CERN-TH.4637/87

A LATTICE CALCULATION OF THE SECOND MOMENT
OF THE PION'S DISTRIBUTION AMPLITUDE

G. Martinelli and C.T. Sachrajda⁺

CERN - Geneva

A B S T R A C T

We calculate $\langle \xi^2 \rangle$, the second moment of the pion's distribution amplitude on a $10^3 \times 20$ lattice, with Wilson fermions in the quenched approximation and at $\beta = 6.0$. We find $\langle \xi^2 \rangle = 0.26 \pm 0.13$, in the lattice renormalisation scheme at $a \approx (1.8 \text{ GeV})^{-1}$. This is in disagreement with the previous lattice determination of this quantity. The reasons for this discrepancy are discussed.

+ On leave of absence from the Department of Physics, The University, Southampton SO9 5NH, United Kingdom.

The non-perturbative information required to predict hard exclusive processes in QCD, such as hadronic form factors or elastic scattering at large momentum transfers, is contained in the quark distribution amplitudes $\Phi(x_i, Q^2)^{1)-5)}$. Q^2 is the square of the momentum transfer and the label i takes values from 1 to n_v , where n_v is the number of valence partons. $\sum_{i=1}^{n_v} x_i = 1$ and in a physical gauge x_i is interpreted as the fraction of momentum carried by the valence parton i . In perturbation theory we are only able to predict the behaviour of Φ with Q^2 (this, of course, is also the case for deep inelastic structure functions), hence for the calculation of Φ itself we have to go beyond perturbation theory. In this letter we shall present the results of a lattice calculation of the second moment of the quark distribution amplitude of the pion.

The moments of the distribution amplitude can be expressed as matrix elements of the lowest twist local operators between the hadron and the vacuum⁶⁾. For the pion, defining

$$\xi = x_q - x_{\bar{q}} \quad (1)$$

we have

$$\langle 0 | O_{\mu_0 \mu_1 \dots \mu_n}(0) | \pi(p) \rangle = \sqrt{2} f_\pi p_{\mu_0} p_{\mu_1} \dots p_{\mu_n} \langle \xi^n \rangle \quad (2)$$

+ terms containing factors of $p^2 g_{\mu_i \mu_j}$,

where f_π is the pion's decay constant $\approx 94 \text{ MeV}$,

$$O_{\mu_0 \mu_1 \dots \mu_n} = (-i)^n \bar{\psi} \gamma_{\mu_0} \gamma_5 \overleftrightarrow{D}_{\mu_1} \dots \overleftrightarrow{D}_{\mu_n} \psi \quad (3)$$

symmetrized over the Lorentz indices, and

$$\langle \xi^n \rangle \equiv \int_0^1 d\xi \xi^n \phi(\xi, Q^2) \quad (4)$$

with the normalisation chosen so that $\langle \xi^0 \rangle = 1$. $\langle \xi^n \rangle$ contains an explicit Q^2 behaviour, where in Eq. (2) Q^2 is the renormalisation scale of the operators $O_{\mu_0 \dots \mu_n}$. Below we shall present our results for $\langle \xi^2 \rangle$ obtained from a lattice calculation. We start, however, with a brief review of previous calculations of this quantity.

The operator matrix elements (2) have been studied in considerable detail using QCD sum rules^{5),7)-9)}. The most complete study, using sum rules for two point functions, is given in Refs. 5) and 7), and the results for the low moments are very close to those one would obtain from the distribution amplitude

$$\phi(\xi, Q^2 \approx (0.5 \text{ GeV})^2) = \frac{15}{4} \xi^2 (1 - \xi^2) \quad (5)$$

We also recall that as $Q^2 \rightarrow \infty$, $\Phi \rightarrow \Phi_{\text{as}} \equiv 3/4 (1 - \xi^2)$. Equation (5) has the interesting feature that Φ is not peaked at $\xi = 0$, but at $\xi \approx \pm 0.7$ where either the quark or the anti-quark carries about 85% of the pion's momentum. In contrast the asymptotic distribution amplitude Φ_{as} is peaked around $\xi = 0$. For the second moment the results are

