation relations

$$[L_{i \delta}, L^{i' \delta'}] = \delta_{i i'} \delta'(\delta - \delta') [L_{i \delta} + L^{i' \delta'}]. \quad (2.22b)$$

We therefore have two commuting sets of conformal operators, one for each value of i , which are usually denoted by L and $\bar{L}$. The curl of a reduced covariant is now defined by

$$\begin{aligned} F_{i \delta, i' \delta'} &= L_{i \delta}(V^{i' \delta'}) - L^{i' \delta'}(V_{i \delta}) \\ &+ \delta_{i i'} \delta'(\delta - \delta') [V_{i \delta} + V_{i' \delta'}]. \end{aligned} \quad (2.23)$$

Finally, we introduce the vielbein $e^{i\delta,j\tau}$, or its inverse $e_{i\delta,j\tau}$ as the dynamical variable. Under reparametrizations, the contravariant vielbein $e^{i\delta,j\tau}$ transforms as in eq. (2.21) with respect to the indices i and δ , and the tangent space indices j and τ are unchanged. The torsion tensor is again the curl of the covariant vielbein, and transformed to the tangent space it reads

$$T_{j''\tau''}^{j'\tau'} = \int d\delta d\delta' \sum_{i\tau} e_{i\delta,j\tau} e^{i'\delta',j'\tau'} [L_{i\delta}(e^{i'\delta',j''\tau''}) - L_{i'\delta'}(e_{i\delta,j''\tau''})] + \int d\delta \sum_{i\tau} e_{i\delta,j''\tau''} [e_{i\delta,j\tau} \partial_\delta (e^{i\delta,j'\tau'}) - e_{i\delta,j'\tau'} \partial_\delta (e_{i\delta,j\tau})]. \quad (2.24)$$

We now have the complete machinery needed to construct the lagrangian.

3. The Open String Lagrangian.

A good candidate for the lagrangian density of the open string is the torsion tensor of eq. (2.14b). However, the lagrangian density should be a scalar, which means that the indices τ, τ' and τ'', τ''' should somehow be contracted. Since, we have not yet introduced an invariance group in the tangent space, there infinitely many ways of doing this. A sufficiently powerful invariance requirement would hopefully fix the lagrangian completely. Here, however, we will approach the problem in an indirect fashion; we will first write down the lagrangian given in (4), and find its symmetry group in the tangent space. The overall symmetry group of the model is then the direct product of the reparametrization group given by (2.13), and the tangent space group to be determined shortly. If one wishes, these steps can then be reversed, and the lagrangian can be fixed by the invariance requirements.

The action given in reference (4) can be written as follows :

$$A = \int D X \mathcal{L}(X), \quad (3.1a)$$

where,

$$\mathcal{L}(X) = \frac{1}{2} D T, \quad (3.1b)$$

and D and T are given by

$$D = \det(e_{\delta,\tau}), \quad (3.2a)$$

$$T = \int \int d\tau d\tau' d\tau'' \delta^{(-)}(\tau', \tau'') T_{\tau', \tau''}^{\tau, \tau'}. \quad (3.2b)$$

The function $\delta^{(-)}$, given by,

$$\delta^{(-)}(\tau', \tau'') = i \sum_{n \neq 1} \left\{ \exp[2\pi i n(\tau' - \tau'')] \right\} - \exp[2\pi i n(\tau'' - \tau')], \quad (3.2c)$$

acts as the "metric" in the tangent space. Since the lagrangian is first order in L 's, one has to introduce a complex structure into the problem through an antisymmetry between the positive and negative fourier components. The function $\delta^{(-)}$ has the needed antisymmetry property and leads to a lagrangian whose free (quadratic) part is satisfactory.

