The Superconducting Super Collider

Equations for Multiparticle Dynamics

Alexander W. Chao
SSC Central Design Group

January 1987

EQUATIONS FOR MULTIPARTICLE DYNAMICS*

Alexander W. Chao
SSC Central Design Group†
c/o Lawrence Berkeley Laboratory
Berkeley, California 94720

January 1987

* Lecture given at Joint US/CERN Accelerator School, Topical course on Frontiers of Particle Beams, South Padre Island, October 1986

† Operated by Universities Research Association, Inc., for the U.S. Department of Energy.

TABLE OF CONTENTS

I. Introduction	1
II. The Vlasov Equation	
Derivation	2
Potential Well Distortion	6
Collective Instabilities	12
III. The Fokker-Planck Equation	
Derivation	13
Potential Well Distortion	20
Linear Coupling	21
Transient Distributions	26
Quantum Lifetimes	30
Normal Modes	43

I. INTRODUCTION

The description of the motion of charged-particle beams in an accelerator proceeds in steps of increasing complexity. The first step is to consider a single-particle picture in which the beam is represented as a collection of non-interacting test particles moving in a prescribed external electromagnetic field. The electromagnetic field generated by the test particles is ignored. Knowing the external field, it is then possible to calculate the beam motion to a high accuracy.

The real beam consists of a large number of particles, typically 10^{11} per beam bunch. It is sometimes inconvenient, or even impossible, to treat the real beam behavior using the single particle approach. One notable important example is the inclusion of the electromagnetic fields associated with these charged particles, which is necessary when one wants to deal with the collective instability effects of an intense beam.

One way to approach this problem is to supplement the single particle picture by another qualitatively different picture. The commonly used tools in accelerator physics for this purpose are the Vlasov¹ and the Fokker-Planck equations.² These equations assume smooth beam distributions and are therefore strictly valid in the limit of infinite number of micro-particles, each carrying an infinitesimal charge. The hope is that by studying the two extremes -- the single particle picture and the picture of smooth beam distributions -- we will be able to describe the behavior of our 10^{11} -particle system.

As mentioned, the most notable use of the smooth distribution picture is the study of collective beam instabilities. However, the purpose of this lecture is not to address this more advanced subject. Rather, it has the limited goal to familiarize the reader with the analytical tools, namely the Vlasov and the Fokker-Planck equations, as a preparation for dealing with the more advanced problems at later times. We will first derive these equations and then illustrate their applications by several examples which allow exact solutions. The actual treatment of instability problems are not included, but the examples selected hopefully still represent useful applications to commonly encountered accelerator problems.

II. THE VLASOV EQUATION

2.1 Derivation

The Vlasov equation is an equation that describes the collective behavior of a multi-particle system under the influence of electromagnetic forces. To construct the Vlasov equation, one starts with the single particle equations of motion

$$\begin{aligned}\frac{dq}{dt} &= f(q,p,t) \\ \frac{dp}{dt} &= g(q,p,t)\end{aligned}\tag{1}$$

where q and p are the coordinate and momentum variables respectively, and the (q,p) plane is the phase space. The state of a particle at a given time t is represented by a point in the phase space. The motion of a particle is described by the motion of its representative point in the phase space. A particle executing a simple harmonic motion, for example, traces out an ellipse in the phase space.

In a conservative deterministic system, particle trajectory in phase space is completely determined by the initial conditions. Two particles having the same initial conditions must have exactly the same trajectory in phase space. It follows that the only way for two trajectories to meet in the phase space at a given time is for them to coincide at all times. In other words, trajectories either completely coincide or will never intersect.

Consider now a distribution of particles occupying an area in the phase space. In order to avoid intersecting with particles on the boundary of the distribution as the distribution evolves in time, particles inside the distribution can not leak out by crossing the boundary. Similarly no particles from outside can leak into the distribution. A moment's reflection indicates that as the distribution evolves, its shape may be distorted but the area must remain constant. This can be put in a mathematical form as follows. If the system is conservative, we have the conditions $f=\partial H/\partial p$ and $g=-\partial H/\partial q$ where H is the Hamiltonian of the system. Therefore

$$\frac{\partial f}{\partial q} + \frac{\partial g}{\partial p} = 0\tag{2}$$

As will be seen in Eq.(3), condition (2) is directly related to the area conservation observed before. In fact, in a non-conservative system, $\partial f/\partial q + \partial g/\partial p$ has the physical meaning of the area shrinking rate.

