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THE MASS GAP OF THE GROSS-NEVEU MODEL FOR ELEMENTARY FERMIONS

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A B S T R A C T

We calculate in perturbation theory, up to the first non-trivial order, the finite size mass gap for the elementary fermions of the Gross-Neveu fermionic model. The connection with the condensate which breaks the discrete chiral symmetry of the model is discussed.

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One of the most interesting two-dimensional fermionic field theories is the Gross-Neveu model, described by the Lagrangian (Euclidean metric)

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m_B)\psi - \frac{1}{2} f_B (\bar{\psi}\psi)^2, \quad (1)$$

where a summation over the flavour index of the fermionic fields ψ^i , $i = 1, \dots, N$ is assumed¹⁾.

This model has asymptotic freedom, dynamical breaking of the discrete chiral symmetry $\psi \rightarrow \gamma_5 \psi$, a non-perturbative mass generation and a rich bound state spectrum²⁾. The model is classically integrable³⁾ and the exact S-matrix for different sectors of the spectrum has been found⁴⁾⁻⁷⁾. In this letter we study the question of the dependence of the dynamically generated mass gap on the coupling constant. We know that for large N the mass gap of the fermions is $(m_B \rightarrow 0)^{1), 8)}$

$$m = \Lambda_{\overline{MS}}, \quad N \rightarrow \infty, \quad (2)$$

if we choose to renormalize the coupling constant in the particular renormalization scheme $\overline{MS}$ ⁹⁾. Here $\Lambda_{\overline{MS}}$ is the invariant scale of the model defined up to two loops as

$$\Lambda_{\overline{MS}} = \mu (b_0 f)^{-b_1/b_0^2} e^{-\frac{1}{b_0 f}}, \quad (3)$$

where b_0, b_1 are the coefficients of the beta function of the model calculated in Ref. 10)

$$\begin{aligned} \beta(f) &= \mu \frac{\partial}{\partial \mu} f = -f^2 (b_0 + b_1 f + \dots), \\ b_0 &= \frac{N-1}{\pi}, \quad b_1 = -\frac{N-1}{2\pi^2}. \end{aligned} \quad (4)$$

For general N , one should calculate the constant of proportionality in the equation

$$m = C \cdot \Lambda_{\overline{MS}}. \quad (5)$$

We shall estimate C using the finite size method introduced by Lüscher¹¹⁾ and justified in 12) and 13). For two-loop calculations in $O(N)$ and CP^{N-1} models, see 14) and 15).

We consider the G-N model in the strip $D = [0, L] \times [-\infty, +\infty]$ with periodic boundary conditions in space and plane wave b.c. in time

$$\psi(x_0, 0) = \psi(x_0, L), \quad \bar{\psi}(x_0, 0) = \bar{\psi}(x_0, L) \quad (6)$$

$$\forall x_0 \in [-\infty, +\infty]$$

We regularize in the ultra-violet the Green's functions of the theory by using extra $-\epsilon$ flat dimensions with plane wave b.c.; that is, the propagator for the massive fermions is

$$S_F(x, y; m) = \frac{1}{L} \sum_{l=-\infty}^{+\infty} \int \frac{d^D k}{(2\pi)^D} \frac{e^{i k(x-y)}}{k + m_B} \quad (7)$$

where $D = 1 - \epsilon$ and

$$k_1 = \frac{2\pi l}{L}, \quad k_0 = \omega.$$

We shall define the dynamically generated mass in the limit $m_B \rightarrow 0$ as the pole of the propagator in momentum space¹⁶⁾. In a previous work, we have shown that the $l = 0$ modes condense and give a well-defined perturbative condensate $\langle \bar{\psi}\psi \rangle$ ⁸⁾. The propagation of massless fermions in such an environment gives a dynamical mass to the fermions which we are going to calculate up to two loops, by looking at the pole of the propagator. For this purpose we calculate the two-point 1-PI function of the fermions

$$\Sigma(k, m_B, \frac{1}{L}) = A(k^2, m_B, \frac{1}{L}) k + m_B B(k^2, m_B, \frac{1}{L}) + \frac{1}{L} C(k^2, m_B, \frac{1}{L}) \quad (8)$$

In Ref. (8), we separate out the piece which vanishes as m_B when $m_B \rightarrow 0$ or as h when $h \rightarrow 0$. The answer we find for A, B, C is that for $m_B \rightarrow 0$

$$A \rightarrow 1 + O(f_B^2), \quad (9a)$$

$$m_B \cdot B \rightarrow 0, \quad (9b)$$

$$C \rightarrow f_B c_1(\frac{1}{L}) + f_B^2 c_2(\frac{1}{L}). \quad (9c)$$

The above behaviour comes out after some tedious but straightforward algebra of the 1-PI diagrams in Figs. 1a-b and 2a-f.