$$\begin{aligned} \langle \xi^2 \rangle &= 0.40 \pm (15-20)\% \text{ at } Q^2 = 1.5 \text{ GeV}^2 \text{ (ref. 5,7)} \\ \langle \xi^2 \rangle &= 0.39 \text{ at } Q^2 = 2.25 \text{ GeV}^2 \text{ (ref. 8)} \\ \langle \xi^2 \rangle &= 0.38 \pm (10-15)\% \text{ at } Q^2 = 0.25 \text{ GeV}^2 \text{ (ref. 9)} \end{aligned} \quad (6)$$

to be compared with $\langle \xi^2 \rangle = 0.2$ at $Q^2 = \infty$. Thus QCD sum rules seem to indicate that the distribution amplitude of the pion is significantly broader than the asymptotic one.

There is also a lattice calculation of $\langle \xi^2 \rangle$ ¹⁰⁾, (on a $6^2 \times 12 \times 18$ lattice, at $\beta = 5.7$, using Wilson fermions in the quenched approximation), yielding the spectacular result

$$\langle \xi^2 \rangle = 1.58 \pm .23 \text{ at } Q = 1 \text{ GeV} \quad (7)$$

in the $\overline{\text{MS}}$ renormalisation scheme. Equation (7) would imply that Φ not only has to have at least one maximum either side of $\xi = 0$, but that it also has to change sign. This result is clearly in conflict with those obtained using QCD sum rules.

From the above discussion, we see that the calculation of $\langle \xi^2 \rangle$ requires the evaluation of the operator matrix element (2), with $n = 2$. Since Lorentz (or Euclidean) invariance is not a symmetry of the lattice, different choices of the indices μ_0, μ_1, μ_2 give rise not only to significant differences in the details of the calculations, but also raise important questions of principle. The operators $O^{(2)}$ in general mix under renormalisation with operators of lower dimension, the mixing coefficients containing power divergences of the form $1/a^2$ and $1/a$ (as can be seen even in first-order perturbation theory). Although for

presently available lattices ($\beta \approx 5.7-6.0$), the one-loop contribution to these mixing coefficients may be relatively small, the perturbative subtraction of these power divergences has been shown in several cases to give a wrong result, as for example in the evaluation of the critical value of the Wilson parameter K , and of $\langle \bar{\psi}\psi \rangle$. By "perturbative subtraction" we mean the subtraction of the matrix elements of the operators of lower dimension (determined from lattice calculations) times the mixing coefficients calculated in perturbation theory. In order to avoid the problem of power divergences, we must take a combination of the operators $O^{(2)}$ which for symmetry reasons cannot mix with lower dimensional ones. This excludes the possibility of taking $O_{\mu\mu\mu}$, i.e. an operator with all three Lorentz indices equal, as was done in Ref. 10). The four operators $O_{\mu\mu\mu}$, $\mu = 1-4$, transform like the $(\frac{1}{2}, \frac{1}{2})$ irreducible representation of the hypercubic group [we are using the notation of Ref. 11) where the character table for this group can be found] and mix under renormalisation with $\bar{\psi}\gamma_4\gamma_5\psi$, $\partial_4\bar{\psi}\gamma_5\psi$ and $\bar{\psi}\sigma_{4\alpha}\gamma_5\psi$, which transform in the same way under space-time symmetries $(\frac{1}{2}, \frac{1}{2})$.

In principle the choice $O_{\mu\nu\rho}$, with μ , ν and ρ all different, is the most attractive. The four operators of this class transform like the $(\frac{1}{2}, \frac{1}{2})$ irreducible representation of the hypercubic group [in the notation of Ref. 11)]. Since no lower dimensional operator transforms in this way, there is no mixing with these operators under renormalisation. On the other hand, in order to get a non-zero matrix element at least two spatial components of p must be non-zero. On currently available lattices, one would expect that in this case the pion's momentum is probably too large to avoid lattice artefacts, an expectation which is borne out by our numerical studies.

