LOOP GROUPS, GRASSMANIANS AND STRING THEORY

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ABSTRACT

The theory of representations of loop groups provides a framework where one can consider Riemann surfaces with arbitrary numbers of handles and nodes on the same footing. Using infinite Grassmannians we present a general formulation of some conformal field theories on arbitrary surfaces in terms of an operator formalism. As a by-product, one can obtain some general results for the chiral bosonization of fermions using the vertex operator representation of infinite dimensional groups. We believe that this set-up provides the natural arena where the recent proposal of Friedan and Schenker of formulating string theory in the universal moduli space can be discussed.

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1. One of the open problems in the theory of strings is how to extend the first quantized formulation [1] to obtain semiclassical or non-perturbative information. In the first quantized approach one computes the partition and correlation functions for a conformally invariant two dimensional field theory on a genus g Riemann surface, and then sums over arbitrary genus. Since one would expect very important non-perturbative effects as a consequence of string dynamics, the perturbative series is unlikely to make much sense. In a recent proposal, Friedan and Shenker [2] try to formulate string theory in some kind of universal moduli space, which would include surfaces of arbitrary genus, with and without nodes, and including infinite genus surfaces as well. In this approach, the generating functional for the string S-matrix is written formally as

$$S = \int_R h(\psi, \psi^*) \quad (1)$$

where R is the "universal moduli" space for Riemann surfaces, and ψ is a section of a holomorphic vector bundle over R . The bundle group would be the modular group of R . In this letter we present a framework where this appealing formulation can be realized more explicitly. Since one of the main motivations for this formulation is provided by the treatment of string theory as a conformal field theory; there are a number of basic issues that should be addressed. In ref.3 one finds a general treatment of conformally invariant field theories on the twice punctured sphere (conformally equivalent to a cylinder). The extension of this analysis to higher genus surfaces encounters a number of problems, for example: i) The treatment of the zero modes of the conformal fields; responsible for the inclusion of moduli and supermoduli parameters in string path integrals [4]. ii). The lack of a natural operator formalism. iii). The meaning of the Virasoro algebra still remains quite obscure... There has been some progress in these problems [5,6], but we are still far from a complete understanding. One would like to have a general framework which treats Riemann surfaces with and without nodes on the same footing, and which incorporates the main features of the operator methods that can be developed in a local coordinate patch, but without losing track of any of the global information implicit in the fact that we want to deal with surfaces of complicated topology. Mathematical objects with these properties appear naturally in the theory of representations of infinite dimensional groups[6]. These objects are the

infinite grassmanians.

Some of the features which make these objects useful for the study of conformal field theories are : i). The collection of all Riemann surfaces; with and without nodes , together with the local data provided by a conformal field theory (for example the line bundle on which the primary fields take their values) , generates a dense set . ii). In analogy with finite dimensional moduli spaces , the infinite grassmannian is quasi-compact , i.e. the only globally defined holomorphic functions are constants.iii).It provides a topology where one may start asking questions concerning the role of infinite genus surfaces , (for example , it is possible to study the limits of sequences of surfaces with growing numbers of handles) .iv). Using representation theory for infinite dimensional groups it is possible to reconstruct the correlation functions for free conformal field theories and ghost systems on arbitrary surfaces , thus providing the analog of an operator formalism for higher genus. The formulae obtained using these methods coincide with the results derived using bosonization techniques [8,9,10] ; thus lending support to the general framework advocated in this letter.

2. In the formulation of conformal field theory on the twice punctured sphere , one can associate an "in" vacuum to the north pole , and an out vacuum to the south pole. Since the north and south poles are related by an SL_2 transformation ; if we use an SL_2 invariant vacuum , we can relate what happens in the north and south poles using standard operator methods. On a higher genus surface Σ , we will try to follow as closely as possible our intuition from the operator approach. In order to make the discussion concrete , we will concentrate on the conformal properties of first order anticommuting b-c systems [3.b] of some given conformal spin j and $1-j$ respectively. Geometrically b and c are sections of holomorphic line bundles L_j , L_{1-j} over Σ . We also pick a point p (the analog of the north pole on the sphere) , and a local coordinate t vanishing at p . Choosing the local parameter conveniently, we can assume that it extends to a local neighborhood U_0 of p homeomorphic to a disk in the complex plane and containing the unit circle $S^1=\{t; |t|=1\}$. In the functional integral approach , the partition function of the b-c system is given in terms of the determinant of the Cauchy-Riemann operator $\bar{\partial}_j$. A convenient way of thinking about $DET \bar{\partial}_j$ is as a line bundle over moduli space : if $Ker \bar{\partial}_j$ and $Cok \bar{\partial}_j$ are the kernel and cokernel of $\bar{\partial}_j$ respectively , $DET \bar{\partial}_j$ is defined as a section of $(\lambda Ker \bar{\partial}_j)^{-1} \otimes (\lambda Cok \bar{\partial}_j)$ (λV is the highest exterior power of

