PROOF OF THE NONEXISTENCE OF A LINEAR SOLUTION FOR THE CR2 INJECTION REGION OF THE CLIC DRIVE BEAM

R. Apsimon, J. Esberg
CERN, Geneva, Switzerland

Abstract
In this paper we present a mathematical proof to show that there exists no linear system of optics which can simultaneously close an orbit bump and correct the dispersion in the CR2 injection region. Due to the requirements of the CR2 injection region, several different trajectories will exist through the injection region which are off-axis; therefore the orbit and dispersion functions need to be corrected. In this paper, we determine the properties of a hypothetical linear lattice which is capable of closing the orbit and dispersion functions and then show that the resulting solutions are either unphysical or trivial.
Proof of the Nonexistence of a Linear Solution for the CR2 Injection Region of the CLIC Drive Beam

R. Apsimon, J. Esberg

Abstract

In this paper we present a mathematical proof to show that there exists no linear system of optics which can simultaneously close an orbit bump and correct the dispersion in the CR2 injection region. Due to the requirements of the CR2 injection region, several different trajectories will exist through the injection region which are off-axis; therefore the orbit and dispersion functions need to be corrected. In this paper, we determine the properties of a hypothetical linear lattice which is capable of closing the orbit and dispersion functions and then show that the resulting solutions are either unphysical or trivial.

1 Introduction

The Compact Linear Collider (CLIC) relies on a two beam acceleration scheme to achieve the 3 TeV centre of mass energy required for the $e^+e^-$ collisions; the details of which can be found in [1]. The main beam accelerators are used to accelerate electrons and positrons from 9 GeV to 1.5 TeV using normal conducting RF cavities. The RF power used to accelerate the main beam is obtained by decelerating a 2.4 GeV, 100 A drive beam. The drive beam consists of 24 bunch trains, each consisting of 2904 bunch with a spacing of 82 ps (∼2.5 cm), the train spacing is 5.8 µs. The drive beam accelerating linac is not capable of producing such a high intensity beam with such a short bunch spacing. The drive beam linac produces 24×24 sub-pulses with a spacing of 2 ns (∼60 cm), a beam current of 4.2 A and a total length of 140 µs. This initial beam time structure is transformed into the required structure by the drive beam recombination system; which is depicted in Figure 1.

The delay loop reduces the bunch spacing by a factor of 2 by deflecting alternate groups of 12 sub-pulses into the delay loop. So for every 24 sub-pulses, the first 12 are deflected into the delay loop while the second group of 12 are undeflected and continue along the shorter beam line. The path length difference is such that at the recombination point at the end of the delay loop, the delayed bunches are interleaved with the undelayed bunches. There are now
Figure 1: A schematic diagram of the CLIC drive beam recombination system [1].

Figure 2: A schematic diagram of the delay loop bunch combination scheme [1].
12×24 sub-pulses with a bunch spacing of 1 ns and also 240 ns spaces between each sub-pulse (Figure 2).

The two combiner rings are used to further reduce the bunch spacing and increase the gaps between the sub-pulses. The sub-pulses are stored in the rings for several turns and each time the stored bunches pass through the injection region, additional bunches are interleaved between the stored bunches. Conventional injection kickers cannot be used in the combiner rings because the kicker would deflect the stored bunches out of the ring when injecting the new bunches. Instead, RF deflectors are used to create a closed orbit bump for the stored bunches while acting as an injection kicker for the injected bunches. Figure 3 show the combiner ring scheme for a combination factor of 4, which is the case of the second combiner ring (CR2).

At the end of the recombination system, there are 2×24 pulses, each with the required 2904 bunches separated by 82 ps and the beam current has increased from 4.2 A to 100 A.

2 CR2 injection

RF deflectors are required for the injection into the combiner rings in order to interleave the bunches. As shown in Figure 3, the stored bunches take different trajectories through the injection region of CR2. With the use of two identical RF deflectors, the local orbit bump can be closed with the use of one or more
quadrupoles between the deflectors. However, as will be shown in this paper, there exists no system of linear optics between the RF deflectors which can correct the dispersion as well as the orbit bump.

