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FUNDAMENTAL DOMAINS AND LIGHT-CONE
STRING FIELD THEORY

J.G. Taylor^{*)}
CERN - Geneva

ABSTRACT

It is shown that light-cone string field theory leads to a very simple specification of the fundamental domain over which the moduli are to be integrated for multiloop string and superstring amplitudes, as well as clarifying related questions.

*) On leave of absence from Department of Mathematics, King's College, Strand, London WC2R 2LS, United Kingdom.

The calculation of multiloop amplitudes for strings and superstrings is presently a much-investigated problem. Great progress has recently been made by Mandelstam¹⁾ in the bosonic case, using the light-cone (LC) gauge, so as to remove unphysical modes and their attendant ghosts. Construction of such amplitudes for superstrings has also been achieved^{2),3)} in the same gauge, starting from light-cone superstring field theory⁴⁾. The expressions for the scattering amplitudes for massless states (and also for massive ones, though these have not yet been worked out as completely as the massless ones in the superstring case) involve integrations over the various parameters describing (a) the interaction positions where pairs of strings fuse or a single string separates into a pair of strings, (b) the widths and twists in the loops, and (c) the positions of the external particles (Koba-Nielsen vertices). These integrations correspond to summing over conformally-inequivalent Riemann surfaces, a process expected on quantum mechanical grounds. One of the crucial questions which arises from these integrations is as to the range of these variables, in other words, the domain D of integration. In particular three questions arise:

- (1) Does LC string quantum field theory reduce, in perturbation theory, to an integral over the moduli space of Riemann surfaces?
- (2) Is the fundamental domain covered once only?
- (3) How is the fundamental domain represented if the answer to (2) is Yes?

This letter attempts to answer these closely-related questions.

The answer to these questions is important, since it is widely assumed [see, for example, Ref. 5)] that (1) and (2) are answered affirmatively, so that at least one example exists of a second-quantized field theory leading to the known string amplitudes. In particular such a feature has been important in guiding the building of string field theories by focusing attention on the introduction of string length in a fundamental manner^{6),7)}, and also in giving a proof of finiteness of the heterotic and type II superstring theories⁸⁾. However, folklore is not the same as proof, so this gap in our justifiable knowledge needs to be filled.

It is to be expected that such has already been done by the founders of light-cone gauge string quantum field theory⁹⁾⁻¹¹⁾. Some attempts along these lines are evident in those references. However, it is clear that Mandelstam⁹⁾ has based his approach on the functional (first-quantized) formulation without attempting to derive it from a second-quantized version. The same attitude appears in the works of Gervais and Sakita. The closest discussion to the

problem arises in Ref. 11). It is shown, in the second paper of this reference, that the LC perturbation expansion gives a set of LC Feynman diagrams which is identical to the set of dual-resonance LC amplitudes with poles in non-overlapping channels¹²⁾. This relation is only established in detail at the tree level, and appeal is made to the theory of equipotentials on Riemann surfaces to justify extension to higher loops. This seems to be begging the answer to (1) above, and there is no discussion of (2), nor of the modular invariance of the amplitudes (which appears to have been a question not fully appreciated at that time). Nor is any appreciation of question (3) evident from the above references, nor from more recent ones on LC string field theory.

Let us try to give an answer to the three questions raised above more in line with current thinking. The basic reduction of the perturbation LC string diagrams, for closed strings, has already been achieved to an integral over interaction times and string widths and twists in the papers in Refs. 2) and 3). The resulting variables [as essentially contained in Ref. 11) and Arfaei's lucid discussion¹³⁾ for tree and loop diagrams] are

- a) the interactions positions $\tilde{p}$
- b) the string loop widths and twists α_i, β_i ($1 \leq i \leq g$, for genus g).

That "genus" and all of the other paraphernalia of Riemann surface theory are available is due to results of Ref. 11), essentially that a LC strip diagram is a canonical form of a Riemann surface.

The appropriate mapping function to proceed further is that of Mandelstam⁹⁾

$$\varphi = F(z) = \sum_{r=1}^L \ell_r \zeta(z, z_r) \quad (1)$$

in terms of the Riemann surface Σ with N punctures at the Koba-Nielsen points z_r , describing the external strings which have LC₊ component of momentum p_r^+ and width $\lambda_r = \pi p_r^+$. The Green's function $\underline{G}$ is the unique one on Σ for which $\text{Re } \underline{G} = G$ is single valued [this is discussed in more detail in Ref. 3), and goes back to Alessandrini¹⁴⁾ and Alessandrini and Amati¹⁵⁾]. Then the variables α_i and β_i are defined by¹⁾⁻³⁾:

$$\begin{aligned}\alpha_i &= \text{Im}[\varphi(\gamma_i P) - \varphi(P)] \\ \beta_i &= \text{Im}[\varphi(\delta_i P) - \varphi(P)]\end{aligned}\quad (1 < i < g) \quad (2)$$

where $\{\gamma_i, \delta_i\}$ is a basis of $H_1(\Sigma, \mathbb{Z})$. Under a change of basis by the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$, (α, β) transforms linearly as a $2g$ -vector. Moreover, since G is modular invariant, then $\tilde{\rho}$ is invariant under such changes of basis. This can be seen directly from the fact that under a modular transformation $z \rightarrow \gamma(z)$, $F(\gamma(z)) = F(z)$. If $\gamma(z)$ is denoted by z_γ , then $\partial F(z_\gamma)/\partial z_\gamma = [\gamma'(z)]^{-1} F'(z)$. Thus if $F'(\tilde{z}) = 0$, so does $\partial F(z_\gamma)/\partial z_\gamma = 0$. But then $\tilde{z}_\gamma$ is also an interaction point for which the value of the mapped point $\tilde{\rho}_\gamma$ is the same as that for $\tilde{z}$, equal to $\tilde{\rho}$:

$$\tilde{\rho}_\gamma = \tilde{\rho} \quad (3)$$

In other words, $\tilde{\rho}$ is a modular invariant, as claimed.