Consider a beam distribution in the rectangular $\Delta q \Delta p$ box drawn at time t with vertices

$$A(q, p)$$

$$B(q + \Delta q, p)$$

$$C(q + \Delta q, p + \Delta p)$$

$$D(q, p + \Delta p)$$

as shown in Fig.1. In general, at an infinitesimal time dt later, the rectangular box deforms into a parallelogram with vertices

$$A' [q + f(q, p, t)dt, p + g(q, p, t)dt]$$

$$B' [q + \Delta q + f(q + \Delta q, p, t)dt, p + g(q + \Delta q, p, t)dt]$$

$$C' [q + \Delta q + f(q + \Delta q, p + \Delta p, t)dt, p + \Delta p + g(q + \Delta q, p + \Delta p, t)dt]$$

$$D' [q + f(q, p + \Delta p, t)dt, p + \Delta p + g(q, p + \Delta p, t)dt]$$

FIG.1. A rectangular box ABCD containing a distribution of particles is drawn in the phase space at time t . As particles move in time, the box deforms. At time $t+dt$ with infinitesimal dt , the rectangle deforms into a parallelogram A'B'C'D', which has the same area as ABCD.

The condition for a conservative system, Eq.(2), implies that the area of the box is conserved, i.e.

$$\begin{aligned} \text{area}(A'B'C'D') &= |\overrightarrow{A'B'} \times \overrightarrow{A'D'}| \\ &= \Delta q \Delta p \left[1 + \left(\frac{\partial f}{\partial q} + \frac{\partial g}{\partial p} \right) dt \right] \\ &= \Delta q \Delta p = \text{area}(ABCD) \end{aligned} \tag{3}$$

Let the number of particles in the box be $N\psi(q,p,t)\Delta q\Delta p$ with N the total number of particles in the system and ψ the distribution density, depending on q , p and t , normalized by

$$\int dq \int dp \psi(q,p,t) = 1. \quad (4)$$

Particles in ABCD at time t must be accounted for in A'B'C'D' at time $t+dt$. This means

$$\psi(q,p,t) \text{ area}(ABCD) = \psi(q+fdt, p+gdt, t+dt) \text{ area}(A'B'C'D'). \quad (5)$$

Equation (3) then gives

$$\begin{aligned} \psi(q,p,t) &= \psi(q+fdt, p+gdt, t+dt) \\ &= \psi + fdt \frac{\partial \psi}{\partial q} + gdt \frac{\partial \psi}{\partial p} + dt \frac{\partial \psi}{\partial t} \end{aligned}$$

or, after cancelling out ψ on both sides,

$$\frac{\partial \psi}{\partial t} + f \frac{\partial \psi}{\partial q} + g \frac{\partial \psi}{\partial p} = 0 \quad (6)$$

Equation (6) is the Vlasov equation -- especially when the forces are electromagnetic in origin. It can also be put in the form

$$\frac{d\psi}{dt} = 0 \quad \text{or} \quad \psi = \text{constant in time} \quad (7)$$

which states that the local particle density does not change with time if the observer moves with the flow of the box. The information of how boxes flow is contained in the functions f and g , which are explicit in Eq. (6) but implicit in Eq. (7).

Strictly speaking, f and g are given by external fields and collisions among discrete particles in the system can not be included in Eq. (6) or (7). On the other hand, if a particle interacts more strongly with the collective fields of the other particles than with its nearest neighbors, Vlasov equation still applies by treating the collective fields on the same basis as the external fields. This in fact forms the basis of treating the collective instabilities using the Vlasov technique³.

When written in the form (7), it is sometimes loosely referred to as the Liouville theorem. However, the Liouville theorem⁴ applies to an ensemble of many systems, each containing many particles. It describes the density conservation of the ensemble in a $2N$ dimensional space (the Γ -space). It applies to situations much more general than that for Eq.(7), such as when collisions among discrete particles are included.

One special case when the Vlasov equation can be solved exactly is when the system can be described by a Hamiltonian $H(q,p)$ which does not have explicit time dependence. Using the properties

$$f = \frac{\partial H}{\partial p} \quad \text{and} \quad g = \frac{-\partial H}{\partial q}, \quad (8)$$

a stationary beam distribution to the Vlasov equation (6) can be found to be

$$\psi(q,p) = \text{any function of } H. \quad (9)$$

In this Hamiltonian system, particles move rapidly in phase space along constant Hamiltonian countours. A stationary distribution (9) is one that does not change in time even as individual particles move within it.

Example 1. For a simple harmonic system with Hamiltonian $H = \omega(q^2+p^2)/2$, the Vlasov equation reads

$$\frac{\partial \psi}{\partial t} + \omega p \frac{\partial \psi}{\partial q} - \omega q \frac{\partial \psi}{\partial p} = 0 \quad (10)$$

Making a transformation to the polar coordinates by

$$q = a \cos \phi, \quad p = a \sin \phi, \quad (11)$$

Eq.(10) can be written as

$$\frac{\partial \psi}{\partial t} - \omega \frac{\partial \psi}{\partial \phi} = 0, \quad (12)$$

which has the general solution

$$\psi(a,\phi,t) = \text{any function of } (a, \phi + \omega t) \quad (13)$$

Once the initial distribution of the beam is given at $t=0$, Eq.(13) means that the distribution at time t in the simple harmonic case is given by rigidly rotating the initial distribution in phase space angle ϕ at a constant angular speed of ω . The stationary distribution is any function of H , or equivalently function of amplitude a , without any ϕ dependence.

