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THE HIERARCHICAL QUARK MASS SPECTRUM AND FLAVOUR MIXING

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ABSTRACT

The observed hierarchy of the quark masses and its interpretation as a result of a specific evolution of a chiral symmetry breakdown is used to derive an associated expansion of the flavour mixing matrix. Requiring that this expansion is reproduced manifestly by the parametrization of the mixing matrix, a specific parametrization is singled out. A strong argument is provided to use this description in the future analysis of the experimental data.

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1. Introduction

The spectrum of quark masses observed in nature exhibits a hierarchical pattern. The ratios of the masses of the quarks belonging to subsequent generations, e.g. the ratios $m_d/m_s \approx 0.05$ or $m_s/m_b \approx 0.03$, are typically small compared to one. Furthermore the weak interaction mixing matrix describing the weak interaction eigenstates in terms of mass eigenstates exhibits another hierarchical pattern in the sense that the transition amplitudes describing the weak transitions between two adjacent generations, e.g. the mixing elements V_{us} or V_{cb} , are small, but the matrix elements for the weak transitions involving generations which are separated by more than one step, e.g. the elements V_{ub} or V_{td} , are significantly smaller than the latter. In [1] it was pointed out by the author that the observed hierarchical pattern can be understood as the result of a specific pattern of chiral symmetries whose breaking would cause the hierarchical tower of masses to appear step by step. Using such an approach one derives that the weak interaction mixing elements are expected to be specific functions of the quark mass eigenvalues in such a way that they are turned off automatically once the masses of a certain generation of quarks are turned off.

In general the mixing is described by a unitary matrix V . In the case of the observed three generations the matrix elements are parametrized by three angles and a phase parameter [2-6]. For an arbitrary number n of generations one has $1/2 n(n-1)$ angles and $1/2 (n-1)(n-2)$ phases, i.e. altogether $(n-1)^2$ parameters. It is well known that there exist many equivalent ways to choose these parameters [7-18]. Recently a specific form of the general mixing matrix has been suggested by Harari and Leurer [19] and by Plankl and the author [20], which has a number of advantages, in particular a one-to-one correspondence between the off-diagonal matrix elements V_{ij} and the mixing angles as well as the possibility for an easy generalization to an arbitrary number of generations.

In this note we should like to point out that the prescription for constructing the general mixing matrix given in [19,20] emerges uniquely if one considers the chiral expansion of the mixing matrix considered below. Instead of deriving our result directly for an arbitrary number of generations, we prefer to consider first the cases of two, three and four generations explicitly. Afterwards the generalized case is discussed.

2. Two generations

The two (lefthanded) weak doublets are denoted by

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad \begin{pmatrix} c \\ s \end{pmatrix}_L.$$

The masses of the quarks of the first generation are small compared to the masses of the c and s quark. If we request that in the limit $m_u = m_d = 0$ the quark mass matrix exhibits a chiral symmetry $U(1)_L \times U(1)_R$ acting on the members of the first generation the weak mixing matrix must be the unit matrix [1]. Subsequently the masses of the u and d quarks are introduced, the chiral symmetry is broken, and the mixing matrix becomes non-trivial. The mixing angle appears as a function of the ratios m_d/m_s and m_u/m_c [1]. In this case there exists, of course, only one possible parametrization of the mixing matrix, apart from a sign ambiguity, the well-known Cabibbo rotation matrix.

3. Three generations

The three weak doublets are denoted by

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad \begin{pmatrix} t \\ b \end{pmatrix}_L.$$

Following [1], we consider first the limit where $m_t, m_b \neq 0$ and all other masses are zero as a result of an underlying $SU(2)_L \times SU(2)_R$ -symmetry acting on the first two generations. In this limit the mixing matrix is the unit matrix. Subsequently the masses of the quarks of the second generation are introduced. The mixing matrix is given by:

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \quad (1)$$

($s_{23} = \sin\theta_{23}$, $c_{23} = \cos\theta_{23}$, θ_{23} : angle referring to the mixing of the second and third generation). As discussed in [1], the angle θ_{23} must be a function of the ratios m_c/m_t and m_s/m_b . Again the parametrization is uniquely fixed.

Finally the masses of the u and d quarks are introduced, and the mixing matrix takes its final form:

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}. \quad (2)$$

The elements V_{us} , V_{ub} , V_{cd} , and V_{td} are functions of the ratios m_d/m_s , m_u/m_c , m_d/m_b and m_u/m_t .