2.1 Requirements of a linear lattice for the injection bump

To prove that there is no linear solution for the injection bump which can simultaneously close both the orbit bump and dispersion, we will first assume that there is a solution, determine the properties of such a lattice and then show that the required properties are either unphysical or trivial. If a linear solution exists, then it must be possible to construct a symmetric lattice which is a solution. To verify this, let us consider a hypothetical asymmetric lattice which is a solution, then the reflection of this lattice must also be a solution. If we connect the original lattice to its reflection, remove the two RF deflectors in the centre and correct the central drift length, then this new lattice will also be a solution and will be symmetric.

Having shown that a symmetric lattice must exist if any solution to the problem exists, we are able to greatly simplify the problem in order to search for a solution. We can now investigate the central region of the hypothetical symmetric lattice and consider what is necessary to obtain a solution. The orbit and dispersion functions will form either a symmetric or anti-symmetric function about the midpoint of the lattice; we will define this as the even and odd parity solutions respectively. It should be noted that the orbit and dispersion functions must have the same parity since the deflection and therefore dispersion contribution of a quadrupole depends on the trajectory through the quadrupole.

The linear optics between the two RF deflectors must consist exclusively of quadrupoles and drift spaces because a dipole would break the transverse symmetry of the injection region; thus equal and opposite trajectories would not both be solutions. Therefore we can assume after the first RF deflector the beam travels through an arbitrary sequence of quadrupoles and drift spaces. The central region can therefore be considered as either a quadrupole singlet or a symmetric doublet because anything more complex can be considered as part of the arbitrary lattice upstream. If we can verify that no solution exists for the singlet and doublet cases then by induction, we can conclude that there is no solution for any symmetric sequence of \(2n+1\) quadrupoles and \(2n\) quadrupoles respectively; hence no symmetric sequence of quadrupoles can be a solution. If no symmetric solution exists then from our earlier statement, no linear solution exists. As a further simplification, the singlet can be considered as a special case of a doublet, where the drift length between the quadrupoles is zero.

3 Beam dynamics

For this paper, we will use the thick lens transfer matrices for the quadrupoles in order to prove conclusively that there is no possible solution. The transfer matrices for focussing and defocussing quadrupoles are given as:
\[
M_{qf} = \begin{pmatrix}
\cos (\sqrt{k_f} l_q) & \sin (\sqrt{k_f} l_q) \\
-\sqrt{k_f} \sin (\sqrt{k_f} l_q) & \cos (\sqrt{k_f} l_q)
\end{pmatrix}
\]
\[
M_{qd} = \begin{pmatrix}
\cosh (\sqrt{k_d} l_q) & \sinh (\sqrt{k_d} l_q) \\
\sqrt{k_d} \sinh (\sqrt{k_d} l_q) & \cosh (\sqrt{k_d} l_q)
\end{pmatrix}
\]

(1)

The transfer matrix for a drift space is given below.

\[
M_{dr} = \begin{pmatrix} 1 & L_{dr} \\ 0 & 1 \end{pmatrix}
\]

(2)

The trajectory downstream of the central region can be defined in terms of the trajectory upstream:

\[
\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = M \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}
\]

(3)

where \( M \) is the transfer matrix for the central region. It should be noted that for any linear transfer matrix, \( M \), describing a conservative system, \( \det (M) = 1 \).

As the beam is travelling off-axis through the quadrupoles, we need to consider the quadrupole contributions to the dispersion function. The dispersion, \( D_q \), and its derivative, \( D'_q \), can be defined as:

\[
D_q = M_{1,2} \int_0^{l_q} \frac{\bar{M}_{1,1}}{\rho (s)} ds - M_{1,1} \int_0^{l_q} \frac{\bar{M}_{1,2}}{\rho (s)} ds
\]
\[
D'_q = M_{2,2} \int_0^{l_q} \frac{\bar{M}_{1,1}}{\rho (s)} ds - M_{2,1} \int_0^{l_q} \frac{\bar{M}_{1,2}}{\rho (s)} ds
\]