the vector space V , and V^{-1} is the vector space dual to V). The first thing we would like to do is to capture the global information contained in $\text{Ker } \bar{\partial}_j$ and $\text{Cok } \bar{\partial}_j$ in terms of local data. To achieve this, let $U_1 = \Sigma - p$; then U_0, U_1 provide an open cover of Σ , intersecting in a disc minus a point. Let $\Gamma(U_j, L)$ and $\Gamma(U_0 \cap U_1, L)$ be the spaces of holomorphic sections of L over U_0, U_1 and $U_0 \cap U_1$ respectively. Given a nowhere vanishing local holomorphic section σ_0 of L_j over U_0 , we can identify sections of L with complex valued analytic functions f via $\sigma = f\sigma_0$. Thus we can identify $\Gamma(U_0, L)$ with convergent power series in U_0 , $f = \sum f_n t^n$, $n=0,1,\dots$, and $\Gamma(U_0 \cap U_1, L)$ with convergent series in $U_0 \cap U_1$, $f = \sum f_n t^n$, $n \in \mathbb{Z}$. If pr denotes the projection from $\Gamma(U_1, L)$ into $\Gamma(U_0 \cap U_1, L) / \Gamma(U_0, L)$, we have the exact sequence

$$0 \rightarrow \text{Ker } \bar{\partial}_j \rightarrow \Gamma(U_1, L) \xrightarrow{\text{pr}} \Gamma(U_0 \cap U_1, L) / \Gamma(U_0, L) \rightarrow \text{Cok } \bar{\partial}_j \rightarrow 0 \quad (2)$$

and we conclude that $\text{Ker } \text{pr} = \text{Ker } \bar{\partial}_j$, $\text{Cok } \text{pr} = \text{Cok } \bar{\partial}_j$; and $\text{DET } \text{pr} = \text{DET } \bar{\partial}_j$. The construction of the infinite grassmannian of interest proceeds as follows[7]: Let $H = L^2(S^1, \mathbb{C})$ be the space of square integrable functions on the circle. A basis for this space is given by $\{t^k, k \in \mathbb{Z}\}$. Set $H = H_+ \oplus H_-$, with H_+ (resp. H_-) spanned by $t^k, k=0,1,2,\dots$ (resp. $t^k, k=-1,-2,\dots$). A linear subspace w of H belongs to the grassmannian Gr if two conditions are met: i). The orthogonal projection pr_- from w to H_- is a Fredholm operator, i.e., it has finite dimensional kernel and cokernel. ii). The orthogonal projection pr_+ from w into H_+ is compact. The index of pr_- is called the virtual dimension of w . Thus Gr splits into an infinite number of components labelled by indpr . Since pr_- must be Fredholm, the full group of invertible operators acting on H does not act on Gr , but there is still a rather large group acting on Gr : GL_{res} . Writing its elements in a decomposition adapted to the splitting $H_+ \oplus H_-$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then a and d must be Fredholm and b, c must be compact. There are several subgroups of GL_{res} which will be important in our discussion: $\Gamma_+, \Gamma_-, \text{GL}(\infty)$ and their central extensions. The group $\Gamma_+ (\Gamma_-)$ is defined as multiplication by nowhere vanishing analytic functions on the circle which extend holomorphically into the disc $|t| \leq 1$ ($|t| \geq 1$) and giving the identity at $t=0$ ($t=\infty$). The group Γ_+ acts on Gr without fixed points. $\text{GL}(\infty)$ is a Kac-Moody group of infinite rank whose elements are infinite dimensional invertible matrices g such that $g-1$ contains only a finite number of non-vanishing entries. With respect to the basis of H we have

