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BRST-HAMILTONIAN APPROACH TO THE FIRST-QUANTIZED SUPERSTRING

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ABSTRACT

The construction of the first-quantized BRST charge for the covariant Green-Schwarz superstring is obtained in the presence of the second-class constraints. The Lagrange multipliers for the second-class constraints are made into dynamical variables by use of the proper-time formalism. Problems with obtaining a Riemann surface description of multiloop amplitudes are briefly considered.

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One of the important questions in the quantization of the covariant Green-Schwarz superstring¹⁾ is that of determining the appropriate canonical structure in a Hamiltonian framework. This may be seen as an independent exercise in its own right, or as a forerunner to a second quantization in a field-theoretic setting. A first step in the analysis has been taken by Hori and Kamimura²⁾, who have given a covariant separation of the constraints into first and second class. They have thus obtained the structure constants of the first-class constraints algebra, as well as giving an explicit form for the second-class constraints. Such a covariant separation is crucial if the covariance property is required to be preserved throughout the analysis under the action of the global ten-dimensional super-Poincaré group. It is the purpose of this paper to extend that analysis²⁾ to a covariantly-quantized version in the Hamiltonian framework. Some aspects of the relation to the light-cone gauge formulation will also be discussed.

The most powerful approach to the quantization of a constrained dynamical system appears to be that of Batalin, Fradkin and Vilkovisky³⁾ (BFV). Additional variables are introduced associated with the constraints L_a as:

- a) Lagrange multipliers λ_a and their conjugate variables π_a ;
- b) ghosts C_a , associated to L_a , and their conjugate variables B_a ;
- c) antighosts $\bar{C}_a$, associated to λ_a , and their conjugate variables $\bar{B}_a$.

It is possible to dispense with the λ -variables and the further variables mentioned above when the constraints are linear in the momenta p , but that is not so when these constraints are quadratic (or worse) in the p 's. In the latter case, the best one can choose is the proper time gauge (PTG), as discussed in detail by Teitelboim⁴⁾ for the particle and for general relativity, for which

$$\dot{\lambda} = 0 \tag{1}$$

where the dot denotes differentiation with respect to the proper time. This PTG still requires the presence of λ , and its associated variables described above, though now being proper-time independent.

The Green-Schwarz (GS) covariant superstring has constraints quadratic in the momenta, so the above approach must be used for quantization. Thus the constraints for the GS theory, in the notation of Ref. 2), are

$$H = \frac{1}{2}(\tilde{\phi}^2 + V^2) = 0 \quad (2)$$

$$T = \tilde{\phi} \cdot V = 0 \quad (3)$$

$$F^A = \zeta^A + i\bar{\theta}^A \gamma_r [\dot{p}_r + (-1)^A (\dot{x}'_r - K_r^A)] = 0 \quad (4)$$

In the above, $A = 1, 2$, denoting the two Majorana-Weyl ten-dimensional spinors, $V_\mu = \dot{X}'_\mu - K_\mu^1 - K_\mu^2$, $\tilde{p}_\mu = p_\mu + K_\mu^1 - K_\mu^2$, $K_\mu^A = i\bar{\theta}^A \gamma_\mu \theta^{A'}$, prime denoting differentiation with respect to σ , and $\tilde{p}_\mu = p_\mu + K_\mu^1 - K_\mu^2$. The system of constraints (2)-(4) is still not first class, but involves mixed first- and second-class constraints. These were separated in Ref. 2) into the second-class constraints

$$G^A = 0 \quad (5a)$$

$$G^1 = F^1 \Gamma_2, G^2 = F^2 \Gamma_1 \quad (5b)$$

with non-zero Poisson bracket

$$\{G_\alpha^A(\sigma), G_\beta^B(\sigma')\} = -4i\delta^{AB}\delta(\sigma-\sigma')(\tilde{p}^2 - V^2)(C\Gamma_A\Gamma_{\bar{A}})_{\alpha\beta} \quad (5c)$$