(4)

where \( \bar{M}_{i,j} = M_{i,j} (s) \) and \( \rho (s) \) is the radius of curvature at a longitudinal position \( s \) in the quadrupole, given as:

\[
\rho (s) = \frac{dL}{ds} = \left( 1 + x'^2 \right)^{\frac{3}{2}}
\]

(5)

Therefore the radius of curvature can be given as:

\[
\rho (s) = \frac{1 + \left( \bar{M}_{2,1} x_0 + \bar{M}_{2,2} x'_0 \right)^2}{x_0 \frac{dM_{2,1}}{dx} + x'_0 \frac{dM_{2,2}}{dx}}
\]

(6)

For the focusing and defocusing quadrupoles the radius of curvature is given respectively as:
\[ \rho_f(s) = -\left(1 + \left(M_{2,1}x_0 + M_{2,2}x'_0\right)^2\right)^{\frac{3}{2}} \]

\[ \rho_d(s) = \left(1 + \left(M_{2,1}x_0 + M_{2,2}x'_0\right)^2\right)^{\frac{3}{2}} \]

\[ \begin{align*}
\rho_f(s) &= -\frac{\left(1 + \left(M_{2,1}x_0 + M_{2,2}x'_0\right)^2\right)^{\frac{3}{2}}}{k_f \left(M_{1,1}x_0 + M_{1,2}x'_0\right)} \\
\rho_d(s) &= \frac{\left(1 + \left(M_{2,1}x_0 + M_{2,2}x'_0\right)^2\right)^{\frac{3}{2}}}{k_d \left(M_{1,1}x_0 + M_{1,2}x'_0\right)}
\end{align*} \]  

\section{Central region}

In total there are only 8 scenarios for the central region which need to be investigated:

- Focussing singlet, symmetric/anti-symmetric bump
- Defocussing singlet, symmetric/anti-symmetric bump
- Focussing doublet, symmetric/anti-symmetric bump
- Defocussing doublet, symmetric/anti-symmetric bump

As previously stated, the singlet can be treated as a special case of a doublet, thus allowing us to study all scenarios at once.

\subsection{Quadrupole doublet}

Rather than using specific matrix elements for a focussing or defocussing quadrupole from Eq. 1, we will write the equations in a more general form.

For the symmetric solution, we require that:

\[ \begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ -x'_0 \end{pmatrix} \]

\[ \begin{pmatrix} D_{x,1} \\ D'_{x,1} \end{pmatrix} = \begin{pmatrix} D_{x,0} \\ -D'_{x,0} \end{pmatrix} \]  

\[ \begin{align*}
\begin{pmatrix} x_1 \\ x'_1 \end{pmatrix} &= \begin{pmatrix} -x_0 \\ x'_0 \end{pmatrix} \\
\begin{pmatrix} D_{x,1} \\ D'_{x,1} \end{pmatrix} &= \begin{pmatrix} -D_{x,0} \\ D'_{x,0} \end{pmatrix}
\end{align*} \]  

Therefore, to generalise the constraints for both parities, we require:
\[
\begin{pmatrix}
  x_1 \\
  x'_1
\end{pmatrix} = \begin{pmatrix}
  \pm x_0 \\
  \mp x'_0
\end{pmatrix}
\]
\begin{equation}
(10)
\end{equation}

Where the top sign in ± or \( \mp \) represents the sign for the symmetric case and the bottom sign for the anti-symmetric case. If we consider the general transfer matrix for a quadrupole to be:

\[
M = \begin{pmatrix}
  M_{1,1} & M_{1,2} \\
  M_{2,1} & M_{2,2}
\end{pmatrix}
\]
\begin{equation}
(11)
\end{equation}

And the transfer matrix of a doublet to be:

\[
N = \begin{pmatrix}
  M_{1,1} & M_{1,2} \\
  M_{2,1} & M_{2,2}
\end{pmatrix} \begin{pmatrix}
  1 & L_{dr} \\
  0 & 1
\end{pmatrix} \begin{pmatrix}
  M_{1,1} & M_{1,2} \\
  M_{2,1} & M_{2,2}
\end{pmatrix}
\]
\begin{equation}
(12)
\end{equation}