introduced, the generators of $GL(\infty)$ are the operators $E_{ij}(t^j)=t^i$; or in matrix form $(E_{ij})_{kl}=\delta_{ik}\delta_{jl}$, for any integers i,j,k,l . We can think of the space H as the space of wave functions for a free field moving on S^1 , where H_- (H_+) represents the space of positive (negative) frequency solutions of the wave equation. Thus we can construct a Fock space F out of H , which admits several useful and equivalent representations. We can represent the elements in the Fock space F as semi-infinite forms $\psi_{i_1} \wedge \psi_{i_2} \wedge \psi_{i_3} \wedge \dots$ for a basis ψ_i of H . If ψ_i corresponds to the basis introduced above for H (the element t^i is a solution to the wave equation with energy $-i$. We will identify ψ_i with the wave function of energy i), the state $\psi_0 \wedge \psi_{-1} \wedge \psi_{-2} \wedge \dots$ can be identified with the filled Dirac sea. The group $GL(\infty)$ acts naturally on F because by definition the elements of this group have well-defined determinants [11]. Similarly, as in field theory we can introduce a set of fermionic creation and annihilation operators ψ_n, ψ_n^* and field operators $\psi(z) = \sum \psi_n \bar{z}^{n-1/2}$, $\psi^*(z) = \sum \psi_n^* \bar{z}^{-n-1/2}$, $n \in \mathbb{Z} + \frac{1}{2}$ so that the semi-infinite forms are represented by acting with creation and annihilation operators on the Dirac vacuum (the vacuum conditions are $\psi_n |0\rangle = \psi_n^* |0\rangle = 0, n \geq 0$). In this way, the generators of $gl(\infty)$ are represented by fermion bilinears $E_{ij} = \psi_i \psi_j^*$. Finally a third useful way of thinking about the same states is in terms of the determinant line bundle of $Gr : DET$. Its fiber at w in Gr is a one dimensional space spanned by elements of the form $\Lambda f = f_1 \wedge f_2 \wedge f_3 \dots$ for a basis of w . Again the group $GL(\infty)$ acts naturally on the space of sections of DET , $\Gamma(DET)$, and we can certainly think of semi-infinite forms as sections of DET . The importance of DET is that it can easily be shown that DET_{pr} is a section of DET^{-1} . The group GL_{res} does not act naturally on DET , because its elements will in general induce changes of basis by means of operators without well defined determinants; nevertheless, for a certain subgroup of GL_{res} , (generated by elements such that b, c are trace class) it is possible to define a projective action on DET [7].

Contact with conformal field theory follows by noting that a local conformal field theory gives a set of geometrical data $(\Sigma, p, t, L, \sigma_0)$. To this data we can associate a point w in Gr , given by the set of analytic functions f on S^1 such that $f\sigma_0$ extends to a section in $\Gamma(U_1, L)$. The converse is not true in general. However, Segal and Wilson show [7] that any point on $Gr_* = \{w \in Gr; \quad t^{-q}H_- \subset w \subset t^q H_- \text{ for some } q > 0\}$ comes from geometrical data. This result is useful because Gr_* is dense on Gr , and it is also the union of a sequence of finite dimensional grassmannians. These two facts together imply that any globally

defined holomorphic function on Gr is a constant ; one of the distinctive features of Gr alluded to in the introduction. As a consequence of these remarks , Gr contains the moduli space of all Riemann surfaces ; including singular ones .

Let us for the time being restrict our attention to the zero component of Gr , $Gr^{(0)}$ (points of Gr of virtual dimension zero). From the exact sequence (1) we know that we are dealing with line bundles of spin $1/2$ over Σ on $Gr_*^{(0)}$. We will later extend the construction to higher spin fields. Using the realization of the space of sections of DET in terms of semi-infinite forms , we can set up an abstract fermionic free field theory on $Gr^{(0)}$. Since we want to have a fermion number current and an energy- momentum tensor with a well-defined action on the vacuum , we introduce the Lie algebra of $A(\infty)$, (a central extension of $gl(\infty)$ [18]) , generated by $E_{ij} = :\psi_i \psi_j^*$: (normal ordering with respect to the Dirac vacuum). The fermionic states can also be represented in terms of sections of DET^{-1} . If w is a point in Gr , we can associate with it the section