Example 2. As a second example, consider a damped simple harmonic motion with

$$f = \omega p \quad \text{and} \quad g = -\omega q - 2\alpha p, \quad (14)$$

where α is the damping rate. A slight modification of the Vlasov equation is required here because

the damping means the conservation of phase space area is violated. Following the derivation from Eq. (3) to Eq. (6) and using $\partial f/\partial q + \partial g/\partial p = -2\alpha$, the modified Vlasov equation reads

$$\frac{\partial \psi}{\partial t} + \omega p \frac{\partial \psi}{\partial q} - (\omega q + 2\alpha p) \frac{\partial \psi}{\partial p} = 2\alpha \psi. \quad (15)$$

Note that a straightforward substitution of f and g into the form (6) would miss the term on the right hand side of Eq. (15). Note also that in this case $\partial f/\partial q + \partial g/\partial p = -2\alpha$ is negative, meaning the area of phase space boxes shrinks with time in the presence of damping.

The general solution of Eq.(15) can be written as

$$\psi(q,p,t) = e^{2\alpha t} \{ \text{any function of } A \text{ and } \Phi \} \quad (16)$$

where

$$A = e^{\alpha t} \left[q^2 + p^2 + \frac{2\alpha qp}{\omega} \right]^{1/2}$$

$$\Phi = \tan^{-1} \left[\frac{\omega p + \alpha q}{q(\omega^2 - \alpha^2)^{1/2}} \right] + (\omega^2 - \alpha^2)^{1/2} t$$

The overall factor $\exp(2\alpha t)$ in Eq.(16) means the distribution is becoming denser as a result of damping. Quantities A and Φ reduce to a and ϕ when $\alpha=0$. The general solution is that of a distribution spiralling inward with time, as one would expect. The only stationary solution is when the entire distribution collapses into a delt-function at the phase space origin.

2.2 Potential Well Distortion

One type of Hamiltonian commonly encountered is that for a system moving in a potential well, in which case,

$$f = \omega p, \quad g = -K(q) \quad (17)$$

and

$$H = \frac{\omega}{2} p^2 + \int_0^q dq' K(q') \quad (18)$$

As before, any function of H expressed above is a possible stationary distribution. The second integral term in Eq.(18) is the potential well term. A simple harmonic system has a parabolic potential well.

Consider now the longitudinal synchrotron oscillation of a particle in a circular accelerator. The physical quantities of interest are the longitudinal displacement z relative to the beam center and the relative energy error $\delta = \Delta E/E$.⁵ The phase space coordinates q and p are related to these quantities by

$$q = z \quad \text{and} \quad p = -\frac{\alpha_c c}{\omega_s} \delta \quad (19)$$

where α_c is the momentum compaction factor, c is the speed of light and ω_s is the synchrotron oscillation frequency. The single-particle equations of motion, in the presence of an arbitrary rf voltage $V_{rf}(q)$, are

$$\dot{q} = \omega_s p \quad \text{and} \quad \dot{p} = -\frac{\omega_s V_{rf}'(q)}{V_{rf}'(0)} \quad (20)$$

where V_{rf}' is the derivative of V_{rf} with respect to the longitudinal coordinate q . For small oscillation amplitudes, the expression for $\dot{p}$ becomes $\dot{p} = -\omega_s q$, the case of simple harmonic motion. A practical case is given by $V_{rf} = V_0 \sin(\omega_{rf} q/c)$. The general stationary distribution is then given by any function of the Hamiltonian

$$H = \frac{\omega_s}{2} p^2 + \frac{\omega_s c^2}{\omega_{rf}^2} \left[1 - \cos\left(\frac{\omega_{rf} q}{c}\right) \right] \quad (21)$$

This Hamiltonian also describes the form of the rf bucket. Figure 2(a) shows a possible stationary distribution covering an area outlined by a constant Hamiltonian contour. Another possible beam distribution with $\psi = \exp(-\text{const.} \times H)$ is shown in Fig. 2(b). One complication associated with this distribution will be discussed when we later examine the quantum lifetime effect in Sec. 3.5.

From Eq.(21), the stationary beam distribution $\exp(-\text{const.} \times H)$ is gaussian in p . In case the bunch length is much shorter than the rf wavelength, i.e. $q \ll c/\omega_{rf}$, the familiar quadratic form of Hamiltonian is re-established, i.e. $H = \omega_s(p^2 + q^2)/2$; the distribution is then also gaussian in q . As the bunch length increases, the bunch shape deviates from gaussian. This phenomenon is called the potential well distortion.

FIG. 2. (a) A possible stationary distribution in the rf bucket is one that conforms to the contours of constant Hamiltonian (21) and has clean distribution boundaries. (b) Another possible stationary distribution is given by, or approximated by, $\exp(-\text{const.} \times H)$. In this case, one has to ask what happens to particles outside the bounded rf bucket. See Sec. 3.5 later.