Given the dispersive contributions, \( D_q \) and \( D'_q \), for a quadrupole from Eq. 4, we can define the contributions for a doublet, \( D_{q,doub} \) and \( D'_{q,doub} \), as:

\[
D_{q,doub} = M_{1,1} D_q + (L_{dr} M_{1,1} + M_{1,2}) D'_q
\]
\[
D'_{q,doub} = M_{2,1} D_q + (L_{dr} M_{2,1} + M_{2,2}) D'_q
\]
\begin{equation}
(13)
\end{equation}

This can be simplified using the results from Eq. 12 to produce:

\[
D_{q,doub} = N_{1,2} \int \tilde{M}_{1,1} \frac{d\rho}{d\rho(s)} ds - N_{1,1} \int \tilde{M}_{1,2} \frac{d\rho}{d\rho(s)} ds
\]
\[
D'_{q,doub} = N_{2,2} \int \tilde{M}_{1,1} \frac{d\rho}{d\rho(s)} ds - N_{2,1} \int \tilde{M}_{1,2} \frac{d\rho}{d\rho(s)} ds
\]
\begin{equation}
(14)
\end{equation}

For the orbit bump, we require:

\[
N_{1,1} x_0 + N_{1,2} x'_0 = \pm x_0
\]
\[
N_{2,1} x_0 + N_{2,2} x'_0 = \mp x'_0
\]
\begin{equation}
(15)
\end{equation}

The solution to these simultaneous equations is:

\[
\frac{x_0}{x'_0} = \frac{N_{1,2}}{\pm 1 - N_{1,1} \frac{N_{1,2}}{N_{2,1}}}
\]
\begin{equation}
(16)
\end{equation}

For the dispersion function we require:

\[
N_{1,1} D_{x,0} + N_{1,2} D'_{x,0} + D_{q,doub} = \pm D_{x,0}
\]
\[
N_{2,1} D_{x,0} + N_{2,2} D'_{x,0} + D'_{q,doub} = \mp D'_{x,0}
\]
\begin{equation}
(17)
\end{equation}
Where $D_{q,doub}$ and $D'_{q,doub}$ are the dispersive contributions from the quadrupole doublet as given in Eq. 14. Eq. 17 can be written out more explicitly as:

\[
\pm D_{x,0} = N_{1,1} D_{x,0} + N_{1,2} D'_{x,0} + N_{1,2} \int \frac{M_{1,1}}{\rho(s)} ds - N_{1,1} \int \frac{M_{1,2}}{\rho(s)} ds
\]

(18)

\[
\mp D'_{x,0} = N_{2,1} D_{x,0} + N_{2,2} D'_{x,0} + N_{2,2} \int \frac{M_{1,1}}{\rho(s)} ds - N_{2,1} \int \frac{M_{1,2}}{\rho(s)} ds
\]

And these equations can be rearranged and the results from Eq. 16 substituted to give the following simultaneous equations:

\[
D_{x,0} = \frac{x_0}{x_0} D'_{x,0} + \frac{x_0}{x_0} \int \frac{M_{1,1}}{\rho(s)} ds - \frac{N_{1,1}}{\rho(s)} \int \frac{M_{1,2}}{\rho(s)} ds
\]

(19)

\[
D_{x,0} = \frac{x_0}{x_0} D'_{x,0} + \left( \frac{x_0}{x_0} \pm \frac{1}{N_{2,1}} \right) \int \frac{M_{1,1}}{\rho(s)} ds + \int \frac{M_{1,2}}{\rho(s)} ds
\]

By solving these simultaneous equations we obtain:

\[
(\mp 1 + N_{1,1}) \int \frac{M_{1,1}}{\rho(s)} ds - N_{2,1} \int \frac{M_{1,2}}{\rho(s)} ds = 0
\]

(20)