$$\Psi_w(w') = \det \langle w, w' \rangle \quad (3)$$

with the determinant defined with respect to a choice of admissible basis. Two bases are admissible if the operator relating them has a determinant. Thus the vacuum section is defined for $w_0 = H_-$. Introducing creation and annihilation operators on $\Gamma(DET^{-1})$, we can construct a mapping from the set of semi-infinite forms into the fermionic Fock space with respect to the vacuum Ψ_{w_0} . If j_n represents the Fourier components of the fermion number current $j(z) = :\psi(z)\psi^*(z):$ the action of Γ_+ and Γ_- in the Fock space can be written in terms of the j_n 's. A generic element of Γ_+ (Γ_-) takes the form $\exp(\sum_{n=1}^{\infty} x_n t^n)$ ($\exp \sum_{n=1}^{\infty} y_n t^{-n}$) where the x 's and y 's are the parameters of $\Gamma_{\pm}$. The corresponding representation in F is respectively given by $H(x) = \sum_{n=1}^{\infty} x_n j_n$, $H(y) = \sum_{n=1}^{\infty} y_n j_{-n}$. In this abstract field theory defined on $Gr^{(0)}$, the proof of bosonization and the computation of fermionic amplitudes can be done in terms of the vertex operator representation of $A(\infty)$, (see [11] for details and references). The first step consists of identifying the space of semi-infinite forms with the space of formal power series $C[x_1, x_2, \dots]$. If $S = \{i_0, i_{-1}, i_{-2}, \dots\}$ represents a Dirac partition ; i.e. an infinite sequence of integers such that $i_0 > i_{-1} > i_{-2} > \dots$ and satisfying $i_n = n$ for $|n|$ large enough , we can construct the state $v_S = \psi_{i_0} \psi_{i_{-1}} \psi_{i_{-2}} \dots$ (recall that in our notation ψ_n is

a solution to the free wave equation with energy n). The associated element in $C[x_1, x_2, \dots]$ is

$$F_S(x_1, \dots) = S_{i_0, i_1+1, i_2+2, \dots}(x) \quad (4)$$

where $S_{\lambda_1, \lambda_2, \dots}(x)$ is a Schur function defined for any finite sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \dots$ as

$$S_{\lambda_0, \lambda_1, \dots}(x) = \frac{1}{z!} (p_{\lambda_i - i + j}(x)) \quad 0 \leq i, j \leq r-1$$

$$\exp \sum_{k=1}^{\infty} z^k x_k = \sum_{n=0}^{\infty} z^n p_n(x), \quad p_0(x) = 1 \quad (5)$$

and r is chosen large enough so that $\lambda_i = 0$ for $i > r-1$. The Schur function for a given set of λ 's is the same as the character formulae for $GL_N(\mathbb{C})$ for big N . On $C[x]$ the generators of Γ are represented by an infinite Heisenberg algebra. For n positive, j_n becomes multiplication by nx_n and j_{-n} becomes $\partial/\partial x_n$, and the rest of the algebra of $gl(\infty)$ is obtained in terms of vertex operators

$$\sum_{i,j} \bar{u}^i \bar{v}^j E_{ij} = -\frac{(uv)^{-1}}{1-u/v} \quad T_0(u, v) = T_0(u) T_0(v)^* \quad (6.a)$$

$$T_0(u, v) = \exp \sum_{j \geq 1} (\bar{u}^j - \bar{v}^j) x_j \cdot \exp - \sum_{j \geq 1} \frac{1}{j} (u^j - v^j) \partial/\partial x_j \quad (6.b)$$

$$T_0(u) = \frac{1}{u} \exp \sum_{j \geq 1} \bar{u}^j x_j \exp \sum_{j \geq 1} \frac{1}{j} u^j \partial/\partial x_j \quad (6.c)$$

$$T_0(u)^* = T(u, -x)$$

for the normal ordered generators of $A(\quad)$, one obtains instead

$$\sum_{i,j} \bar{u}^i \bar{v}^j \hat{E}_{ij} = -\frac{(uv)^{-1}}{1-u/v} (T_0(u, v) - 1) \quad (7)$$

The hermitian form in Fock space satisfying $j_n^+ = j_{-n}$ induces a hermitian form in $C[x]$:

$$\langle f(x_1, \dots), g(x_1, \dots) \rangle = \left\{ \left(\frac{\partial}{\partial x_1}, \frac{1}{2} \frac{\partial}{\partial x_2}, \dots, \frac{1}{n} \frac{\partial}{\partial x_n}, \dots \right) g(x_1, \dots) \right\} \Big|_{x_i=0} \quad (8)$$

thus, if Ψ_ω is any section of DET^{-1} , the correspondence just described implies that

$$\langle \Psi_0, \Psi_\omega \rangle = \langle 1, F_{S(\omega)}(x) \rangle = F_{S(\omega)}(0) \quad (9)$$

and $S(\omega)$ is the Dirac partition associated to ω . Notice that (6.a) is nothing but the bosonic representation of $\Psi(u)\Psi^*(v)$. Therefore matrix elements of fermionic operators between any state and the vacuum can be represented as

$$\langle \Psi_0, \Psi(u)\Psi^*(v)\Psi_\omega \rangle = \left. T_0(u)T_0^*(v) F_{S(\omega)}(x) \right|_{x=0} \quad (10)$$