There is another reason for the Hamiltonian to deviate from the quadratic form and thus to cause potential well distortion.^{6,7} A particle propagating in an accelerator leaves behind a wake electromagnetic field due to its interaction with the surrounding environment. For relativistic particles, this wake field vanishes in front of the particle but retards all particles trailing it. Let the retarding effect be characterized by a wake function $W(z)$, where z is the distance lag from the particle to a trailing particle, and $W(z)=0$ if $z<0$. The single particle motion can be written as

$$\begin{aligned} f(q,p) &= \omega_s p \\ g(q,p) &= -\omega_s q + N\omega_s \int_q^\infty dq' \rho(q') W(q'-q) \end{aligned} \quad (22)$$

where the integral specifies the retarding (accelerating if negative) voltage seen by a particle at longitudinal location q due to the wake produced by all particles in front of it; $N\rho(q')$ is the charge density at location q' . Equation (22) satisfies the area conservation condition (2). The corresponding Hamiltonian is

$$H = \frac{\omega_s}{2}(q^2 + p^2) - N\omega_s \int_0^q dq'' \int_{q''}^\infty dq' \rho(q') W(q'-q'') \quad (23)$$

The stationary solution to the Vlasov equation must be a function of H . The complication here compared with the case of rf potential well distortion is that the complicated q -dependence of H now involves p , which in turn depends on the stationary distribution itself. Clearly some self-consistency is necessary to solve the problem. Below we will give three examples for which the problem can be simplified for solution.

Example 3 Although the q -dependence is complicated, the distribution can maintain its gaussian distribution in p , i.e.

$$\psi(q,p) = \frac{1}{\sqrt{2\pi} \sigma_p} \exp\left(\frac{-p^2}{2\sigma_p^2}\right) \rho(q) \quad (24)$$

where σ_p , the rms value of p , is related to the rms relative energy spread by $\sigma_p = c\alpha_c \sigma_g / \omega_s$ as dictated by Eq.(19). The gaussian form and the value of σ_p are arbitrary in the sense that any well-behaved function of H can be a stationary solution of the Vlasov equation. On the other hand, we will show later that if the multi-particle system has to be described by the Fokker-Planck equation instead of the Vlasov equation, the stationary beam distribution has to assume the gaussian form (24) with a very specific value of σ_p .

Equation (24) matches the stationary solution

$$\psi(q,p) = Q \exp\left(\frac{-H}{\omega_s \sigma_p^2}\right) \quad (25)$$

where Q is a normalization factor determined by condition (4). Substituting (23) into (25) and noting that the line charge density is by definition given by

$$\rho(q) = \int_{-\infty}^{\infty} dp \psi(q,p) \quad (26)$$

we obtain a transcendental equation for $\rho(q)$, i.e.

$$\rho(q) = Q \sqrt{2\pi} \sigma_p \exp\left[\frac{-q^2}{2\sigma_p^2} + \frac{N}{\sigma_p^2} \int_0^q dq'' \int_{q'}^{\infty} dq' \rho(q') W(q'-q'')\right] \quad (27)$$

Together with the normalization condition (4), which now reads $\int_{-\infty}^{\infty} \rho(q) dq = 1$, Eq.(27) can in principle be solved numerically for the line charge density $\rho(q)$ once the wake function $W(z)$ is

known and σ_p specified. In the limit of small beam intensity $N \rightarrow 0$, the solution reduces to the bi-gaussian form. For high beam intensities, the line charge density deforms from gaussian shape.

Example 4 Another soluble example applies to the case when $W(z) = S\delta'(z)$ where $\delta(z)$ is the delta-function. Under this condition, the retarding wake at location q can be related to the local derivative of the line density, i.e.

$$\int_q^\infty dq' \rho(q') W(q'-q) = -S \frac{d\rho(q)}{dq} \quad (28)$$

Physical effects that approximately produce such retarding wake are those due to the space charge Coulomb repulsion and due to capacitive or inductive impedances.⁸ In this example, we assume the distribution of the beam bunch has the ansatz form

$$\psi(q,p) = C \left[\left(\frac{L_0}{2} \right)^2 - \frac{p^2}{K} - Kq^2 \right]^{1/2} \quad \text{if } \frac{p^2}{K} + Kq^2 < \left(\frac{L_0}{2} \right)^2$$

$$0 \text{ otherwise} \quad (29)$$

$$C = \frac{12}{\pi L_0^3}$$

where the coefficient C is to satisfy the normalization condition (4). As we will see, the parameter K specifies the degree of distortion of beam distribution from the ideal case in the absence of wake fields. The form (29) maintains an elliptical form in the phase space even in the presence of wake fields; it is chosen in such a way that the area covered by the beam in the phase space (the emittance) is independent of K , i.e., independent of the wake field.

Distribution (29) has a parabolic line density

$$\rho(q) = \frac{\pi C}{2} K^{1/2} \left[\left(\frac{L_0}{2} \right)^2 - Kq^2 \right] \quad (30)$$

The beam has a total length of $L = L_0 K^{-1/2}$. The unperturbed beam length is L_0 .