By expressing the matrix elements of $N$ in terms of $M$ and using the fact that $\det(M) = 1$, Eq. 20 can be simplified for the symmetric and anti-symmetric cases respectively:

\[
M_{2,1} \left( (L_{dr} M_{1,1} + 2M_{1,2}) \int \frac{M_{1,1}}{\rho(s)} ds - (L_{dr} M_{2,1} + 2M_{1,1}) \int \frac{M_{1,2}}{\rho(s)} ds \right) = 0
\]

\[
(2M_{1,1} + L_{dr} M_{2,1}) \left( M_{1,1} \int \frac{M_{1,1}}{\rho(s)} ds - M_{2,1} \int \frac{M_{1,2}}{\rho(s)} ds \right) = 0
\]

(21)

For the symmetric case in Eq. 21, we can express it as:

\[
M_{2,1} \left( 2D_{q} + L_{dr} D'_{q} \right) = 0
\]

(22)

Where $D_{q}$ and $D'_{q}$ are the dispersive contributions for one of the quadrupoles in the doublet, or half of a quadrupole for the singlet. Therefore, either $M_{2,1} = 0$ or $2D_{q} + L_{dr} D'_{q} = 0$; the latter implies that $D'_{q} \propto D_{q}$. We can consider $D_{q}$ and $D'_{q}$ in Eq. 4 as vectors in an abstract coordinate system where $\int_{0}^{l_{q}} \frac{M_{1,1}}{\rho(s)} ds$ and $\int_{0}^{l_{q}} \frac{M_{1,2}}{\rho(s)} ds$ are the coordinate bases. If $D'_{q} \propto D_{q}$, then the corresponding vectors in our abstract coordinate system must be parallel; therefore $\frac{M_{1,2}}{M_{1,1}} = \frac{M_{2,2}}{M_{2,1}}$. This implies that $\det(M) = 0$, but this is a contradiction because we know that $\det(M) = 1$. Therefore $2D_{q} + L_{dr} D'_{q} = 0$ is not possible except for the trivial case and $M_{2,1} = 0$ is the only solution for the symmetric case.

For the focusing quadrupoles, $M_{2,1} = 0$ has solutions at $\sqrt{k_{f} l_{q}} = m \pi$. For the defocusing quadrupoles, $M_{2,1} = 0$ only has the solution $\sqrt{k_{d} l_{q}} = 0$, which is trivial and can be neglected.
For the anti-symmetric case, Eq. 21 can be simplified with the use of Eq. 4 and the fact that $M_{1,1} = M_{2,2}$:

$$(2M_{1,1} + L_{dr}M_{2,1}) D'_q = 0$$

(23)

Therefore either $D'_q = 0$ or $2M_{1,1} + L_{dr}M_{2,1} = 0$.

For the defocusing singlet, there are no real solutions to $2M_{1,1} + L_{dr}M_{2,1} = 0$ and for the doublet there are no solutions for $l_q, L_{dr} > 0$. For the other solution, $D'_q = 0$, the only solution is the trivial case when $\sqrt{k_{fl}}l_q = 0$.

For the focusing singlet, solutions for $2M_{1,1} + L_{dr}M_{2,1} = 0$ occur at $\sqrt{k_{fl}}l_q = (2m + 1)\pi/2$ and $D'_q = 0$ when $\sqrt{k_{fl}}l_q = m\pi$; these solution sets can be combined to give $\sqrt{k_{fl}}l_q = m\pi/2$. For the doublet both $D'_q = 0$ and $2M_{1,1} + L_{dr}M_{2,1} = 0$ have non-trivial solutions only if $\tan(\sqrt{k_{fl}}l_q) = \sqrt{k_{fl}}L_{dr}/2$.