This leaves the final possibility $O_{\mu\nu\nu}$ with $\mu \neq \nu$. The twelve operators in this set transform like the reducible $(\frac{1}{2}, \frac{1}{2}) + 8$ representation of the hypercubic group. In order to eliminate mixing with lower dimensional operators we take a linear combination of these operators which transforms like the eight-dimensional irreducible representation. A convenient choice is $O_{411}-O_{433}$. Thus one can eliminate the problem of power divergences, at the price of having to introduce just one non-zero spatial component of momentum ($p_1 \neq 0$, $p_2=p_3=0$ say). It seems to us that the third option is the optimal one on currently available lattices; indeed it is probably the only acceptable one. For the remainder of this letter we will consider matrix elements of the operator $O \equiv O_{411}-O_{433}$, with $p_1 = \pi/5a$, the smallest non-zero value on our $10^3 \times 20$ lattice, and $p_2 = p_3 = 0$. We then have

$$\langle 0 | O(0) | \pi(p) \rangle = \sqrt{2} f_\pi p_4 p_1^2 \langle 3^2 \rangle \quad (8)$$

In order to "measure" $\langle \xi^2 \rangle$ we consider the following two ratios of correlation functions:*)

$$C_1(t) \equiv \frac{\sum_x e^{i p \cdot x} \langle 0 | O(x) \bar{\psi}(0) \gamma_5 \psi(0) | 0 \rangle}{\sum_x e^{i p \cdot x} \langle 0 | \bar{\psi}(x) \gamma_4 \gamma_5 \psi(x) \bar{\psi}(0) \gamma_5 \psi(0) | 0 \rangle} \quad (9a)$$

and

$$C_2(t) \equiv \frac{\sum_x e^{i p \cdot x} \langle 0 | O(x) \bar{\psi}(0) \gamma_4 \gamma_5 \psi(0) | 0 \rangle}{\sum_x e^{i p \cdot x} \langle 0 | \bar{\psi}(x) \gamma_4 \gamma_5 \psi(x) \bar{\psi}(0) \gamma_4 \gamma_5 \psi(0) | 0 \rangle} \quad (9b)$$

where the summations are over the lattice points in time slice t , both in the numerators and the denominators. If t is sufficiently large so that the correlation functions are dominated by the single pion intermediate state, we have

$$C_1(t) = C_2(t) = \text{constant} = p_1^2 \langle 3^2 \rangle. \quad (10)$$

Note that Eq. (10) is valid even if we do not explicitly symmetrize on the Lorentz indices in O_{411} and O_{433} , and below we shall not symmetrize.

Our calculations were performed on a $10^3 \times 20$ lattice at $\beta = 6.0$ using Wilson fermions in the quenched approximation. We used 15 configurations at three different values of the Wilson parameter K , ($K = 0.1515, 0.1530, 0.1545$) and extrapolated to the physical value $K = 0.1564$. Further details of the simulation and of the generation of the quark propagators can be found in Ref. (13). For the discrete form of the derivatives we take the symmetric prescription

*) f_π itself has also been calculated with the lattice used in this paper¹³⁾, with the result $f_\pi = 110 \pm 20$ MeV.

$$\partial_\mu f(x) = \frac{1}{2a} [f(x+a\hat{\mu}) - f(x-a\hat{\mu})] \quad (11a)$$

where $\hat{\mu}$ is a unit vector in the μ direction, and for the covariant derivative

$$D_\mu f(x) = \frac{1}{2a} [U_\mu(x) f(x+a\hat{\mu}) - U_\mu^\dagger(x-a\hat{\mu}) f(x-a\hat{\mu})] \quad (11b)$$

where $U_\mu(x)$ is the gauge group matrix defined on the link from x to $x+a\hat{\mu}$.

Our first task is to demonstrate the existence of a plateau for $C_1(t)$ and $C_2(t)$, i.e. that for sufficiently large t and $20-t$ (recall that our lattice has 20 points in the time direction) these quantities are independent of t . As will be seen below the errors in the measurements of $C_1(t)$ and $C_2(t)$ are relatively large. For this reason we verify the dominance of the single pion contribution to the unitarity sum by studying the correlation functions with 0 replaced by O_{411} or O_{433} . As an illustration of the general features of the results, we plot in Fig. 1 the results for $C_2(t)$ with 0 replaced by O_{411} and O_{433} with $K = 0.1530$. This is a typical example. The results for $K = 0.1515$ have smaller errors and those for $K = 0.1545$ larger ones. The errors for $C_1(t)$, (with 0 replaced by O_{411} and O_{433}) are comparable to those for $C_2(t)$. The errors were estimated by using the procedure described in Ref. 13) of dividing the 15 configurations for each value of K into 3 sets of 5 configurations and treating the result from each set as independent.