To be self-consistent, the distribution has to be a function of the Hamiltonian. This means, using Eq.(23),

$$N \int_0^q dq'' \int_{q''}^\infty dq' \rho(q') W(q'-q'') = -(K^2 - 1) \frac{q^2}{2} \quad (31)$$

Substituting Eqs.(28) and (30) into (31) then leads to the self-consistency condition

$$12 NS K^{3/2} + (K^2 - 1) L_0^3 = 0$$

or, in terms of the total beam length L , a fourth order equation⁸

$$L^4 - 12 NSL - L_0^4 = 0 \quad (32)$$

In the limit of weak beam intensity, $L=L_0$. As beam intensity increases, the beam shape remains parabolic, but its length changes, either shortens or lengthens according to the sign of S . Figure 3(a) illustrates the phase space distribution of the unperturbed case ($K=1$). Figure 3(b) shows the cases of shortened ($K>1$) and lengthened ($K<1$) bunches for this example.

XBL 872-8931

XBL 872-8925

FIG. 3. (a) Unperturbed beam distribution ($K=1$) in phase space. (b) Potential-well distorted beam distribution for example 4. (c) Potential-well distorted beam distribution for example 5. In (b) and (c), the cases of bunch shortening ($K>1$) and lengthening ($K<1$) are shown. (d) Bunch length L/L_0 as a function of wake strength NS/L_0^3 for the two examples. Solid curve is the solution to Eq. (32). Dashed curve is the solution to Eq. (33).

Example 5 In the previous example, the beam area in phase space is kept constant as beam intensity is varied, leading to a fourth order algebraic equation for the parabolic bunch length. The condition of constant phase space area corresponds to the case when the accelerator operator carefully matches the injected beam to the distorted potential well so that there is no increase in emittance. A somewhat different condition is to maintain a constant spread in p while letting the beam length vary, as the case would be for a Fokker-Planck system (see discussions in Section 3.2) such as an electron beam in a storage ring. On the other hand, this example gives only a qualitative illustration of bunch lengthening for an electron beam because a Fokker-Planck system dictates a solution of a gaussian form instead of the form discussed in this example. Equation (29) of the previous example now becomes

$$\psi(q,p) = C \left[\left(\frac{L_0}{2} \right)^2 - p^2 - Kq^2 \right]^{1/2} \quad \text{if } p^2 + Kq^2 < \left(\frac{L_0}{2} \right)^2$$

$$0 \quad \text{otherwise} \quad (33)$$

$$C = \frac{12 K^{1/2}}{\pi L_0^3}$$

The total spread in p is equal to L_0 , while the total beam length is $L_0 K^{-1/2}$. Following a similar procedure as the previous example, we obtain a cubic equation for L , i.e.,⁸

$$L^3 - L_0^2 L - 12 NS = 0. \quad (34)$$

Again, $L=L_0$ when $N=0$ as it should. The phase space distribution of this beam is illustrated in Fig. 3(c). Bunch length as function of wake field strength is shown in Fig. 3(d) for both examples 4 and 5. Note that there is no physical solution of Eq. (34) when $NS/L_0^3 < 0.032$.

2.3 Collective Instabilities

In the previous section, we have studied the stationary solutions of the Vlasov equation for a multi-particle system whose single particle motion is described by Eq.(22). This leads to the phenomenon of potential well distortion. It turns out that a wealth of beam dynamical properties

can be studied by the corresponding time-dependent Vlasov equation, which reads

$$\frac{\partial \psi}{\partial t} + \omega p \frac{\partial \psi}{\partial q} + \left[-\omega q + N \omega \int_q^\infty dq' W(q'-q) \int_{-\infty}^\infty dp' \psi(q', p', t) \right] \frac{\partial \psi}{\partial p} = 0 \quad (35)$$

Equation (35), or its variations, is the starting point of most literatures on collective beam instabilities.³ It is a partial-differential-integral equation, and even worse, it is nonlinear in ψ , the function we are solving for. General time-dependent solutions are very difficult to find and one relies on making drastic approximations and being content with limited success.

One drastic approximation that researchers often make is to assume the beam distribution ψ is always close to some stationary solution ψ_0 . Consequently, we have

$$\psi(q, p, t) = \psi_0(q, p) + \Delta\psi(q, p, t), \quad (36)$$

where

$$|\Delta\psi| \ll \psi_0.$$

Substituting Eq.(36) into (35) and keeping only to first order terms in $\Delta\psi$ give an equation that is linear in $\Delta\psi$. The problem is therefore shifted to the study of infinitesimal deviations from an assumed equilibrium. What we learn will be whether the assumed stationary distribution ψ_0 is stable under infinitesimal perturbations. This is a limited success because there are an infinite number of possible stationary distributions to start the problem with. Nevertheless, this is an important research area which is outside the scope of this lecture.