From Eqs. (9a)-(9c), we deduce that up to order f_B^2 , the fermion self mass Σ has a zero which is the pole of the propagator at

$$K = f_B C_1\left(\frac{1}{L}\right) + f_B^2 C_2\left(\frac{1}{L}\right). \quad (10)$$

This pole defines the dynamically generated mass which is renormalized only through the coupling constant renormalization

$$m(L) = -f_B C_1\left(\frac{1}{L}\right) - f_B^2 C_2\left(\frac{1}{L}\right). \quad (11)$$

Only the zero space-momentum modes in the diagrams 1a-b contribute to the coefficient $C_1(1/L)$ ($D=1-\epsilon$)

$$\begin{aligned} C_1\left(\frac{1}{L}\right) &= -\frac{2N-1}{L} \int \frac{d^D \omega}{(2\pi)^D} \frac{m_B}{\omega^2 + m_B^2} \\ &= -\frac{2N-1}{2L} \left[1 + \frac{\epsilon}{2} (\ln \pi - \gamma_E - 2 \ln m_B) \right]. \end{aligned} \quad (12)$$

Here $\gamma_E = 0.5772$ is the Euler-Macaroni constant and we keep the term proportional to ϵ , which is infra-red divergent, to get the correct result at the order f_R^2 , after coupling constant renormalization.

The order f_B^2 coefficient $C_2(1/L)$ gets a contribution from all the six diagrams 2a-f. The contributions of the diagrams 2a, 2b, 2c, 2e come out proportional to the same integral

$$(2a) = -4N^2 I, \quad (13a)$$

$$(2b) = 2N I, \quad (13b)$$

$$(2c) = (2b), \quad (13c)$$

$$(2e) = -I, \quad (13d)$$

where ($D = 1-\epsilon$),

$$I = \frac{1}{2} \frac{\partial}{\partial m_B} \left\{ \frac{1}{L} \sum_{\omega=0}^{\omega=\infty} \int \frac{d^D \omega}{(2\pi)^D} \frac{m_B}{\omega^2 + m_B^2} \right\}^2 \quad (14)$$

$$= \frac{1}{4\pi L} \left[\frac{1}{\epsilon} + \frac{1}{2} (\ln L^2 + \gamma_E - \ln 4\pi) + \frac{1}{2} (\ln \pi - \gamma_E - 2 \ln m_B) \right]. \quad (15)$$

Miraculously enough the diagrams 2e and 2d give a similar contribution

$$(2d) = (2c), \quad (13e)$$

$$(2f) = (2e). \quad (13f)$$

The total contribution to C_2 therefore is:

$$C_2 = - \frac{(2N-1)(N-1)}{2\pi L} \left[\frac{1}{\epsilon} + \frac{1}{2} (\ln L^2 + \gamma_E - \ln 4\pi) + \frac{1}{2} (\ln \pi - \gamma_E - 2 \ln m_B) \right]. \quad (16)$$

Finally we renormalize the coupling constant¹⁰⁾

$$f_B = f \left[1 - \frac{N-1}{\pi} f \frac{1}{\epsilon} \right], \quad (17)$$

and we check that the $1/\epsilon$ term in (16) as well as the infra-red term cancel correctly

$$m(L) = f \frac{2N-1}{2L} + f^2 \frac{(2N-1)(N-1)}{4\pi L} (\ln \mu^2 L^2 + \gamma_E - \ln 4\pi). \quad (18)$$

The similarity with the mass gap of the $O(2N)$ non-linear sigma model^{11),14)} is striking. There is also a correspondence of the G-N and $O(2N)$ beta functions up to two loops. The G-N model has up to two loops, a fixed point at $f = 2\pi$ while $O(2N)$ at $f = -2\pi$.