[where $h_A = \frac{1}{2}(1+\gamma^{11})$ is the ten-dimensional chirality projector, $\bar{A} = 1$ if $A = 2$, $\bar{A} = 2$ if $A = 1$, and $\Gamma_A = \gamma^\mu(\tilde{p}_\mu + (-1)^A V_\mu)$], and the first-class constraints

$$\tilde{F}^A = 0 \quad (6a)$$

$$\tilde{F}^1 = F^1 \Gamma_1, \tilde{F}^2 = F^2 \Gamma_2 \quad (6b)$$

By translation of the Lagrange multipliers, the constrained Hamiltonian H is purely first class, with

$$\mathcal{H} = \int d\sigma [\lambda_0 \tilde{H} + \lambda_1 \tilde{T} + \tilde{F}^A \tilde{\Lambda}^A] \quad (7)$$

where

$$\tilde{H} = \frac{1}{2}(\dot{p}^2 + \dot{x}^2) - \zeta^1 \theta^{1'} + \zeta^2 \theta^{2'} \quad (8a)$$

$$\tilde{T} = p \cdot X' + \zeta^1 \theta^{1'} + \zeta^2 \theta^{2'} \quad (8b)$$

and $\tilde{F}$ is given by (4) and (6b).

The system may now be quantized by applying the BFM approach³⁾ to the set of constraints $\tilde{H}$, $\tilde{T}$, $\tilde{F}^A$, and the second-class constraint approach of Senjanovic⁵⁾ to the constraints G^A of (5). It is to be noted that Senjanovic only discussed bosonic constraints explicitly, but the case of fermionic second-class constraints can also be handled in a similar manner, leading to a factor in the functional measure for quantization given by

$$\prod_{A,\alpha} \delta(G_{\alpha}^A) [\det \{G_{\beta\gamma}^B, G_{\gamma\delta}^C\}]^{-\frac{1}{2}} \quad (9)$$

Thus the only change in the measure (9) from the bosonic to fermionic case is to invert the square root of the determinant of the second-class constraints.

The application of the BFM method to the constraints $\tilde{H}$, $\tilde{T}$, $\tilde{F}^A$ corresponds to introducing the sets of variables:

$$\{\lambda_R, \pi_R, C_R, B_R, \bar{C}_R, \bar{B}_R\} \quad (10)$$

where R denotes $\tilde{H}$, $\tilde{T}$, or $\tilde{F}^A$, which themselves will be denoted by $\tilde{F}_R$, and

$$\{\lambda_R, \pi_S\} = \delta_{RS} \cdot \delta = \{C_R, B_S\} = \{\bar{C}_R, \bar{B}_S\} \quad (11)$$

with δ denoting $\delta(\sigma_R - \sigma_S)$, and R, S denoting one of $\tilde{H}$, $\tilde{T}$, or $\tilde{F}^A$. The structure constants of the algebra of $\tilde{H}$, $\tilde{T}$, $\tilde{F}^A$ gives from²⁾ the Poisson brackets:

$$\begin{aligned}
 \{\tilde{H}(\sigma), \tilde{H}(\sigma')\} &= (\tilde{T}(\sigma) + \tilde{T}(\sigma')) \delta'(\sigma - \sigma') \\
 \{\tilde{H}(\sigma), \tilde{T}(\sigma')\} &= (\tilde{H}(\sigma) + \tilde{H}(\sigma')) \delta'(\sigma - \sigma') \\
 \{\tilde{T}(\sigma), \tilde{T}(\sigma')\} &= (\tilde{T}(\sigma) + \tilde{T}(\sigma')) \delta'(\sigma - \sigma') \\
 \{\tilde{H}(\sigma), \tilde{F}^A(\sigma')\} &= (-1)^A (\tilde{F}^A(\sigma) + \tilde{F}^A(\sigma')) \delta'(\sigma - \sigma') \\
 \{\tilde{T}(\sigma), \tilde{F}^A(\sigma')\} &= (\tilde{F}^A(\sigma) + \tilde{F}^A(\sigma')) \delta'(\sigma - \sigma') \\
 \{\tilde{F}^A_\alpha(\sigma), \tilde{F}^B_\beta(\sigma')\} &= -4i \delta^{AB} \delta(\sigma - \sigma') [(\tilde{H} + (-1)^A \tilde{T})(C \chi_A \gamma_\mu)_\alpha{}_\beta \\
 &\quad - (-1)^A (2 \tilde{F}^A_\alpha \bar{\theta}^{\beta A'} + \tilde{F}^A \delta^\mu{}_\alpha \theta^{\beta A'} (C \chi_A \gamma_\mu)_\alpha{}_\beta)]
 \end{aligned} \tag{12}$$