Let us investigate the invariance properties of the action given by (3.2). The invariance under reparametrizations is obvious : the tensor $T_{\tau''}^{\tau'}$ transforms like a scalar under this group, and the determinant D compensates for the jacobian resulting from the coordinate transformations. This is because the tangent space indices τ, τ' and τ'', τ''' do not transform at all under reparametrizations, and they can be contracted in any fashion without violating the reparametrization invariance. This is in contrast to the treatment given in (4), where the tangent space indices did

transform under reparametrizations, and therefore the invariance group was broken down to a subgroup. Having decoupled the tangent space from the coordinate space, the symmetry group of the tangent space itself can now be investigated separately. For this purpose, let us consider the transformations given by

$$e'_6 \tau \rightarrow \int d\tau' K(\tau, \tau') e'_6 \tau', \quad (3.3a)$$

$$e_6 \tau \rightarrow \int d\tau' K^{-1}(\tau', \tau) e_6 \tau',$$

where,

$$\int d\tau' K(\tau, \tau') K^{-1}(\tau', \tau'') = \delta(\tau - \tau''). \quad (3.3b)$$

There are two distinct set of transformations which leave the action invariant. The first set consists of those K 's which commute with the metric δ^- :

$$\int d\tau' \{ K(\tau, \tau') \delta^-(\tau', \tau'') - \delta^-(\tau, \tau') K(\tau', \tau'') \} = 0,$$

or in an obvious matrix notation,

$$K \delta^- - \delta^- K = 0. \quad (3.4)$$

This condition is necessary to have invariance in indices τ' and τ'' . Since the index τ is integrated, a further condition is needed:

$$\int d\tau K(\tau, \tau') = \det(K) \quad (3.5)$$

for all τ' , so that the left hand side of this equation is independent of τ' . These conditions would be sufficient if K did not depend on X , but since we are considering gauge transformations, K is allowed to depend on X . As a result, under the transformation (3.3), the action changes by an additive term which must be set equal to zero:

$$\iint d\epsilon d\tau e'_6 \tau \text{Tr} \{ \delta^-(\tau') K L_6(K^{-1}) \} = 0. \quad (3.6)$$

Eqs (3.4), (3.5) and (3.6) are the necessary and sufficient conditions for K to be a symmetry transformation. These equations can be simplified as follows. Eq. (3.4) implies that K can be written in block diagonalized form as follows:

$$K = K^{(+)} \oplus K^{(0)} \oplus K^{(-)}, \quad (3.7a)$$

which, expressed in terms of fourier components, reads

$$K(\tau, \tau') = \sum_{n \neq j, m=1}^{\infty} K_{n,m}^{(+)} \exp(2\pi i(n\tau - m\tau')) + K^{(0)} + \sum_{n \neq j, m=1}^{\infty} K_{n,m}^{(-)} \exp(2\pi i(m\tau' - n\tau)). \quad (3.7b)$$

Using eq. (3.7), condition (3.5) can be simplified to

$$\det(K^{(+)} \det(K^{(-)}) = 1. \quad (3.8)$$

Now let us consider eq. (3.6). For this equation to be valid for an arbitrary $e'_6 \tau$, we need the stronger condition

$$\text{Tr} \{ \delta^-(\tau') K L_6(K^{-1}) \} = 0. \quad (3.9a)$$

Substituting (3.7) in this equation, one obtains

$$L_6 \{ \text{Tr} [\log K^{(+)} - \log K^{(-)}] \} = 0, \\ \text{or } K^{(+)} = \text{const. } K^{(-)}. \quad (3.9b)$$

Without loss of generality, this constant can be set equal to unity. Combining eq. (3.8) and (3.9b) we arrive at the final result

$$\det(K^{(+)} = \det(K^{(-)}) = 1. \quad (3.10)$$

A transformation of the form (3.7b) with satisfies (3.10) is then a tangent space symmetry.

In addition to the transformations described above, there is a second set of transformations that leave the action invariant. They owe their existence to the fact that δ^- has no zero mode. Consider the following special form for K :

$$K(\tau, \tau') = \delta(\tau - \tau') + K(\tau), \quad (3.11)$$

$$K^{-1}(\tau, \tau') = \delta(\tau - \tau') - K(\tau),$$

where $\int dr k(T) = 0$. Using the fact that δ^- has no zero mode, after a straightforward calculation, one can establish that (3.11) leaves the action invariant.