Of course, many ways exist to parametrize this matrix in terms of three angles and one phase. However one description is singled out by the following argument. The transition elements V_{cb} and V_{ts} depend in the limit $m_u = m_d = 0$ on mass ratios, involving the masses of the quarks of the second and third generation. Once the u and d masses are introduced, these matrix elements which do not involve the u and d quarks will remain essentially unchanged, apart from very small corrections due to unitarity. If we request that the parametrization of the matrix reflects this inertia of the V_{cb} and V_{ts} matrix elements, we must describe them in terms of the same angle θ_{23} , and not in terms of a combination of angles. One way to achieve this is to take

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{23} & S_{23} \\ 0 & -S_{23} & C_{23} \end{pmatrix} \begin{pmatrix} C_{13} & 0 & S_{13}^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -S_{13}e^{i\delta_{13}} & 0 & C_{13} \end{pmatrix} \begin{pmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (3)$$

$$= \begin{pmatrix} C_{12}C_{13} & S_{12}C_{13} & S_{13}e^{-i\delta_{13}} \\ -C_{23}S_{12} - C_{12}S_{23}S_{13}e^{i\delta_{13}} & C_{12}C_{23} - S_{12}S_{23}S_{13}e^{i\delta_{13}} & C_{13}S_{23} \\ S_{12}S_{23} - C_{12}C_{23}S_{13}e^{i\delta_{13}} & -C_{12}S_{23} - C_{23}S_{12}S_{13}e^{i\delta_{13}} & C_{13}C_{23} \end{pmatrix}$$

where S_{ij} , C_{ij} ($i, j = 1, 2, 3$) stand for $\sin\theta_{ij}$, $\cos\theta_{ij}$ respectively, and θ_{ij} can be viewed as the mixing angles describing the mixing between generation i and j . The index (13) of the phase δ_{13} indicates that this phase appears always in association with the angle θ_{13} . The parametrization eq. (3) has been discussed by Chau and Keung [5] and the author [6] following earlier work by Maiani [3].

Introducing the symbols $\tilde{R}_{ij}$ for the 3×3 (complex) rotation matrices in eq. (3), we can write:

$$V = R_{23} \cdot \tilde{R}_{13} \cdot R_{12}. \quad (4)$$

Besides this order of multiplication one can consider five other permutations e.g. $R_{23} R_{12} R_{13}$. Furthermore the phase parameter δ could appear in any of the three rotation matrices.

All these representations have the property that the chiral limit $m_u = m_d = 0$ can be realized simply by turning off the rotations R_{12} and R_{13} , and leaving the matrix R_{23} unchanged.

The parametrization given by Kobayashi and Maskawa [2] may serve as an example of a description which shows a "bad" behaviour in the chiral limit $m_u = m_d = 0$. This parametrization which will not be given here explicitly can be written as

$$V_{KM} = R_{23}(\theta_2) R_{12}(\theta_1) R_{23}(\theta_3) \quad (5)$$

(we neglect the phase). The limit $m_u = m_d = 0$ is achieved by turning off the rotation R_{12} , in which case one obtains

$$V_{KM} = R_{23}(\theta_2 + \theta_3), \quad (6)$$

i.e. the sum of the angles θ_2 and θ_3 is a function of the ratios m_s/m_b and m_c/m_t . Splitting up the rotation between the second and third generation into two parts as described in eq. (5) is, of course, allowed in a mathematical sense, but in our view rather awkward, since it introduces artificially features of the first chiral symmetry breaking generating m_u and m_d in the elements V_{cb} and V_{ts} , i.e. into places where they are not supposed to appear. Therefore the specific KM representation or similar ones, e.g. $V = R_{12} R_{13} R_{12}$, should not be used to describe the weak mixing.

One should use only those parametrizations for which the various stages of the mass generations by a sequence of chiral symmetry breaking can be carried through in such a way that the various rotation angles are either set to zero or remain invariant once the masses of a certain generation are set to zero. This singles out the representation given in eq. (3), or its five partners. In [6] the author has given phenomenological arguments related to the actual observed magnitudes of the various matrix elements why the specific form, given in eqs. (3,4) is the most suitable one to describe the weak mixing.