The next step is to construct the BRST nilpotent charge Q . The structure coefficients C_{ab}^c of the first-class constraint algebra (12) are variable-dependent, and it is necessary to determine the rank of the system following the analysis of Henneaux⁶⁾. The rank of a constrained dynamical system is given by the powers of the antighosts B_R needed in the complete definition of Q (with $\varepsilon_R = 0$ or 1 according to whether $\tilde{F}_R$ is bosonic or fermionic) as

$$\begin{aligned}
 Q &= C_R \tilde{F}_R - \frac{1}{2} (-1)^{\varepsilon_R} C_R C_S C_{SR}^T B_T + C_R C_S C_T C_{RST}^{UV} B_V B_U + \\
 &\quad + \dots + (B)^n \text{ term} + \bar{B}_R \pi_R
 \end{aligned} \tag{13}$$

and summation over repeated indices is now assumed. The coefficients C_{RST}^{UV} , C_{RSTU}^{VWX} , ... are determined by a system of equations guaranteeing the nilpotency of Q . Thus for C_{RST}^{UV} there is the equation⁶⁾

$$\begin{aligned}
 2 C_{RST}^{UV} \tilde{F}_V &= (-\frac{1}{2} (-1)^{\varepsilon_S} \{ C_{RS}^U, \tilde{F}_T \} (-1)^{(\varepsilon_T+1)\varepsilon_U} + \\
 &\quad + \frac{1}{2} (-1)^{\varepsilon_U + (\varepsilon_T+1)\varepsilon_V + \varepsilon_C + \varepsilon_V} C_{RS}^V C_{TV}^U)_A
 \end{aligned} \tag{14}$$

where $(\)_A$ denotes antisymmetrization on the variables RST . This equation (14) and similar equations for the higher-order coefficients C_{RSTU}^{VWX} , etc., are obtained from the Jacobi identity.

It is now possible to show that the rank of the GS superstring is one. This

may be done rather rapidly by taking each of the non-zero structure constants C_{RS}^T from (12) and calculating the right-hand side of (14), using (12) to obtain the first term. It is then clear that the resulting expressions cannot contain any term proportional to $\tilde{F}$. In particular, there is no term containing the θ momentum ζ on the right-hand side of (14), so that by the definitions of the $\tilde{F}_R$'s by (6b), (8a) and (8b) it is not possible to have non-zero values of C_{RST}^{UV} . This can be made more precise by showing that there is exact cancellation on the right-hand side of (14), as there must be by the theorems (3.3) and (3.4) of Henneaux⁶⁾.

The expression for Q is therefore

$$Q = \int_0^\pi d\sigma \left[C_R \tilde{F}_R - \frac{1}{2} (-1)^{\Sigma_R} C_R C_S C_{SR}^T B_T + \bar{B}_R \pi_R \right] \quad (15)$$

Choice of the gauge-fixing function ψ as

$$\psi = \int_0^\pi d\sigma (B_R \lambda_R + \bar{C}_R \chi_R) \quad (16)$$

leads to the Hamiltonian

$$H_1 = \{Q, \psi\} = \int_0^\pi d\sigma \left[\lambda_R \tilde{F}_R + B_R \bar{B}_R + \pi_R \chi_R + C_R \bar{C}_S \{ \tilde{F}_R, \chi_S \} - \frac{1}{2} \lambda_R C_S C_{RS}^T B_T \right] \quad (17)$$

[where signature factors still need to be included in (17)]. Thus H_1 of (17) extends that of (7) by the contributions of the additional BFV variables in an explicit manner.