III. FOKKER-PLANCK EQUATION

3.1 Derivation

The Vlasov equation describes the multi-particle motion in a deterministic system. The system is no longer deterministic when a source of noise is introduced. In the presence of noise, the single particle equation of motion becomes

$$\begin{aligned} \frac{dq}{dt} &= f(q, p, t) \\ \frac{dp}{dt} &= g(q, p, t) + \sum_k \epsilon_k \delta(t-t_k) \end{aligned} \quad (37)$$

where ϵ_k is the noise represented as sudden changes in momentum of the particle at times given by

t_k . The noise is such that the sudden changes have an average value of zero. A nonzero average, if exists, can be transformed away by properly redefining the coordinates.

One commonly encountered case of a multi-particle system with noise is that of an electron beam in a storage ring. The noise comes mainly from emission of quantized photons as the electron circulates around the storage ring. In addition, the synchrotron radiation also gives rise to a radiation damping to the system.⁹ In contrast, the Vlasov equation is most relevant for a proton beam, whose synchrotron radiation is generally ignorable.

Consider again the box and its evolution in the (q,p) phase space shown in Fig.1. In that description, dt has been regarded as truly infinitesimal. In the presence of noise, this can no longer be done. The time increment, now designated as Δt , has to be large enough that there are at least several ϵ_k kicks occurring within it. If we designate the total kick accumulated in time Δt as E , and the probability density of having a particular accumulated kick be of the magnitude E as $P(E)$, we have

$$E = \sum_k \epsilon_k \text{ from } t \text{ to } t+\Delta t$$

with

$$\begin{aligned} \int P(E) dE &= 1 \\ \int E P(E) dE &= 0 \\ \int E^2 P(E) dE &= \langle \dot{N} \epsilon^2 \rangle \Delta t \end{aligned} \quad (38)$$

where $\langle \dot{N} \epsilon^2 \rangle$ is the rate of increase of the second moment with $\dot{N}$ interpreted as the average rate of occurring of the sudden changes.

The deterministic evolution shown in Fig. 1 holds only when $\epsilon=0$. In general, condition (5) has to be modified to become

$$\int dE P(E) \psi(q, p-E, t) \text{ area}(ABCD) = \psi(q+f\Delta t, p+g\Delta t, t+\Delta t) \text{ area}(A'B'C'D'). \quad (39)$$

The right hand side is the number of particles found in area $A'B'C'D'$ at time $t+\Delta t$. The same number has to be accounted for by the weighted average on the left hand side.

Equation (3) gives

$$\text{area}(A'B'C'D') = \text{area}(ABCD) \left[1 + \left(\frac{\partial f}{\partial q} + \frac{\partial g}{\partial p} \right) \Delta t \right]$$

The other quantities in Eq.(39) can be expanded to first order in Δt to give

$$\psi(q+f\Delta t, p+g\Delta t, t+\Delta t) = \psi + f\Delta t \frac{\partial \psi}{\partial q} + g\Delta t \frac{\partial \psi}{\partial p} + \Delta t \frac{\partial \psi}{\partial t}$$

and

$$\begin{aligned} \int dE P(E) \psi(q, p-E, t) &= \int dE P(E) \left[\psi - E \frac{\partial \psi}{\partial p} + \frac{E^2}{2} \left(\frac{\partial^2 \psi}{\partial p^2} \right) + \dots \right]_{q, p, t} \\ &= \psi(q, p, t) + \frac{1}{2} \langle \dot{N} \epsilon^2 \rangle \Delta t \frac{\partial^2 \psi}{\partial p^2} + \dots \end{aligned}$$

which, when substituted into Eq.(39), gives the Fokker-Planck equation:²

$$\frac{\partial \psi}{\partial t} + f \frac{\partial \psi}{\partial q} + g \frac{\partial \psi}{\partial p} = - \left(\frac{\partial f}{\partial q} + \frac{\partial g}{\partial p} \right) \psi + \frac{1}{2} \langle \dot{N} \epsilon^2 \rangle \frac{\partial^2 \psi}{\partial p^2} + \dots \quad (40)$$

Additional terms on the right hand side involve higher moments $\langle \dot{N} \epsilon^m \rangle$ of the noise. These terms often do not contribute much and are ignored (See Example 6 later). Compared with the Vlasov equation, the Fokker-Planck equation contains an additional damping term and a noise term (the first and second terms on the right hand side of Eq.(40), respectively). Note that the damping term is proportional to ψ while the diffusion term is proportional to the second derivative of ψ . The damping term has appeared in Eq. (15) when we discussed the damped simple harmonic example. Note also that it is the diffusion rate of the second moment of the noise that enters the equation. Different noises give the same average beam distribution, regardless of their detailed spectrum, as long as they have the same $\langle \dot{N} \epsilon^2 \rangle$.

For damped simple harmonic motion,

$$f = \omega p, \quad g = -\omega q - 2\alpha p, \quad (41)$$

the Fokker-Planck equation reduces to

$$\frac{\partial \psi}{\partial t} + \omega p \frac{\partial \psi}{\partial q} + (-\omega q - 2\alpha p) \frac{\partial \psi}{\partial p} = 2\alpha \psi + D \frac{\partial^2 \psi}{\partial p^2} \quad (42)$$

where we have defined a diffusion coefficient $D = \langle \dot{N} \epsilon^2 \rangle / 2$.