What are the consequences of the symmetries we have been discussing so far? We have seen that there are two different types of symmetry transformations: The reparametrizations given by (2.13) and the tangent space transformations given by (3.7b), (3.10) and (3.11). These two classes of transformations have rather different consequences. The reparametrization invariance is the mechanism behind ghost elimination; it allows the choice of a gauge which decouples ghosts. The tangent space symmetries, on the other hand, imply that some of the field components are redundant and do not appear in the action. We will now examine these issues in the context of the free (quadratic) action. To define the free part of the action, one has to develop a perturbation expansion by setting

$$e^{\epsilon, \tau} = \delta(\epsilon - \tau) + E^{\epsilon, \tau}, \quad (3.12)$$

and expanding in powers of E . We note that the zeroth order term $\delta(\epsilon - \tau)$ is not invariant under any subgroup of the symmetries we have been considering, which means that they are spontaneously (and completely) broken. We now linearize the theory by expanding to lowest non-trivial order in E , and for the sake of convenience, we shall express the results in terms of the Fourier components given by

$$E^{\epsilon, \tau} = \sum E^{n, m} \exp[2\pi i(n\epsilon + m\tau)], \quad (3.13)$$

$$L_\epsilon = \sum L_n \exp(2\pi i n \epsilon),$$

$$f(\epsilon) = \sum f_n \exp(2\pi i n \epsilon),$$

where f is the infinitesimal reparametrization function of (2.3a). When linearized, (2.13) becomes

$$E^{n, m} \rightarrow E^{n, m} + L_m f_n - 2\pi i(n+2m)f_{n+2m}. \quad (3.14)$$

These are the linear gauge transformations introduced by Banks and Peskin (10) and by Neveu, Schwarz and West (15, 16). The linearized form of the first set of the tangent space transformations can be obtained by considering the infinitesimal version of (3.7b). We then have

$$E^{n, m} \rightarrow E^{n, m} + H_{(t)}^{n, m} + H_{(-)}^{n, m} + H_0 \delta_{n,0} \delta_{m,0} \quad (3.15a)$$

where H 's satisfy the following conditions:

$$H_{(t)}^{n, m} = 0, \quad (3.15b)$$

unless $n \geq 1$ and $m \leq -1$.

$$H_{(-)}^{n, m} = 0, \quad (3.15b)$$

unless $n \leq -1$ and $m \geq 1$. Also, there are two trace conditions:

$$\sum_{n=1}^{\infty} H_{(t)}^{n, -n} = \sum_{n=1}^{\infty} H_{(-)}^{-n, n} = 0. \quad (3.15c)$$

These last two conditions are the infinitesimal versions of (3.10). The second set of transformations given by (3.11) can be linearized to give

$$E^{0, n} \rightarrow E^{0, n} + K_n. \quad (3.16)$$

It is now easy to see that one can use (3.15) to set all the components of $E^{n,m}$, where n and m have opposite signs, equal to zero, except for the two traces

$$\psi = \sum_{n=1}^{\infty} E^{n,-n}, \quad \psi^* = \sum_{n=1}^{\infty} E^{-n,n}. \quad (3.17)$$

The traces can be represented by some arbitrary choice of fields; for example, $E^{1,-1}$ and $E^{-1,1}$. Then all the other $E^{n,m}$, where n and m are of opposite signs, including $E^{0,0}$, can be set equal to zero. Similarly, (3.16) can be used to eliminate all the field components of the form $E^{0,n}$. To see how these invariances manifest themselves, we need the quadratic part of the lagrangian:

$$\begin{aligned} \delta q = & i \psi^* (L_0 - 1) \psi + i \sum_{n=1}^{\infty} \{ E^{n,0} L_{-n} (\psi^*) \\ & - E^{-n,0} L_n (\psi) - 4\pi i n E^{n,0} E^{-n,0} \} \\ & + i \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \{ E^{-n,m} L_n (E^{m,0}) - E^{n,m} L_{-n} (E^{-m,0}) \\ & - E^{-m,-n} (L_0 - 1) E^{n,m} + 2\pi i (n+m) E^{-n,m} E^{m,n} \\ & - 2\pi i (2n+m) [E^{n+m,0} E^{-n,m} + E^{-n-m,0} E^{n,m}] \}. \end{aligned} \quad (3.18)$$

This is the lagrangian of reference (15) and also of (4). Notice that the fields that can be eliminated by the tangent space transformations, discussed in the last paragraph, are all absent from the lagrangian. For example, $E^{n,m}$ with opposite signs for n and m , as well as $E^{0,n}$, do not appear, and only the trace terms ψ and ψ^* are present. This is clearly so because these field components are gauge artifacts and their absence from the lagrangian is required by the tangent space symmetries. This fills a gap in reference (4), where no explanation was provided for the missing field components.