In particular we expect all of these formulae to reproduce the field theoretic results obtained on the twice punctured sphere. Here we find that (6-10) do not completely yield the answer that one would expect. The reason is very simple. If one computes for example $\langle \Psi_0, \Psi(u)\Psi^*(v)\Psi_0 \rangle$ one gets

$$\left. T_0(u)T_0^*(v)1 \right|_{x_i=0, i>0} = \frac{-1}{1-u/v} \cdot \frac{1}{uv} \quad (11)$$

therefore, even though one gets the correct singularity, the residue at the pole is incorrect. This is easily fixed by including one more variable on the vertex operators but not on the Schur functions. The new variable x_0 corresponds to enlarging Γ_+ so that $g(t) \in \Gamma_+$ can now take any non-zero value at $t=0$. From the fermionic Fock space point of view this means that we include in $H(x)$ a term $x_0 j_0$ which counts the fermionic charge of the state where it acts. If we now define

$$T(u, x) = \exp\left(\sum_{n=0}^{\infty} x_n \bar{u}^n\right) \exp\left(-\ln u \frac{\partial}{\partial x_0} - \sum_{n=1}^{\infty} \frac{1}{n} u^n \frac{\partial}{\partial x_n}\right) \quad (12)$$

and in the hermitian form (8) one requires at the end that x_0 vanishes along with the other x_i 's. Using (12) one now gets the desired result

$$\left. T(u)T^*(v) \cdot 1 \right|_{\substack{x_i=0 \\ i \geq 0}} = \frac{1}{u-v} \quad (13)$$

and the rest of the correlation functions at tree level then follow from standard vertex operator manipulations.

The reason why this abstract fermionic system is interesting is because to the data $(\Sigma, p, t, L_{1/2}, \sigma_0)$ we can associate a point in $Gr_*^{(0)}$. What is more interesting yet, is that using Krichever's construction of solutions to the K-P hierarchy [12], the Schur function $F_{S(w)}(x)$ for a point $w \in Gr_*^{(0)}$ is given in by the Riemann theta function of the surface Σ it represents with characteristics determined by the spinor bundle $L_{1/2}$. In this case the Schur function is known as a τ -function for the K-P hierarchy, and it is explicitly given by [13]

$$F_S(x) = \exp \sum_{n,m=1}^{\infty} Q_{nm} x_n x_m \vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left(\sum_{n=1}^{\infty} \vec{A}_n x_n \mid \Omega \right) \quad (14)$$

The various factors in (14) are defined as follows : if $\omega_i(t)dt$ denotes the abelian differentials on Σ normalized as

$$\int_{a_i} \omega_j = \delta_{ij} \quad \int_{b_i} \omega_j = \Omega_{ij} \quad (15)$$

with respect to a canonical homology basis ; the coefficients A appearing in (14) are given by the coefficients of the Taylor expansion of the ω 's around p

$$\omega_i = (A_{i1} + A_{i2}t + \dots) dt \quad (16)$$

and the quadratic form Q_{nm} is constructed using the normalized differentials of the second kind. If η_n is the differential of the second kind with zero a -periods and with a single pole at P of order $n+1$. η_n is normalized so that close to p its integral takes the form

$$\int^P \eta_n = \bar{t}^n - 2 \sum_{m=1}^{\infty} Q_{nm} t^m / m \quad (17)$$

recalling that the differentials of the second kind are obtained by taking logarithmic derivatives of the prime form of the surface (see for example [14]), it is easy to check that the fermion propagator is given by

$$\frac{\langle \psi_{w_0}, \psi(z) \psi(w)^* \psi_w \rangle}{\langle \psi_{w_0}, \psi_w \rangle} = \frac{\vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] \left(\int_w^z \vec{\omega} \mid \Omega \right)}{\vartheta \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix} \right] (0 \mid \Omega) E(z, w)} \quad (18)$$

and similarly, if we consider multiple point functions we obtain an answer involving prime forms and theta functions