The first thing to know about Eq.(42) is that it has an equilibrium solution which is described by a bi-gaussian distribution,

$$\psi_0 = \frac{\alpha}{\pi D} \exp \left[-\frac{\alpha}{D} (q^2 + p^2) \right] \quad (43)$$

The rms expectation values of q and p are $\sigma_q = \sigma_p = (D/2\alpha)^{1/2}$. The equilibrium is reached as a result of balancing the damping and diffusion effects, and gaussian distribution is the only distribution that achieves such a balance at all points in the phase space. As we will see repeatedly later, in attempts to solve the Fokker-Planck equation under more complicated conditions, it is often convenient to start with an ansatz of a gaussian form, or at least with a gaussian factor. One exception to this rule is described in Example 14.

The beam distribution in an electron storage ring therefore, unlike that of the proton case, tends to be gaussian. The damping and diffusion coefficients in an electron storage ring depend on which dimension of motion is being considered. Table 1 below gives the expressions for α , D and the rms beam dimensions for the horizontal betatron x -motion, the vertical betatron y -motion and the longitudinal synchrotron z -motion in terms of other storage ring parameters.⁹ The table applies to an ideal planar storage ring for which there is no coupling among these motions, particularly that between x - and y -motions. The case with linearly coupled x - and y -motions will be considered later in Sec. 3.3.

TABLE I. The dynamical quantities q , p and the radiation damping and quantum diffusion coefficients for the three dimensions of motion in an electron storage ring.

	horizontal betatron (x)	vertical betatron (y)	longitudinal synchrotron (z)
ω	ω_x	ω_y	ω_s
q	$-\frac{c}{\omega_x} x'$	y	z
p	x	$\frac{c}{\omega_y} y'$	$-\frac{\alpha_c c}{\omega_s} \delta$
α	$\frac{U_0}{2E_0 T}$	$\frac{U_0}{2E_0 T}$	$\frac{U_0}{E_0 T}$
D	$\left(\frac{\rho}{E_0 v_x^2}\right)^2 D_u$	$\frac{(c/\omega_y E_0)^2 D_u}{2\gamma^2}$	$\left(\frac{\alpha_c c}{\omega_s E_0}\right)^2 D_u$
rms	$\sigma_x = \frac{c}{\omega_x} \sigma_{x'}$ $= \frac{1}{v_x^2} \sqrt{C_q \gamma^2 \rho}$	$\sigma_y = \frac{c}{\omega_y} \sigma_{y'}$ $= \frac{1}{v_y} \sqrt{\frac{C_q \rho}{2}}$	$\frac{\omega_s}{\alpha_c c} \sigma_z = \sigma_\delta$ $= \sqrt{\frac{C_q \gamma^2}{2\rho}}$

In Table 1, $U_0 = (4\pi/3)(r_0/\rho)mc^2\gamma^4$ is the synchrotron radiation loss per particle per turn, $D_u = (1/2)\langle N u^2 \rangle = (55/48\sqrt{3})r_0(\hbar)mc^4\gamma^7/\rho^3$ is the photon emission diffusion rate with u the photon energy, $C_q = (55/32\sqrt{3})(\hbar)/mc = 3.84 \times 10^{-13} \text{m}$ is a fundamental constant defined for convenience. Other quantities are: r_0 is the classical radius of the particle, m is the rest mass, $E_0 = mc^2\gamma$ is the energy of the particles, α_c is the momentum compaction factor, T is the revolution time, ρ is the bending radius of the storage ring, assumed to be uniform, $v_{x,y} = \omega_{x,y}\rho/c$ are the horizontal and vertical betatron tunes. We have used the average value of β -functions $\beta_{x,y} = \rho/v_{x,y}$ and the average value of the dispersion function $\eta = \rho/v_x^2$ in presenting the table. The partition numbers⁹ are taken to be 1, 1 and 2 for the x -, y - and z -motions. Note that for x -motion, the displacement x is identified with p , rather than q , because the synchrotron radiation noise occurs in x rather than in x' .

Example 6 To estimate the effects due to the higher order moments ignored in Eq.(42), we note that the next even order term on the right hand side of Eq.(40) is the fourth order term $(1/24) \langle \dot{N} \epsilon^4 \rangle (\partial^4 \psi / \partial p^4)$. A perturbation calculation shows that the relative change in the equilibrium distribution from Eq.(43) due to this extra term is of the order of

$$\frac{\Delta \psi}{\psi_0} \approx \frac{\langle \dot{N} \epsilon^4 \rangle}{12 \sigma_q^2 \langle \dot{N} \epsilon^2 \rangle} F \left(\frac{q^2 + p^2}{2 \sigma_q^2} \right) \quad (44)$$

where $F(x) \approx -3x$ for small x and $\ln(x)$ for large x . For synchrotron radiation noise, the fraction in front is of the order of $(u/E_0 \sigma_\delta)^2$, which in practice is about 10^{-6} with u a typical photon energy. We conclude that the correction term is not important except for extremely large amplitudes. It should be pointed out, however, the distribution at large amplitudes may be sensitive to the higher moments, especially when the noise has a long tail in ϵ . In particular, if the noise has a power law spectrum, the moments diverge beyond a certain order and the expansion used in our analysis breaks down. The beam distribution then will have a power law tail of the same order as the noise tail.