It is now possible to insert H into the classical action to obtain

$$S_{cl} = \int_0^\pi d\sigma \left[p \dot{X} + \zeta^A \dot{\theta}^A + B_R \dot{C}_R + \bar{B}_R \dot{\bar{C}}_R + \pi_R (\dot{\lambda}_R - \chi_R) - \lambda_R \tilde{F}_R - B_R \bar{B}_R - C_R \bar{C}_S \{ \tilde{F}_R, \chi_S \} + \frac{1}{2} \lambda_R C_S C_{RS}^T B_T \right] d\sigma \quad (18)$$

Transition amplitudes are then to be obtained by integration over all the variables $p, X, \zeta, \theta, C, B, \bar{C}, \bar{B}, \lambda, \pi$ with phase factor $\exp[iS_{cl}]$, and an additional

factor given by the second-class constant measure (9). The first quantization has thus been achieved in full BFV Hamiltonian form, and depends on the arbitrary gauge-fixing function ϕ , so on the canonical gauge choice $\chi_R = \chi_R(p, X, \zeta, \theta)$.

Let us consider in more detail the second-class constraints (5). They may be solved for ζ^1 and ζ^2 as

$$\zeta^1 = -i \bar{\theta}^1 \gamma^r [\dot{p}_r - \dot{x}'_r - K^1_r] + \eta^1 \quad (19a)$$

$$\zeta^2 = -i \bar{\theta}^2 \gamma^r [\dot{p}_r + \dot{x}'_r - K^2_r] + \eta^2 \quad (19b)$$

where the spinors η^1 and η^2 are constrained by

$$\eta^1 \gamma^r [\tilde{p}_r + V_r] = 0 \quad (19c)$$

$$\eta^2 \gamma^r [\tilde{p}_r - V_r] = 0 \quad (19d)$$

and the term in the classical action for $\lambda_{\tilde{F}} \tilde{F}$ reduces to

$$2(\lambda^1 \tilde{p} \eta^1 + \lambda^2 \tilde{p} \eta^2) \quad (20)$$

whilst the terms in $\lambda_{\tilde{H}} \tilde{H}$ and $\lambda_{\tilde{T}} \tilde{T}$ involving ζ reduce to

$$\begin{aligned} & -\eta^1 \theta^{1'} + \eta^2 \theta^{2'} - i \bar{\theta}^1 \gamma^r (\dot{p}_r - \dot{x}'_r) \theta^{1'} + i \bar{\theta}^2 \gamma^r (\dot{p}_r - \dot{x}'_r) \theta^{2'} \\ & - i [(K^1_r)^2 - (K^2_r)^2] \end{aligned} \quad (21a)$$

and

$$\begin{aligned} & \eta^1 \theta^{1'} + \eta^2 \theta^{2'} - i \bar{\theta}^1 \gamma^r (\dot{p}_r - \dot{x}'_r) \theta^{1'} - i \bar{\theta}^2 \gamma^r (\dot{p}_r - \dot{x}'_r) \theta^{2'} + \\ & + i [(K^1_r)^2 - (K^2_r)^2] \end{aligned} \quad (21b)$$

respectively. The terms involving η^A are purely quadratic in their variables, and in spite of the constraints (19) it appears that there will be a difficulty in developing a covariant second-quantized field theory of the GS superstring without removal of the quartic terms in θ , as will be discussed briefly at the

end of this letter. Thus it is necessary to determine if it is possible to choose a gauge in which these terms automatically cancel.

It is known that there are no such quartic terms in θ in the light-cone (LC) gauge. In order to obtain that gauge, a limiting procedure must be followed in which a rescaling occurs⁶⁾. The momentum conjugate to λ is rescaled as $\pi \rightarrow \beta\pi^*$, and the antighost $\bar{C} \rightarrow \beta\bar{C}^*$, whilst $\chi = \beta^{-1}\chi^*$, where χ^* is the new canonical gauge-fixing function. Since the super-Jacobian of the transformation is 1, and physical results are independent of β , then the limit $\beta \rightarrow 0$ may be taken. On so doing, and integration taken over π , the gauge constraint $\delta(\chi)$ ensues. Integration over the Lagrange multipliers then leads to the usual Faddeev-Popov expression⁷⁾. If this is taken to be the non-covariant choice

$$\chi^A = (\gamma^+ \theta^A) = 0 \quad (22)$$

then the quartic terms $(K^1)^2, (K^2)^2$ disappear. It is not even necessary to impose the bosonic LC constraints

$$\begin{aligned} \chi^+ &= \chi^+ - (x^+ + p^+ \tau) = 0 \\ \chi^- &= p^+ = 0 \end{aligned} \quad (23)$$

If these are chosen, then only $\eta^1 = \eta^2 = 0$ is allowed as a solution of (19) and, as is well known, complete gauge fixing and ghost decoupling has occurred. Thus we conclude that the partial gauge choice (22) removes the quartic terms in θ .