$$\frac{\langle \psi_{\omega_0}, \psi(z_1) \psi^*(w_1) \dots \psi(z_n) \psi^*(w_n) \psi_{\omega} \rangle}{\langle \psi_{\omega_0}, \psi_{\omega} \rangle} = \frac{\prod_{i < j} E(z_i, z_j) \prod_{k < l} E(w_k, w_l)}{\vartheta[\frac{\alpha}{p}](0|\Omega) \prod_{i,k} E(z_i, w_k)} \vartheta[\frac{\nu}{p}]\left(\sum_i z_i - \sum_k w_k\right) \quad (19)$$

as expected from the general results of bosonization in higher genus surfaces [8-10]. Note however that (19) follows from straightforward vertex operator algebra, and that we can deal with chiral fermions from the beginning. Thus by working with the abstract field theory provided by the grassmannian, we can use operator methods, to obtain the desired correlation functions on a Riemann surface, by computing matrix elements between the vacuum ψ_{ω_0} and the state ψ_{ω} representing in Gr the geometrical data. It is in this sense that one can develop an operator formalism for higher genus surfaces.

3. Before extending the previous results to higher spin fields on Σ , we would like to give a geometrical picture of what is going on, and understand where the Virasoro algebra appears. Since in our construction we always need a triple (Σ, p, t) , let $\hat{M}_g$ be the moduli space of such triples for genus g surfaces. If M_g denotes the moduli space for curves, there is a natural projection map $p: \hat{M}_g \rightarrow M_g$. The universal curve $\hat{C}$ over $\hat{M}_g$ is a complex manifold fibered over $\hat{M}_g$, whose fiber at $\hat{m}$ is a copy of the triple (Σ, p, t) parametrized by $\hat{m}$. If $\hat{C}$ is the universal curve over M_g , then $p^*(\hat{C}) = \hat{C}$. Along the fibers of $\hat{C}$ we can consider the line bundle of fields of spin j . These fit together to give a line bundle over $\hat{C}$: $\hat{\omega}^j$ ($\hat{\omega}$ stands for the line bundle of differentials). At $\hat{m} \in \hat{M}_g$, we have the space of sections $\Gamma(\hat{C}_{\hat{m}} - p, \hat{\omega}^j|_{\hat{C}_{\hat{m}}})$. The local coordinate t gives a trivializing section $\sigma_0 = dt^j$, and thus we can get a point in Gr as explained before. The holomorphic map $k_j: \hat{M}_g \rightarrow \text{Gr}^{(i)}$

($i = (2j-1)(g-1) = \text{index of } \bar{\partial}_j$) obtained this way will be called the Krichever map and the set of points obtained on Gr the Krichever locus. The family version of the exact sequence (1) implies that the pull back $k_j^*(\text{DET})$ is a line bundle over $\hat{M}_g$ whose fiber at $\hat{m}$ is $(\text{DET } \bar{\partial}_j)^{-1}$. Since $\text{DET } \bar{\partial}_j$ only involves coordinate independent information on Σ , it follows that $k_j^*(\text{DET})$ is trivial along the fibers of p , and we have $k_j^*(\text{DET}) \simeq p^*(\lambda_j)$ where $\lambda_j = (\text{DET } \bar{\partial}_j)^{-1}$ is the determinant line bundle for spin j fields over M_g . Using the Grothendieck-Riemann-Roch theorem [15], or using the

Quillen metric on λ_j [16], we can compute the first Chern class of $k_j^*(\text{DET})$. The result is

$$c_1(k_j^*(\text{DET})) = c_1(\vartheta^* \lambda_j) = (6j^2 - 6j + 1) \vartheta^*(c_1(\lambda_1)) \quad (20)$$

again by (1) we have the identification $k_j^*(\text{DET}) \simeq \vartheta^*(\text{DET } \bar{\partial}_j)$. The disadvantage of the Krichever map k_j introduced is that it maps M to different components of Gr according to the value of j . We know from bosonization in higher genus surfaces [8-10], that the partition and correlation functions for a general $b-c$ system can be expressed in terms of either spin 1/2 or spin 0 correlation functions. This is achieved in this context by introducing a modified Krichever map [17]. The universal curve $\hat{C}$ has a natural section $s: \hat{M}_g \rightarrow \hat{C}$ given by $s(\Sigma, p, t) = p \in \Sigma$, the image $\Delta = s(\hat{M}_g)$ is a divisor on $\hat{C}$. The line bundle $\hat{\omega}^j(-n\Delta)$ (the family of j -differentials vanishing to at least order n at p), has degree $2j(g-1)-n$. Thus if we choose $n=n_j = (2j-1)(g-1)$, we have that $\mathcal{L}_j = \hat{\omega}^j(-n_j\Delta)$ has degree $g-1$ and zero index, and the Krichever map $k(\mathcal{L}_j): \hat{M}_g \rightarrow \text{Gr}^{(0)}$ given by $\mathcal{L}_j$ with $\sigma_0 = t^{n_j} dt^j$ takes values on $\text{Gr}^{(0)}$. It is possible to show that [17]