The two features of Fig. 1 which we particularly wish to stress, are the following:

- 1) The plateaus for both O_{411} and O_{433} are clearly present, and in all cases the plateau region can be taken to be $5 < t < 15$ [as was also the case in Ref. 12)]. As t approaches its central value of 10, the numerators and denominators in $C_1(t)$ and $C_2(t)$ both decrease [for $C_1(t)$ they even change sign], so that the errors increase.
- 2) Although the measurements of O_{411} and O_{433} are as accurate as could be hoped for, their difference, which is the quantity in which we are interested is relatively very small. Thus necessarily there will be a large relative error on the measurement of $\langle \xi^2 \rangle$. The non-perturbative renormalisation, i.e. the subtraction of O_{433} from O_{411} has proved to be extremely significant. In this respect we expect the computation of O_{412} on a large lattice, with p_1 and p_2 non-

zero, to have smaller statistical error. In spite of the large relative errors, the existence of such clear plateaus for O_{411} and O_{433} implies that the calculation of $\langle \xi^2 \rangle$ is meaningful.

For each value of t in the plateau region we combine the data for t with that for $20-t$ (these points should be equivalent on our lattice), compute $\langle \xi^2 \rangle$ for each value of K and then extrapolate to the physical limit. An example of such an extrapolation is shown in Fig. 2. The results are as follows:

$$\langle 3^2 \rangle_L = .26 \pm .10 \text{ using } C_1(t) \text{ and data from } t=5 \text{ and } 15, \quad (12a)$$

$$\langle 3^2 \rangle_L = .27 \pm .13 \text{ using } C_1(t) \text{ and data from } t=6 \text{ and } 14, \quad (12b)$$

$$\langle 3^2 \rangle_L = .05 \pm .23 \text{ using } C_2(t) \text{ and data from } t=5 \text{ and } 15, \quad (12c)$$

$$\langle 3^2 \rangle_L = .18 \pm .23 \text{ using } C_2(t) \text{ and data from } t=6 \text{ and } 14, \quad (12d)$$

The results for $t > 7$ are consistent with Eq. (12), but have very large errors. The subscript L is to remind us that these results are in the lattice renormalisation scheme at $a \approx (1.8\text{GeV})^{-1}$; we will discuss the connection with continuum renormalisation schemes below. In combining the results of Eq. (12) to obtain a single number, it must of course be remembered that they are correlated. Preferring to err on the side of caution, we take as our final result,

$$\langle 3^2 \rangle_L = 0.26 \pm .13 \quad (13)$$

It is clear that our result disagrees with that of the previous lattice calculation, Eq. (7). [In Ref. 10) the result in the lattice renormalisation scheme is $\langle \xi^2 \rangle_L = 1.37 \pm 0.20$ at an a of 1 GeV^{-1} .] It should be noted that we work on a different lattice and at $\beta = 6.0$ rather than 5.7 . In addition, as stressed above, our main theoretical objection to the calculation of Ref. 10), is that there is no way of checking directly whether the one-loop perturbative subtraction of power divergences was correct or not. In our case the non-perturbative subtraction of power divergences produced a significant numerical

cancellation, which then was responsible for the large errors in Eqs. (12) and (13).*) In Ref. 10) neither the subtraction of the traces, nor the perturbative removal of the remaining power divergences seems to spoil the numerical accuracy of the result.

Before comparing our results with those obtained using QCD sum rules, we have to state ours in some continuum renormalisation scheme ($\overline{\text{MS}}$ say), using

$$\langle \xi^2 \rangle_{\overline{\text{MS}}} = \left(\frac{Z_0}{Z_A} \right) \langle \xi^2 \rangle_L \quad (14)$$

where Z_0 is the renormalisation constant for the operator O (or more precisely the ratio of renormalisation constants in the $\overline{\text{MS}}$ and lattice schemes) and Z_A is the renormalisation constant of the axial current $\bar{\psi}\gamma_4\gamma_5\psi$. In perturbation theory $Z_A = 0.87$ and although we have not calculated Z_0 explicitly, the tadpole dominance of lattice perturbation theory [see, e.g., Refs. 12) and 14)] leads us to expect confidently that $Z_0 \approx 1.13$ (with an estimated uncertainty of 0.05)**). Thus

$$\langle \xi^2 \rangle_{\overline{\text{MS}}} \approx 1.3 \langle \xi^2 \rangle_L \quad (15)$$