5 Results

From the results above, we can conclude that there are no non-trivial solutions for any of the defocusing quadrupole cases. Thus we can investigate the possible solutions for the focusing quadrupoles. There are three distinct possible solutions for the focusing quadrupole cases, which are:

- $\sqrt{k_{fl}}l_q = m\pi$ for the symmetric focusing cases
- $\sqrt{k_{fl}}l_q = m\pi/2$ for the anti-symmetric focusing singlet
- $\tan(\sqrt{k_{fl}}l_q) = \sqrt{k_{fl}}L_{dr}/2$ for the anti-symmetric focusing doublet

5.1 $\sqrt{k_{fl}}l_q = m\pi$ for the symmetric focusing cases

If $\sqrt{k_{fl}}l_q = m\pi$, then the transfer matrix $N$ in Eq. 12 can be expressed as:

$$N = \begin{pmatrix} 1 & L_{dr} \\ 0 & 1 \end{pmatrix}$$

(24)

Thus the requirements to produce a symmetric dispersion function become:

$$D_{x,0} + L_{dr}D'_{x,0} + D_{q,doub} = D_{x,0}$$
$$D'_{x,0} + D'_{q,doub} = -D'_{x,0}$$

(25)

Therefore we obtain $\frac{D_{q,doub}}{L_{dr}} = \frac{D'_{q,doub}}{2}$, which implies that det $(N) = 0$ which is contradictory as we know that det $(N) = 1$. Hence there are no non-trivial solutions for the focusing cases for $\sqrt{k_{fl}}l_q = m\pi$. 

10
5.2 $\sqrt{k_f l_q} = m\pi/2$ for the anti-symmetric focusing singlet

For the focusing singlet, we obtain the transfer matrix:

$$N = \left( \begin{array}{c}
\cos (2\sqrt{k_f l_q}) & \sin(2\sqrt{k_f l_q}) \\
-\frac{1}{2\sqrt{k_f}} & \cos(2\sqrt{k_f l_q})
\end{array} \right)$$  \hspace{1cm} (26)

If $\sqrt{k_f l_q} = (2m + 1) \pi/2$, then Eq. 26 becomes $-I$, where $I$ is the identity matrix. By considering the resulting orbit and dispersion functions, we obtain:

$$-x_0 = -x_0'$$
$$-x_0' = x_0'$$
$$-D_{x,0} + D_q = -D_{x,0}$$
$$-D_{x,0}' + D_q' = D_{x,0}'$$  \hspace{1cm} (27)

which gives the result $D_q = x_0' = 0$. Similarly, if $\sqrt{k_f l_q} = m\pi$, we obtain the result $D_q' = x_0 = 0$.

If $D_q = 0$ then from Eq. 4, it can be shown that:

$$M_{1,2} \int_0^{l_q} \frac{M_{1,1} \rho(s)}{\rho(s)} ds = M_{1,1} \int_0^{l_q} \frac{M_{1,2} \rho(s)}{\rho(s)} ds$$  \hspace{1cm} (28)

As $\sqrt{k_f l_q} = (2m + 1) \pi/2$, $M_{1,1} = 0$ and $M_{1,2} = -1$, which implies that $\int_0^{l_q} \frac{M_{1,1} \rho(s)}{\rho(s)} ds = 0$. However, if we substitute the results from Eq. 7 then we obtain the integral:

$$\int_0^{l_q} \frac{M_{1,1} \rho(s)}{\rho(s)} ds = -k_f x_0 \int_0^{(2m+1)\pi/2} \frac{\cos^2(\sqrt{k_f s})}{(1+k_f x_0^2 \sin^2(\sqrt{k_f s}))^2} ds$$

$$= (2m + 1) \frac{K(-k_f x_0^2) - E(-k_f x_0^2)}{\sqrt{k_f x_0}}$$  \hspace{1cm} (29)

Where $K$ and $E$ are complete elliptic integrals of the first and second kind respectively. Eq. 29 only equates to zero when $x_0 = 0$; therefore $\sqrt{k_f l_q} = (2m + 1) \pi/2$ leads to the trivial solution that $x_0 = x_0' = 0$.