At this point, we will briefly review the argument

given, for example, in reference (10), which shows that unphysical fields decouple. Our purpose is to provide the background for the slightly more complicated case of the closed string. One has to decide first what the physical field is, and in this case, we know that ψ (or ψ^*) is the physical field. This choice must be justified by showing that ψ propagates with the correct equation of motion, is free of ghosts and decouples from the remaining Stueckelberg fields. All these nice things happen only when the gauge is properly chosen. Under the reparametrization transformations, the physical field transforms as

$$\psi \rightarrow \psi + \sum_{n=1}^{\infty} L_{-n} [f_n(x)]. \quad (3.19a)$$

This enables one to impose the gauge condition

$$L_n \psi = 0 \quad \text{for } n \geq 1. \quad (3.19b)$$

From the standard string lore, we know that in this gauge ψ is ghost free. Also, the terms that couple ψ and ψ^* to the Stueckelberg fields $E^{n,0}$ in (3.18) vanish in this gauge. With little bit more work, one can show that this decoupling carries over to the equations of motion, and therefore ψ satisfies the standard free string equation

$$(L_0 - 1) \psi = 0. \quad (3.20)$$

In closing the section, we would like to point out that in a semiclassical treatment like ours, one has to face several singular and somewhat ill-defined manipulations. These manipulations must eventually be justified by a careful operator treatment. For example, the replacement

$$L_0 \rightarrow L_0 - 1 \quad (3.21)$$

can be justified by a careful definition of this singular operator with the aid of a cutoff (17). There is also a singular linear term in the lagrangian proportional to

$$\Delta = \int d\tau' d\tau'' \xi^{(-)}(\tau', \tau'') \bar{\xi}(\tau' - \tau'') (1 + E^{\tau''0}), \quad (3.22)$$

which we have eliminated by shifting ψ by (an infinite) constant. Despite these singular manipulations, one can directly show that the action given by (3.18) is invariant under the reparametrization (3.14) in the critical space-time dimension of 26 (10,15).

4. The Closed String Lagrangian.

The machinery needed to construct the closed string lagrangian was already developed at the end of section 2. From the torsion tensor (2.24), we can write down a lagrangian which is the direct analogue of eq. (3.2) :

$$\mathcal{L}_1 = \frac{1}{2} D T_1, \quad (4.1a)$$

where,

$$D = \det(e_{i6,j}\tau)$$

, and

$$T_1 = \int \int \int d\tau d\tau' d\tau'' \xi^{(-)}(\tau, \tau'') \sum_{i,j} T_{ij}^{(-)}(\tau, \tau') \quad (4.1b)$$

We have the subscript 1 labeling the lagrangian to distinguish it from an additional term $\mathcal{L}_2$ that will be introduced later.

We will first, however, study the symmetry properties of (4.1). By construction, it is invariant under reparametrizations, and furthermore, the tangent space transformations (3.7) and (3.8) can easily be generalized to the present case. Starting with the transformation

$$e^{i6,j}\tau \rightarrow \int d\tau' \sum K_{ij}(\tau, \tau') e^{i6,j'}\tau', \quad (4.2)$$

one can repeat the analysis starting with eq. (3.4). The analogues of eq's (3.4), (3.5) and (3.9a) easily follow :

$$[K, \xi^{(-)}] = 0, \quad (4.3a)$$

$$\sum_i \int d\tau K_{ij}(\tau, \tau') = \det(K), \quad (4.3b)$$

$$\tau' [\xi^{(-)} K L_{i6} (K^{-1})] = 0. \quad (4.3c)$$

Eq. (4.3a) is again solved by block diagonalization.

$$K = K^{(+)} \oplus K^{(0)} \oplus K^{(-)}, \quad (4.4a)$$

where $K^{(\pm)}$ and $K^{(0)}$ have the same mode expansions as in (3.7b). Eq. (4.3c) again leads to the constraint

$$\det[K^{(+)}] = \det[K^{(-)}]. \quad (4.4b)$$

So far, everything is in perfect analogy to the open string case. The only difference is that the dynamical fields and also the transformation matrices carry two extra indices i and j , each of which take on two values. However, the analogy breaks down when one tries to generalize the second set of transformations given by (3.11). These transformations simply do not have any generalization to the closed string case.