The renormalization factor of 1.3 brings the central value of our result very close to those found using QCD sum rules [Eq. (13) now becomes $\langle \xi^2 \rangle_{\overline{\text{MS}}} = 0.34 \pm 0.17$] although of course our error is very large. Whether perturbation theory is enough to calculate Z_0/Z_A sufficiently accurately at $\beta = 6.0$ is one of the systematic uncertainties of this calculation. A non-perturbative calculation of Z_A on the same lattice gave a result of $Z_A \approx 0.7^{13)}$ as compared to the perturbative calculation of 0.87, but there is no equivalent calculation of Z_0 .

*) Even the one-loop perturbative contribution to the power divergences of O_{411} is much larger than the matrix element of the operator O .

**) Notice that since the renormalisation constants are at most logarithmically divergent, they can be computed in perturbation theory for sufficiently large $\beta^{15)}$.

We end with a brief recap of our conclusions. The main result of this paper is Eq. (13). The relatively large statistical error (corresponding to 15 configurations) is due to the significant cancellation which occurred when the power divergences were subtracted non-perturbatively. Our result is in disagreement with that found in the previous lattice calculation¹⁰⁾, however, we do not accept that the treatment of power divergences in Ref. 10) was valid. Our value of $\langle \xi^2 \rangle$ is consistent with those found using QCD sum rules. Clearly it would be desirable to reduce the error, which can be done either by increasing the statistics or by working on a large lattice with the operator O_{412} , which is free of power divergences.

ACKNOWLEDGEMENTS

We acknowledge helpful discussions with B. Gavela, A. Kronfeld and L. Maiani and the support for computing time by INFN and CNR.

REFERENCES

- 1) S.J. Brodsky and G.P. Lepage, Phys. Lett. 87B (1979) 359; Phys. Rev. D22 (1980) 2157.
- 2) A.V. Efremov and A.V. Radyushkin, Phys. Lett. 94B (1980) 245; Riv. Nuovo Cimento 3 (1980) 1.
- 3) A. Duncan and A. Mueller, Phys. Rev. D21 (1980) 1636; Phys. Lett. 98B (1980) 159.
- 4) G.R. Farrar and D.R. Jackson, Phys. Rev. Lett. 43 (1979) 246.
- 5) V.L. Chernyak and A.R. Zhitnitsky, Physics Reports 112 (1984) 173.
- 6) S.J. Brodsky, Y. Frishman, G.P. Lepage and C.T. Sachrajda, Phys. Lett. 91B (1980) 239.
- 7) V.L. Chernyak and A.R. Zhitnisky, Nucl. Phys. B201 (1982) 492.
- 8) M.J. Lavelle, Z. Phys. C29 (1985) 203.
- 9) D.A. Johnston and H.F. Jones, Imperial College Preprint, TP/85-86/28 (1986).
- 10) S. Gottlieb and A.S. Kronfeld, Phys. Rev. D33 (1985) 227.
- 11) J. Mandula, G. Zweig and J. Goevarts, Nucl. Phys. B228 (1983) 91.
- 12) A.S. Kronfeld and D.M. Photiadis, Phys. Rev. D31 (1985) 2939.
- 13) L. Maiani and G. Martinelli, Phys. Lett. 178B (1986) 265.
- 14) G. Martinelli and Y.C. Zhang, Phys. Lett. B123 (1983) 433; Phys. Lett. B125 (1983) 77.
- 15) M. Bochicchio et al., Nucl. Phys. B262 (1985) 331.

FIGURE CAPTIONS

Fig. 1

Demonstration of the existence of a plateau for the correlator $C_2(t)$ with 0 replaced by O_{411} [Fig. 1(a)] and O_{433} [Fig. 1(b)] at $K = 0.1530$.

Fig. 2

Example of the extrapolation to the physical limit of $\langle \xi^2 \rangle$. This figure is obtained using $C_1(t)$ at $t = 5$ and 15. The three points with their errors are correlated and the final error on $\langle \xi^2 \rangle$ is obtained using the clustering procedure discussed in the text.

Figure 1

Figure 2