For the case where $\sqrt{k_f l_q} = m\pi$, except for the trivial case where $\sqrt{k_f l_q} = 0$, the quadrupole can be divided into three smaller quadrupoles such that:

$$N = \left( \begin{array}{c}
\cos (2m\pi) & \sin(2m\pi) \\
-\frac{1}{2\sqrt{k_f}} & \cos(2m\pi)
\end{array} \right) = \left( \begin{array}{c}
\cos \left( \frac{\pi}{2} \right) & \sin \left( \frac{\pi}{2} \right) \\
-\frac{1}{2\sqrt{k_f}} & \cos \left( \frac{\pi}{2} \right)
\end{array} \right) \times$$

$$\left( \begin{array}{c}
\cos \left( (2m-1)\pi \right) & \sin(2m-1)\pi) \\
-\frac{1}{2\sqrt{k_f}} & \cos((2m-1)\pi)
\end{array} \right) = \left( \begin{array}{c}
\cos \left( \frac{\pi}{2} \right) & \sin \left( \frac{\pi}{2} \right) \\
-\frac{1}{2\sqrt{k_f}} & \cos \left( \frac{\pi}{2} \right)
\end{array} \right)$$  \hspace{1cm} (30)
We have proven that the only anti-symmetric solution for the central quadrupole is the trivial case where \( x = x' = 0 \). Hence if we determine an initial trajectory which can produce this result after passing through the first quadrupole, we find:

\[
\begin{pmatrix}
0 & \frac{1}{\sqrt{k_f}} \\
-\sqrt{k_f} & 0
\end{pmatrix}
\begin{pmatrix}
x_0 \\
x'_0
\end{pmatrix}
= \begin{pmatrix}
\frac{x'_0}{\sqrt{k_f}} \\
-\sqrt{k_f}x_0
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\] (31)

Therefore the only solution for the anti-symmetric focusing singlet is the trivial case where \( x_0 = x'_0 = 0 \).

5.3 \( \tan \left( \sqrt{k_f} l_q \right) = \sqrt{k_f L_{dr}} / 2 \) for the anti-symmetric focusing doublet

From Eq. 12, using the fact that \( \det (N) = 1 \), the transfer matrix for a doublet can be expressed as:

\[
N = \begin{pmatrix}
(2M_{1,1} + L_{dr}M_{2,1})M_{1,1} - 1 & (2M_{1,1} + L_{dr}M_{2,1})M_{1,2} + L_{dr} \\
(2M_{1,1} + L_{dr}M_{2,1})M_{2,1} & (2M_{1,1} + L_{dr}M_{2,1})M_{1,1} - 1
\end{pmatrix}
\] (32)

From Eq. 22, this can be simplified to:

\[
N = \begin{pmatrix}
-1 & L_{dr} \\
0 & -1
\end{pmatrix}
\] (33)

From this, we obtain the requirements for the anti-symmetric dispersion function:

\[
-D_{x,0} + L_{dr}D'_{x,0} + D_{q,doub} = -D_{x,0} \\
-D'_{x,0} + D'_{q,doub} = D'_{x,0}
\] (34)

Which implies that \( -\frac{D_{q,doub}}{L_{dr}} = D_{x,0} = \frac{D'_{q,doub}}{2} \), and therefore \( D_{q,doub} \propto D'_{q,doub} \). But we have previously shown that this implies that \( \det (N) = 0 \), which is contradictory because we know that \( \det (N) = 1 \); therefore there are no non-trivial solutions for the anti-symmetric focusing doublet.

6 Summary

In this paper we have shown that there is no possible linear solution to simultaneously close orbit and dispersion functions. We showed that if a solution exists then it must be possible to create a symmetric lattice which is also a solution. For a symmetric lattice, both the orbit and dispersion must be either symmetric or anti-symmetric about the midpoint of the lattice. This allows us to investigate just the central region of the lattice to determine whether a solution is possible. By considering a quadrupole singlet at the centre of the injection region, we are able to consider any lattice consisting of an odd number
of quadrupoles. Similarly by considering a doublet at the centre, we are able to consider any lattice consisting of an even number of quadrupoles. By considering a quadrupole singlet as the special case of a quadrupole doublet with a drift length \( L_{dr} = 0 \), we are able to investigate all cases and show that no non-trivial solutions exist.

References