We proceed to construct the function^{11),14)} $[z=m(L), L]$

$$\frac{m(L)}{\Lambda_{MS}} = F(z) = K e^{a_{00}/2b_0 z} z^{\frac{b_1}{2b_0} + 1}, \quad (19)$$

where

$$K = \left(\frac{b_0}{a_{00}} \right)^{b_1/b_0^2} e^{a_{10}/a_{00} b_0}, \quad (20)$$

and a_{00} , a_{11} , a_{10} are defined as

$$m(L) = \frac{1}{L} \left[a_{00} f + f^2 (a_{11} \ln \mu^2 L^2 + a_{10}) \right]. \quad (21)$$

In our case

$$a_{00} = \frac{2N-1}{2}, \quad a_{11} = \frac{(2N-1)(N-1)}{4\pi}, \quad (22a)$$

$$a_{10} = \frac{(2N-1)(N-1)}{4\pi} (\gamma_E - \ln 4\pi). \quad (22b)$$

The minimum of the function $F(z)$ gives an estimate of the constant C ¹¹⁾

$$m(L) \underset{L \rightarrow \infty}{=} C \Lambda_{\overline{MS}}. \quad (5)$$

To get a more precise estimate, in Ref. 16), we take into account finite volume corrections^{12),13),17)}. If we redefine $\Lambda_{\overline{MS}} = \Lambda_{\overline{MS}} e^{\frac{1}{2}(\ln 4\pi - \gamma_E)}$, then

$$\frac{m(L)}{\Lambda_{\overline{MS}}} = G(z) = \frac{2}{4\pi} e^{\frac{\pi}{z}} e^{\gamma_E} \left[\frac{2N-1}{2(N-1)} \frac{\pi}{z} e^{\frac{\pi}{z}} \right]^{\frac{1}{2(N-1)}}. \quad (23)$$

The minimum of $G(z)$ in

$$z_{\min} = \pi \frac{2N-1}{2N-3}, \quad (24)$$

and it has the value

$$C = \frac{m(L)}{\Lambda_{\overline{MS}}} \Big|_{\min} = \frac{2N-1}{4(2N-3)} \left[\frac{2N-3}{2(N-1)} \right]^{\frac{1}{2(N-1)}} e^{\frac{2N-3}{2(N-1)} + \gamma_E}. \quad (25)$$

Finally we come to the relation of the dynamical mass and the condensate $\langle \bar{\psi}\psi \rangle$ calculated in Ref. 8) to be:

$$\langle \bar{\psi}\psi \rangle = \frac{N}{L} \left[1 + \frac{2N-1}{4\pi} f (\ln \mu^2 L^2 + \gamma_E - \ln 4\pi) \right]. \quad (26)$$

In Ref. 18), Politzer found that in the short-distance expansion of the fermion propagator, in a chirally asymmetric vacuum, the dynamically generated mass is given [here $\dim(\bar{\psi}\psi)=1$] by

$$m(L) \underset{L \rightarrow 0}{=} c_{\bar{\psi}\psi}(f; L, \mu) \langle 0 | \bar{\psi}\psi | 0 \rangle (\mu, L) \\ + \text{less singular terms in } L \rightarrow 0 . \quad (27)$$

The coefficient function starts at order f and satisfies the RGE

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(f) \frac{\partial}{\partial f} - \gamma_{\bar{\psi}\psi} \right] c_{\bar{\psi}\psi} = 0 , \quad (28)$$

so that $m(L)$ is RGE invariant. In our framework, we can isolate $C_{\bar{\psi}\psi}$ by using the expression for $\langle 0 | \bar{\psi}\psi | 0 \rangle$ given in (26). We find that

$$c_{\bar{\psi}\psi} = \frac{2N-1}{2N} f \left[1 - \frac{1}{4\pi} f (\ln \mu^2 L^2 + \gamma_E - \ln 4\pi) \right], \quad (29)$$

and we check RGE (28).

In the large N limit we must rescale $f \rightarrow f/N$ and the coefficient function becomes

$$c_{\bar{\psi}\psi} = \frac{f}{N} ,$$

so that

$$m(L) \underset{N \rightarrow \infty}{=} f \frac{\langle \bar{\psi}\psi \rangle}{N}$$

as is shown in Refs. 1) and 8).

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1a

1b

2a

2b

2c

2d

2e

2f