The conclusion is therefore that [after a little further analysis of the amplitude²⁾] the measure

$$\prod_a d^2 \tilde{\rho}_a \prod_{i=1}^g d\alpha_i d\beta_i$$

and the integrand are separately modular invariant. This result answers question (1) positively [the measure quoted above is derivable directly from Ref. 1)], as is spelt out in detail on pp. 5-6 of the first paper of Ref. 2)].

The answer to question (2) is of a somewhat more unexpected nature. It is that the variables $\tilde{\rho}$, being modular invariant, are already "on" moduli space $\mathcal{M}_g$. The variables α_i, β_i are therefore the only ones which need discussing. But they are required, by the geometrical structure of the LC strip diagram, to be such as only to allow the "slits", corresponding to the loops or handles, to lie inside the strip diagram. The transformations of $\text{Sp}(2g, \mathbb{Z})$ send the slits out of the LC strip, as can be seen directly, for example, at one loop. Thus the range of these variables is a torus, $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ (with suitable normalization for α_i 's with respect to the set of positive α_i 's and for β_i 's so as to be angular variables).

This feature is discussed in detail in the first paper of Ref. 2) on p. 31, using the short-string limit. The answer to question (2) is therefore "once", as expected.

The answer to the above questions has already given a partial answer to the third question. For the variables $\tilde{p}$ do not need any further specification than lying in the LC strip domain, whilst the (α_i, β_i) lie in a torus $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$. They are non-analytic variables on modular space closely related to the analytic ones of Ahlfors¹⁶⁾. This would appear to be a complication if analytic co-ordinates were desired, but the divergence analysis⁸⁾ can be performed directly for the heterotic superstring by using the beautiful results of Belavin and Knizhnik for the bosonic string¹⁷⁾; such analytic co-ordinates do not seem to be essential for a divergence analysis⁸⁾ of type II superstrings.

The remaining variables which need to be analyzed to give an answer to question (3) are the source variables z_r ($2 \leq r \leq N-1$). It proved convenient, in the analysis in the second paper in Ref. 2) of the reduction of the one-loop "raw" superstring Feynman amplitudes to the known form, to use as fundamental variables $u_r = u(z_r)$, where $u(z)$ is the first Abelian integral on the torus $u(z) = 1/2\pi i \log z$. The range of integration for the variables u_r then becomes

$$\mathcal{F}_1: 0 \leq \text{Im} u_r \leq \text{Im} \tau, |\text{Re} u_r| \leq 1/2 \quad (4)$$

where τ is the usual modular parameter for the torus with $\text{Im} \tau > 0$. This leads to the possibility of describing the domain of integration in the higher genus case by the variables $u_{ir} = u_i(z_r)$, $1 \leq i \leq g$, where u_i are the g first Abelian integrals on a Riemann surface of genus g . The corresponding condition to (4) would be that, for each r , the g -component vector $\underline{u}_r$, with i^{th} component $u_i(z_r)$, belongs to the lattice generated by the period matrix. Thus the extension of (4) is

$$\mathcal{F}_g: \underline{u}_r = \underline{\lambda} + \pi \underline{\mu} \quad (5)$$

where π is the period matrix of the surface and $\underline{\lambda}, \underline{\mu} \in [0, 1]^g$.

This definition of $\mathcal{F}_g$ seems satisfactory in that under a modular transformation

$$\pi \rightarrow (A\pi + B)(C\pi + D)^{-1} \quad (6)$$

where $\begin{pmatrix} A & C \\ B & D \end{pmatrix} \in \text{Sp}(2g, \mathbb{Z})$ is a modular transformation, the $2g$ -component vector $(\underline{\lambda}, \underline{\mu})$ may be shown to transform under the modular matrix $\begin{pmatrix} A & B \\ -C & D \end{pmatrix}$. Thus $\mathcal{F}_g$ is transformed by such a modular transformation into another fundamental domain of the Jacobian of the Riemann surface, which is $\mathbb{C}^g/L$, where L is the lattice $L = \{\underline{m} + \tau \underline{n}\}, \underline{m}, \underline{n} \in \mathbb{Z}^g$.

Therefore the complete answer to (3) is given by $\{\tilde{\rho}\}$ and $\{\alpha_i, \beta_i\}$ lying over the LC strip diagram, and for each such choice, with associated period matrix, the source variables lie in $\mathcal{F}_g$ (or one of its modular transforms). It is interesting to note that an identical prescription of the fundamental domain arises⁷⁾ when a Hamiltonian BRST-approach is used to introduce string lengths in the gauge-covariant string field theory presently being developed^{6),7)}.

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