There is also a serious problem with (4.1), which does not have a counterpart in the open string case. If we expand the vielbein by

$$e^{i6,j}\tau = \delta_{ij} \delta(\tau - \tau') + \sum_{n,m} E_{i,j}^{n,m} \exp[2\pi i(n\tau + m\tau')], \quad (4.5)$$

in powers of E , the quadratic lagrangian has the following form :

$$\mathcal{L}_q = i \psi^* (L_{1,0} + L_{2,0} - 2) \psi + \mathcal{L}_r, \quad (4.6a)$$

where,

$$\psi = \sum_n \sum_i E_{ij} \eta_j^n, \quad (4.6b)$$

and $\mathcal{L}_r$ is the rest of lagrangian, which we do need in explicit form at the moment. This lagrangian would give the correct equation of motion for ψ , if ψ decoupled from the unphysical fields. Unfortunately, it is not possible to choose a gauge which results in complete decoupling. Rather than explain the source of this difficulty here, we will first introduce the additional term which cures it, and explain later why this term is necessary. This term is given by

$$\mathcal{L}_2 = \frac{1}{2} D T_2, \quad (4.7a)$$

where,

$$T_2 = \iiint d\tau d\tau' d\tau'' \sum_{ijk} \frac{1}{2} \varepsilon^{ij} c_k T_{k\tau}^{i\tau, j\tau'} \quad (4.7b)$$

and ε^{ij} is the usual antisymmetric symbol, and $C_{1,2}$ are real constants. The total lagrangian is given by

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2. \quad (4.7c)$$

At first sight, the form of this term seems somewhat arbitrary; however, it simply reintroduces the zero modes in τ' and τ'' projected out by the metric $g_{\tau'\tau''}^{(-1)}$. In that case, could one not have introduced a similar term for the open string? The answer is no, since without the indices i and j , the integral vanishes because of the antisymmetry between τ

and τ' . This also shows that the antisymmetrization of the indices i and j is no restriction; these indices are already antisymmetric. It is then clear that T_2 is the most general expression one can construct out of the zero modes in variables τ , τ' and τ'' , of the torsion tensor. It depends on two arbitrary constants C_1 and C_2 , which is disturbing, since we know that the correct string action cannot have such an arbitrariness. However, these constants can be scaled to any fixed value by merely redefining the fields. As a convenient choice, we will take

$$C_1 = C_2 = 1. \quad (4.8)$$

To see that such a choice can always be made, we redefine the vielbein by

$$e_{i\tau, j\tau'} \rightarrow \sum_j A^{jj'} e_{i\tau, j'\tau'}, \quad (4.9)$$

where A is a numerical (X independent) two by two matrix. If we further impose on A the condition that

$$\sum_i A^{ij} = \det(A), \quad (4.10)$$

then the main term $\mathcal{L}_1$ is invariant under this redefinition. However, the subsidiary term $\mathcal{L}_2$ is not invariant, and the constants C_i transform according to

$$C_i \rightarrow \sum_j (A^{-1})^{ij} C_j. \quad (4.11)$$

The constraint (4.10) can be solved in terms of two arbitrary constants a and b :

$$A^{11} = a, \quad A^{12} = a - 1, \quad A^{21} = b, \quad A^{22} = 1 + b. \quad (4.12)$$

By a proper choice of these constants, the C 's can be adjusted to any desired value so long as

$$C_1 + C_2 \neq 0. \quad (4.13)$$

We shall assume this condition is satisfied, and discuss its significance later. From now on, we shall fix the C's by eq. (4.8).

It is remarkable that even with the additional term included, the tangent space symmetry discussed earlier is still valid with a slight modification. To insure the invariance of δ_2 , we have to supplement the conditions (4.4a) and (4.4b) by an additional constraint :

$$K_{ij}^{(0)} = \delta_{ij}. \quad (4.14)$$

The full lagrangian is then invariant under the transformations defined by (4.4a), (4.4b) and (4.14). As a result, the field components $E_{ij}^{n,m}$, where $|n| \geq 1, |m| \geq 1$ and n and m are of opposite sign, are redundant, with the exception of the trace given by (4.6b). This means that the only combination of these field components that appear in the quadratic lagrangian are ψ and ψ^* in analogy with the open string. However, since the second set of transformations, given by (3.11), have no counterpart for the closed string, the field components $E_{i,j}^{0,n}$, including $E_{i,j}^{0,0}$, are also present in the lagrangian. The four field components $E_{i,j}^{0,0}$, whose analogue $E^{0,0}$ was absent from the open string lagrangian, will play an important role in what follows.