The possibility of making a covariant gauge choice to remove the quartic terms in θ seems very problematic. One might try to use the projectors Γ_A mentioned at the beginning, but these apparently cannot be manipulated in the same manner as (22) to achieve the desired result. We have therefore to accept that in a covariant approach to the GS string there will be θ^4 terms at the first-quantized level.

We now consider how to manipulate the second-class constraint factor (9) into a Hamiltonian form. The δ -function can evidently be expressed in the usual fashion:

$$\int \mathcal{D}\lambda \exp[-i \int \lambda_A^\alpha G_\alpha^A d\sigma] \quad (24)$$

Since the quantum brackets of the second-class constraints with H_1 of (17) are zero (being obtained by replacing the Dirac brackets with commutator brackets,

and the former - hence the latter - being zero between first- and second-class constraints), then the constraints (24) should only be imposed at a given proper time τ . This leads us to investigate the interesting possibility that the proper time approach allows the Lagrange multipliers for the second-class constraints to become dynamical variables. To see this, we emphasize the time independence of the second-class constraints remarked on above. This necessitates the introduction of a factor $\delta(\dot{\lambda}_A^\alpha)$, so that, using a similar representation to (24), we have a net factor arising from the δ -function of constraints as

$$\int \delta \pi_A^\alpha \delta \lambda_A^\alpha \exp[i \int d\tau d\sigma (\pi_A^\alpha \dot{\lambda}_A^\alpha - \lambda_A^\alpha G_A^\alpha)] \quad (25)$$

Finally, the factor from the determinant in (9) can be written, by the same argument about changing multipliers into dynamical variables, as

$$\int \exp[i \int d\tau d\sigma (b_{A\alpha} \dot{c}_{A\alpha} - c_{A\alpha} \{G_{A\alpha}, G_{B\beta}\} c_{B\beta})] \delta b_{A\alpha} \delta c_{B\beta} \quad (26)$$

where $\{G_{A\alpha}, G_{B\beta}\}$ is given by (5c), and the variables c, b are bosonic (so are also ghosts). Thus finally the system has been transformed into one of Hamiltonian form, with net Hamiltonian

$$H_2 = H_1 + \lambda_A^\alpha G_A^\alpha + c_{A\alpha} \{G_{A\alpha}, G_{B\beta}\} c_{B\beta} \quad (27)$$

Now that the system is in Hamiltonian form, a second-quantized version may be constructed, using as configuration space variables

$$Z(\sigma) = \{X^\mu(\sigma), \theta_A^\alpha(\sigma), C_R(\sigma), \bar{C}_R(\sigma), \lambda_R(\sigma), \lambda_A^\alpha(\sigma), c_{A\alpha}(\sigma), \bar{c}_{A\alpha}(\sigma)\}$$

with conjugate momenta

$$\Pi(\sigma) = \{P_\mu(\sigma), \pi_A^\alpha(\sigma), B_R(\sigma), \bar{B}_R(\sigma), \pi_R(\sigma), \pi_A^\alpha(\sigma), b_{A\alpha}(\sigma), \bar{b}_{A\alpha}(\sigma)\}$$

The resulting superstring action in gauge-fixed form, following the stimulating suggestion of Neveu and West⁸⁾ for the particle and its bosonic string development^{9),10)}, will thus be

$$S_0^{(f)} = \int \mathcal{L} dz d\tau \bar{\Phi}^* [i\partial/\partial\tau - H_2(z, -i\delta/\delta z)] \Phi \quad (28)$$

The arguments used in the second paper of Ref. 9) might indicate, on extending (28) to the interacting case, that perturbation theory should lead a formulation in terms of Riemannian surfaces with the Lagrange multipliers and proper times of the interaction point corresponding to the moduli of the surfaces. However, this transition, pioneered in the light-cone gauge by Kaku and Kikkawa¹¹⁾, would seem to be possible only if quadratic expressions in the co-ordinates Z in H_2 occur. Thus it may well be that the Riemann surface formulation will only be available for the spinning (NSR) string and not the GS string. This will be an interesting problem to determine in the future.

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