4. Four Generations

We denote the weak doublets by

$$\begin{pmatrix} u \\ d \end{pmatrix}_L \quad \begin{pmatrix} c \\ s \end{pmatrix}_L \quad \begin{pmatrix} t \\ b \end{pmatrix}_L \quad \begin{pmatrix} h \\ l \end{pmatrix}_L$$

The masses m_h and m_l are supposed to be heavy enough such that $m_t/m_h \ll 1$, $m_b/m_l \ll 1$. Four different steps are needed to generate the full mass spectrum. At the first step I only m_h and m_l become massive, and there is no flavor mixing. At step II m_t and m_b are introduced. The flavor mixing can be described in terms of a rotation between the third and fourth generation

$$V = R_{34}. \quad (7)$$

The associated rotation angle θ_{34} is a function of the ratios m_t/m_h and m_b/m_l .

At the next step III m_c and m_s are introduced. The flavor mixing can be described by a unitary 3×3 matrix (three angles, one phase). One way to describe the matrix is to use the analog of eq. (4) and write:

$$V = R_{34} \cdot \tilde{R}_{24} R_{23}. \quad (8)$$

The angles θ_{24} and θ_{23} are functions of the ratios m_c/m_t and m_s/m_b . Besides the possibility eq. (8) there exist five other possibilities to parametrize the matrix. We propose to use the description above.

The next step consists of turning on the masses m_u and m_d . The mixing matrix is a unitary 4×4 matrix which can be parametrized in terms of six angles and three phases. It can be obtained from the matrix eq. (8) by turning on the (complex) rotations between the first and the other generations, e.g. by writing

$$V = R_{34} \tilde{R}_{24} R_{23} \tilde{R}_{14} \tilde{R}_{13} R_{12}. \quad (9)$$

The angles θ_{14} , θ_{13} and θ_{12} must be functions of the ratios m_u/m_t , m_u/m_c , m_d/m_b and m_d/m_s . It is worthwhile to emphasize that the four steps in the mass generation can be followed directly by looking at the evolution of the mixing matrix:

$$\begin{aligned}
 \text{I:} & \quad I \\
 \text{II:} & \quad R_{34} \\
 \text{III:} & \quad (R_{34}) (\tilde{R}_{24} R_{23}) \\
 \text{IV:} & \quad (R_{34}) (\tilde{R}_{24} R_{23}) (\tilde{R}_{14} \tilde{R}_{13} R_{12}).
 \end{aligned} \tag{10}$$

The mixing matrix obtained at the next step is simply the one obtained at the previous step, multiplied by a number of (complex) rotation matrices. The parametrization is such that the rotation angles appearing step by step are functions of the corresponding quark masses turned on at the particular step. In this sense the parametrization given in eq. (9) is unique, apart from permutations of the rotations appearing at each step (e.g. writing $R_{23} \cdot \tilde{R}_{24}$ instead of $\tilde{R}_{24} R_{23}$). However the order of the various blocks of rotation matrices, indicated by the parentheses in the last row of eq. (10), cannot be changed. Note that only the matrices R_{ij} with $|i-j| > 1$ are written with a $\sim$ - superscript, i.e. they involve a complex phase, following the prescription given in [19,20].

The matrix (9) is the one proposed in [19,20]. There the matrix $V = R_{34} \tilde{R}_{24} \tilde{R}_{14} R_{23} \tilde{R}_{13} R_{12}$ was introduced. Since the commutator $[\tilde{R}_{14}, R_{23}]$ vanishes, this matrix is identical to the one given in eq. (9). The complete form of the 4×4 mixing matrix was written down in [20] and will not be repeated here.

5. Flavor Mixing for n Generations

The (lefthanded) weak doublets are denoted by

$$\begin{pmatrix} U_1 \\ D_1 \end{pmatrix}_L, \quad \begin{pmatrix} U_2 \\ D_2 \end{pmatrix}_L, \quad \dots \quad \begin{pmatrix} U_n \\ D_n \end{pmatrix}_L,$$

the mass eigenvalues by m_{U_i}, m_{D_i} ($i = 1, 2, \dots, n$). We suppose that the physical quark masses obey a hierarchical order:

$$\frac{m_{U_{i-1}}}{m_{U_i}} \ll 1, \quad \frac{m_{D_{i-1}}}{m_{D_i}} \ll 1, \tag{11}$$

in accordance with observation, as far as the first three generations are concerned.