We shall now examine the question of gauge choice and the decoupling of the physical field in the context of the free lagrangian. Our initial choice for the physical field is ψ , given by (4.6b), again in analogy with the open string. To investigate the question of decoupling, we only need to know that part of the lagrangian which depends on ψ and ψ^* :

$$\begin{aligned} \delta_2 = & i\psi^*(L_{1,0} + L_{2,0} - 2)\psi + iF^*(L_{1,0} + L_{2,0} - 2)\psi \\ & - iF(L_{1,0} + L_{2,0} - 2)\psi^* + G^*(L_{1,0} - L_{2,0})\psi \\ & + G(L_{1,0} - L_{2,0})\psi^* + i \sum_{n=1}^{\infty} \sum_{i,j} (E_{ij}^{n,0} L_{i,-n}\psi - E_{ij}^{-n,0} L_{i,n}\psi) \\ & + \delta r, \end{aligned} \quad (4.15a)$$

$$\begin{aligned} F = & \frac{1}{2} \sum_i E_{i,i}^{0,0} - \frac{1}{4} \sum_{i,j} E_{ij}^{0,0} \\ & + \frac{1}{4} \sum_{i,j} (E_{j,1}^{0,0} - E_{j,2}^{0,0}), \end{aligned} \quad (4.15b)$$

$$G = \frac{1}{4} \sum_j (E_{1,j}^{0,0} - E_{2,j}^{0,0}) - \frac{1}{4} \sum_{i,j} E_{ij}^{0,0}.$$

L_r generically contains only those terms which do not depend on ψ or ψ^* and therefore are of no interest for the present problem. Since the operator

$$L - 2 = L_{1,0} + L_{2,0} - 2, \quad (4.16)$$

is the inverse of the closed string propagator, we immediately see that ψ mixes with F . The true physical field χ is the field that diagonalizes (4.15a), and it is given by

$$\chi = \psi + F. \quad (4.17)$$

Written in terms of χ , the lagrangian reads,

$$\begin{aligned} \delta_2 = & i\chi\chi^*(L-2)\chi + G^*(L_{1,0} - L_{2,0})\chi \\ & + G(L_{1,0} - L_{2,0})\chi^* + i \sum_{n=1}^{\infty} \sum_{i,j} (E_{ij}^{n,0} L_{i,-n}\chi^* \\ & - E_{ij}^{-n,0} L_{i,n}\chi) + \delta r, \end{aligned} \quad (4.18)$$

where again δr contains only the χ -independent terms. We now need the gauge transformation properties of χ . Since under the linearized reparametrizations, the field components

$$E_{ij}^{n,m} \rightarrow E_{ij}^{n,m} + L_{j,m} f_i^n - 2\pi i(n+2m) \delta_{ij} f_i^{n+m}, \quad (4.19a)$$

it then follows that

$$\begin{aligned} \chi &\rightarrow \chi + \sum_i \sum_{n=1}^{\infty} L_{i,-n} f_i^n \\ &+ \frac{1}{4}(L_{1,0} - L_{2,0}) [f_1^0 - f_2^0 + i f_1^0 + i f_2^0]. \end{aligned} \quad (4.19b)$$

Now, as in the case of the open string, we can choose the functions f_i^n with $n \geq 1$, so that χ satisfies the gauge conditions

$$L_{i,n} \chi = 0 \quad \text{for } n \geq 1. \quad (4.20a)$$

However, we still have f_1^0 and f_2^0 at our disposal. These functions can be used to impose one more gauge condition on χ :

$$(L_{1,0} - L_{2,0}) \chi = 0. \quad (4.20b)$$

This gauge choice is possible since the operator $L_{1,0} - L_{2,0}$ is invertible acting on those χ which contain no eigenvector of this operator with zero eigenvalue. Any such χ is in the orthogonal complement of the space defined by (4.20b), and it can be set equal to zero by choosing