$$k(\mathcal{L}_j)^*(\text{DET}) \simeq \vartheta^*(\lambda_j) \otimes s^* \hat{\omega}^{-\alpha} \quad (21)$$

$$\alpha = (2j-1)^2 g(g-1)/2$$

This result agrees with the general bosonization rules of [8-10] in a particular limit as will be discussed later. A geometric realization of the Virasoro algebra for any conformal field theory on Σ can now be described. At $\hat{m} \in \hat{M}$ we have the Lie algebra of meromorphic vector fields on $U_0 \cap U_1: \mathcal{D} = T(u_0 \cap u_1, \hat{\omega}_{\hat{m}}^{-1})$, ($\hat{\omega}_{\hat{m}}^{-1}$ are holomorphic vectors along the curve labelled by $\hat{m}$). A basis for this algebra is given by $X_n = -t^{n+1} d/dt$ satisfying $[X_n, X_m] = (n-m)X_{n+m}$. For surfaces of genus $g \geq 1$ we can single out the following subspaces [19]:

- i). $\mathcal{D}_1 = T(u_1, \hat{\omega}_{\hat{m}}^{-1})$. The subalgebra of vector fields which induce holomorphic automorphisms of $U_1 = \hat{C}_{\hat{m}} - p$. Its action does not change any of the data registered in M .
- ii). $\mathcal{D}_0 = T(u_0, \hat{\omega}_{\hat{m}}^{-1})$. The subalgebra of vector fields extending holomorphically to U_0 . Those vector fields vanishing at p induce infinitesimal variations in the local parameter t , while the constant vector X_{-1} induces infinitesimal translations of p . Hence $\mathcal{D}_0$ can be identified with the vertical tangent space $T_{\hat{m}}^v \hat{M}_g$ to the fiber of p at $\hat{m}$.

iii). The remaining linear space $\mathcal{D}/\mathcal{D}_0 \oplus \mathcal{D}_1$ is isomorphic to $H^1(\hat{\mathcal{C}}_{\hat{m}}, \hat{\omega}_{\hat{m}}^{-1})$ the tangent space to moduli space at $m=p(\hat{m})$.

Summarizing, we see that at any m there is an isomorphism $T_{\hat{m}} \hat{M}_g \cong \mathcal{D}/\mathcal{D}_0$. Letting $\hat{m}$ vary in $\hat{M}_g$, the collection $X_n(\hat{m}) = -t^{n+1} \frac{d}{dt} \big|_{\hat{m}}$ gives rise to a vector field on $\hat{M}_g$. Its action on $\hat{M}_g$ and on the determinant line bundle can be studied directly on the Krichever locus of $Gr^{(0)}$ by pushing $X_n(\hat{m})$ forward using $k(\mathcal{Y}_j)$. To construct this vector field $k(\mathcal{Y}_j)_* (X_n)$, notice that at any $\hat{m}$ the algebra $\mathcal{D}$ acts naturally on the space $T^*(u_0 \cap u_1, \hat{\omega}_{\hat{m}}(-n_j p))$ by the Lie derivative along the fibers of $\hat{\mathcal{C}}$. If $\sigma = t^k \sigma_0$, ($\sigma_0 = t^{n_j} dt^j$) is a local section of $\hat{\omega}_{\hat{m}}^1(-n_j p)$ we have

$$\mathcal{L}_{X_n} \sigma = (k + n_j + (n+1)j) t^{k+n} (t^{n_j} dt^j) \quad (22)$$