Following [1] we consider various hypothetical limits in which the masses of the first k generations ($k = 1, 2, \dots, n$) are set to zero. In this case the mass term is required to obey a chiral $U(k) \times U(k)$ symmetry. As a result the flavor mixing matrix, denoted by $V^{(k)}$, is diagonal for the first k generations:

$$V^{(k)} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & V_{k+1,k+1} & \dots & \dots & V_{k+1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & V_{n,k+1} & \dots & \dots & V_{n,n} \end{pmatrix} \quad \begin{matrix} 1 \\ 2 \\ 3 \\ \vdots \\ k \\ k+1 \\ \vdots \\ n \end{matrix} \quad (12)$$

Thus a nontrivial mixing takes place only among the massive generations $k+1, \dots, n$. We arrive at the physical mixing matrix V by a succession of n steps:

Step 1: The masses of the n^{th} generation are introduced. The chiral symmetry reduces from $U(n)_L \times U(n)_R$ to $U(n-1)_L \times U(n-1)_R$. The flavor mixing matrix $V^{(n-1)}$ is diagonal.

Step 2: The masses of the $(n-1)^{\text{th}}$ generation are introduced. The chiral symmetry reduces to $U(n-2)_L \times U(n-2)_R$. The flavor mixing $V^{(n-2)}$ is simply a rotation among the generations $(n-1)$ and (n) , described by one angle $\theta_{n-1,n}$, which must be a function of the ratios $m_{U_{n-1}}/m_{U_n}$ and $m_{D_{n-1}}/m_{D_n}$: $V^{(n-2)} = R_{n-1,n}$.

Step 3: The $(n-2)^{\text{th}}$ generation becomes massive. The chiral symmetry reduces to $U(n-3)_L \times U(n-3)_R$. The flavor mixing is described by the 3×3 submatrix:

$$V^{(n-3)} = R_{n-1,n}(\tilde{R}_{n-2,n} R_{n-2,n-1}). \quad (13)$$

The mixing angles $\theta_{n-2,n}$ and $\theta_{n-2,n-1}$ are functions involving the masses of the quarks of the $(n-2)^{\text{th}}$ generation.

The procedure can be repeated a number of steps until we arrive at step k , where the masses of the generation $n-k+1$ are introduced. The mixing matrix is given by

$$V^{(k)} = (R_{n-1,n}) (\tilde{R}_{n-2,n} R_{n-2,n-1}) \cdots (\tilde{R}_{n-k+1,n} \tilde{R}_{n-k+1,n-1} \cdots R_{n-k+1,n-k}). \quad (14)$$

The new mixing angles $\theta_{n-k+1,n-1} \dots \theta_{n-k+1,n-k}$ appearing at this step are functions involving the masses of the k^{th} generation. Finally we arrive at the last step n , where the masses of the first generation are introduced. The full mixing matrix is given by

$$V = V^{(n)} = \underset{1}{(R_{n-1,n})} \underset{2}{(\tilde{R}_{n-2,n} R_{n-2,n-1})} \underset{3}{(\tilde{R}_{n-3,n} \tilde{R}_{n-3,n-1} R_{n-3,n-2})} \cdots \underset{n}{(\tilde{R}_{1n} \tilde{R}_{1n-1} \cdots R_{12})}. \quad (15)$$

In eq. (15) we have denoted by the parentheses and the numbers below them those blocks of matrices appearing at each step of the mass generation.

The order of the rotation matrices given in eq. (15) is different from the one proposed in [19,20]. However a closer inspection shows that due to the vanishing of the commutators $[R_{ij}, R_{kl}]$ ($i \neq k, j \neq l$) the two parametrizations are, in fact, identical.

Conclusions:

In our view the mass spectrum of the quarks and the phenomenon of weak interaction mixing must be closely related to each other. The observed hierarchical spectrum of the quark masses can be viewed as the result of a cascade-like breaking of the chiral symmetry in the space of the generations which would be exact if all quark masses were zero, as discussed in [1]. Step by step the masses of the various generations are introduced. Along with the mass generation goes the evolution of the flavor mixing matrix which starts out as the unit matrix and develops more and more elements as more generations become massive. The actual parametrization of the mixing matrix is, of course, not of direct relevance for the underlying physics. However if we require that the expansion of the mixing matrix is manifestly described by introducing more mixing parameters and not changing the ones introduced before, the

parametrizations given in eqs. (4,9,15) are singled out. We believe that this is a strong argument, apart from the convenience stressed in [19,20], to use these parametrizations. Finally we stress that these parametrizations reflect our view that the weak interaction mixing is indeed a cascade-like process which starts with the heaviest generation and proceeds downward step by step. This view can be realized in a simple way by special forms of the quark mass matrix, e.g. the one given in [21,22]. Of course, physics is developing in the opposite direction by proceeding from the light generations to the heavier ones. But it is time to realize that the pattern of the mass generation of the quarks and presumably also of the leptons can only be understood by going the other way.

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