$$f_1^0 - f_2^0 + i f_1^0 + i f_2^0 = -4 (L_{1,0} - L_{2,0})^{-1} \chi, \quad (4.21)$$

and therefore, the gauge condition (4.20b) follows. Looking back at (4.18), we see that, with the gauge choice given by eq's. (4.20a) and (4.20b), the physical fields χ and χ^* completely decouple from the unphysical (Stueckelberg) fields G and $F_{i,j}^{n,0}$. It is also clear that the choice of the physical field is unique; for example, our old choice ψ would simply not lead to decoupling. Furthermore, the gauge condition (4.20b) is also necessary to achieve decoupling; it is rather gratifying to see this constraint emerge in a

natural fashion.

We end this section with a brief discussion of why the term in (4.7c) is needed, or more generally, why one cannot set

$$C_1 + C_2 = 0, \quad (4.22)$$

in violation of (4.13). Let us remember that, in order to impose the gauge condition (4.20b), one has to be able to solve eq. (4.21) in terms of real functions f_1^0 and f_2^0 . These functions are real, because they are zero modes of the real reparametrization functions $f_i(\delta)$, whereas χ , which appears on the right side, is complex. However, because of the appearance of "i" on the left hand side, there is no difficulty in finding f 's that satisfy this equation. If, however, both c 's are set equal to zero, the left hand side of eq. (4.21) becomes pure real, and then this equation cannot be satisfied. In the more general case of c 's that satisfy (4.22), the real and imaginary parts of the left hand side are related by a numerical factor, and again for a general χ , there is no solution. This explains the necessity of adding the term $\mathcal{L}_2$, with c 's not constrained by (4.22), to the lagrangian.

5. Conclusions

The main purpose of this paper was to clarify the gauge invariance properties of the lagrangian for the open string proposed in reference (4), and to extend it to the closed string. This being accomplished, the most pressing problem for future research is the question of gauge fixing. In particular, it is clear that anticommuting fields should be incorporated into the formalism from the beginning. After the gauge is fixed, one can then investigate the interacting part of the lagrangian to see whether it reproduces the

correct string amplitudes. It is clear that the program initiated in (4) and in this paper is still in its infancy, and a lot still remains to be done.

Acknowledgements

I would like to acknowledge an interesting discussion with John G. Taylor. I would also like to thank Professors Henri Ruegg and Maurice Jacob for hospitality at the University of Geneva and at CERN respectively while this work was in progress.

6. References

1. M. Kaku and K. Kikkawa, Phys. Rev. D10 (1974) 1110.
2. C. Marshall and P. Ramond, Nucl. Phys. B85 (1975) 375. For a more recent treatment, see C.R.R. Smith and J.G. Taylor, King's College, London, preprint (1986).
3. K. Bardakci, Nucl. Phys. B271 (1986) 561.
4. K. Bardakci, Berkeley Preprint (1986), to appear in Nucl. Phys. B.
5. See, for example, reference (12) and the discussion in reference (13).
6. C. Becchi, A. Rouet and R. Stora, Phys. Lett. 52B (1974) 344.
7. M. Kato and K. Ogawa, Nucl. Phys. B212 (1983) 443.
8. W. Siegel, Phys. Letters 149B (1984) 157, 151B (1985) 391, 396.

9. W. Siegel and R. Zwiebach, Nucl. Phys. B263 (1986) 105.
10. T. Banks and M. E. Peskin, Nucl. Phys. B264 (1986) 513; T. Banks, M.E. Peskin, C.R. Preitchoff, D. Friedman and E. Martinec, SLAC preprint (1985).
11. E. Witten, Nucl. Phys. B268 (1986) 253, and Princeton Preprint (1986).
12. J. Lykken and S. Raby, Los Alamos, preprint (1986).
13. T. Banks, SLAC preprint (June 1986).
14. H. Aratyn and A.H. Zimmerman, Phys. Letters 168B (1986) 130 and 168B (1986) 75.
15. A. Neveu, J.H. Schwarz and P.C. West, Phys. Letters 164B (1985) 51.
16. A. Neveu, H. Nicolai and P. West, Phys. Letters 168B (1986) 192, Nucl. Phys. B278 (1986) 601, and CERN preprints (1986).
17. L. Brink and H.B. Nielsen, Phys. Letters 45B (19873) 332.