inducing an infinitesimal action on the basis of H $\delta_n t^k = (k + n_j + (n+1)j) t^{k+n}$. This gives a representation of $\mathcal{D}$ in GL_{res} . If we compute the action of this algebra DET , one obtains a central extension for the image of the Virasoro algebra proportional to $6j^2 - 6j + 1$ as should be expected from (20). Making contact with the bosonization formulae [8-10] is slightly more difficult in this case; and it is easy to understand why. In the approach of ref.8 one starts with a line bundle ξ over Σ of degree $> 2g-2$ and a set of points $P_1, \dots, P_k$, $k = \text{ind } \partial_j$ in general position. This means that $\xi(-\sum P_i)$ is a line bundle of degree $g-1$ without a holomorphic section. In the construction of ref. [17], all of these points are identified with p . Since the P 's are used in the field theoretic approach to insert b-fields and remove the zero modes, it is clear that in order to compare the formulae obtained from $k(\mathcal{Y}_j)$ on $\hat{M}_g$ and those obtained by bosonization, we have to take the limiting case where all the P 's coincide. The section of $\hat{\omega}^\alpha$, $\alpha = (2j-1)^2(g-1)g/2$ that one has to pull-back to the surface in (21), [17] is the one whose locus of zero modes coincides with the Weierstrass points of ω . This gives the clue for the construction of the Schur function in this case. Even though there is yet no complete derivation from the grassmannian point of view of the Schur function that we will write down, the proof should be a consequence of extending the construction of [17] so that $n_j p$ is replaced by a divisor of degree n_j in general position. Combining the arguments presented for spin 1/2 together with the results of [8-10] we obtain a prescription for the computation of correlation functions for a b-c system of spin j .

$$\left\langle \prod_{i=1}^n b(z_i) \prod_{j=1}^m c(w_j) \right\rangle \quad n-m = n_j \quad (23)$$

on a general surface in terms of the vertex operator representation of $A(\infty)$. Let ψ_ω be the section of DET^{-1} associated to the geometrical data embodied in the b-c system. Since ψ_ω has fermion number charge $-n_j$, choose n_j points on the surface (or add n_j formal parameters to $C(x)$) $u_1, u_2, \dots, u_{n_j}$. Define the Schur function as

$$\Gamma^*(u_1) \dots \Gamma^*(u_{n_j}) F(x) \quad (24)$$

where $F(x)$ is the right hand side of (14), and the characteristics depend on how we chose to parametrize holomorphic line bundles on Σ (see for instance [8] for some useful choices). Now (23) becomes

$$\prod_{i=1}^m \Gamma(z_i) \prod_{l=1}^n \Gamma^*(w_l) \prod_{k=1}^{n_j} \Gamma^*(u_k) F(x) \Big|_{x=0} \quad (25)$$

comparing (25) with the chiral bosonization formulae results in [8-10] (in ref.[8] only non-chiral bosonization formulae are derived. It is however straightforward to extract their "meromorphic" square root, and one obtains the results of [9,10]), we find that after some theta function gymnastics they agree. In particular if $\phi_i(z)$, $i=1, \dots, n_j$ are the holomorphic j -differentials, then (25) coincides with the holomorphic square root of

$$|\det \phi_i(z_j)|^2 \cdot \frac{\det' \bar{\partial}_j^+ \bar{\partial}_j}{\det \langle \phi_i | \phi_j \rangle} \left(\frac{\det' \bar{\partial}_0^+ \bar{\partial}_0}{\int \Gamma_g \det \text{Im} \Omega} \right)^{1/2} \quad (26)$$

if we take the limit as the u_i 's become equal to p , both (25-26) vanish. The leading non-vanishing terms are proportional to the wronskian of the holomorphic j -differentials at p . When p moves over Σ , the divisor of zeros of the wronskian is given by the Weierstrass points on the surface as implied by (21). As in the case of spin 1/2, the rather simple prescription embodied in (25) allows us to compute arbitrary correlation functions of any free b-c system on arbitrary surfaces using the vertex operator representation of $A(\infty)$. The only non-trivial step in this operator construction of the correlation function (23) is the identification of the state ψ_ω and the corresponding Schur function associated to the geometrical information conveyed by the b-c system.

We believe that the approach advocated in this letter provides the natural framework to formulate conformal field theory and string theory on higher genus surfaces using operator methods. Further details will be presented elsewhere [21].

While this work was being completed, we received the work of N. Ishibashi, Y. Matsuo and H. Ooguri [22], which also analyzes some conformal theories on higher genus surfaces using similar techniques. We have also received a recent paper by M. Dugan and H. Sonoda [23] which also present a derivation of the chiral bosonization formulas for b-c systems using field theoretic methods. We would like to thank P. Ginsparg for bringing this reference to our attention.

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