Example 7 An alternative way, without referring to the Fokker-Planck equation, of examining the contribution of the 4th moment is as follows. The sudden kicks from the noise send the q and p coordinates to execute damped simple harmonic oscillation according to

$$q(t) = \sum_{\substack{k \\ t > t_k}} \epsilon_k e^{-\alpha(t-t_k)} \sin \omega(t-t_k) \quad (45)$$

and $p(t)$ has a similar expression as $q(t)$ with sine replaced by a cosine. Equation (45) provides the link between the beam distribution moments and the noise spectrum. For example, it gives

$$\langle q^2 \rangle = \langle p^2 \rangle = \left\langle \frac{1}{2} \sum_{\substack{k \\ t > t_k}} \epsilon_k^2 e^{-2\alpha(t-t_k)} \right\rangle \quad (46)$$

assuming no correlation among different noise kicks. A moment's reflection shows that the summation over k is to be replaced by

$$\sum_k \rightarrow \dot{N} \int_0^\infty d(t-t_k)$$

Equation (46) then gives the expected result

$$\langle q^2 \rangle = \langle p^2 \rangle = \frac{\langle \dot{N} \epsilon^2 \rangle}{4\alpha} = \frac{D}{2\alpha}$$

One can proceed further to compute the 4th beam distribution moment in a similar way and find

$$\langle p^4 \rangle = \langle q^4 \rangle = 3 \left(\frac{D}{2\alpha} \right)^2 + \frac{3}{32} \frac{\langle \dot{N} \epsilon^4 \rangle}{\alpha} \quad (47)$$

The first term on the right hand side is what is expected if the beam distribution is strictly gaussian. the second term gives the modification due to the 4th moment of the noise spectrum. the procedure can be continued to higher moments.

Example 8 One effect ignored so far for the multi-particle system is the spin of the particles. To make sure the system is described by classical statistics rather than Fermi statistics, we need to show that the number of quantum states far exceeds the number of particles for all practical cases.

The beam occupies a volume in the 6-dimensional phase space of the order of

$$V = \sigma_x \sigma_{x'} \sigma_y \sigma_{y'} \sigma_\delta \sigma_z P_o^3 \quad (48)$$

where P_o is the beam momentum in the laboratory frame. The number of quantum states available for the protons is

$$N_q = \frac{2V}{h^3} \quad (49)$$

where a factor of 2 is for the two spin states.

The particles under consideration for the Fokker-Planck equation are most likely electrons, but in this example we will consider the case of a 20 TeV proton beam for the Superconducting Super Collider¹⁰, with parameters $\gamma = 2 \times 10^4$, $\rho = 10$ km, $\alpha_c = 0.0002$, $v_x = v_y = 80$, $\omega_s = 40$ s⁻¹.

Assuming the proton beam has been stored sufficiently long in the storage ring so that the stationary distribution has been reached, we obtain from Table 1 that $\sigma_\delta = 2 \times 10^{-6}$, $\sigma_x = 5$ μ m, $\sigma_y = 0.01$ μ m, $\sigma_z = 3$ mm, $\sigma_{x'} = 0.04$ μ rad, $\sigma_{y'} = 0.0001$ μ rad. This in turn gives $N_q = 2 \times 10^{21}$.

Since the number of particles considered is about 10^{11} , the number of quantum states exceeds the number of protons per bunch by about 10 orders of magnitude; there is no problem ignoring the Fermi statistics.

3.2 Potential-Well Distortion

It turns out one can do better than solving just the case of damped simple harmonic motion. A more general case occurs when the single particles move in a potential well which in general is nonlinear, i.e.

$$f(q,p,t) = \omega p, \quad g(q,p,t) = -K(q) - 2\alpha p. \quad (50)$$

The corresponding Fokker-Planck equation is

$$\frac{\partial \psi}{\partial t} + \omega p \frac{\partial \psi}{\partial q} + [-K(q) - 2\alpha p] \frac{\partial \psi}{\partial p} = 2\alpha \psi + D \frac{\partial^2 \psi}{\partial p^2} \quad (51)$$

It is remarkable that the stationary solution to this equation can be found for arbitrary $K(q)$. The solution, as can be shown by direct substitution, is⁷

$$\psi_0 = Q \exp \left\{ -\frac{\alpha}{D} \left[p^2 + \frac{2}{\omega} \int_0^q dq' K(q') \right] \right\} \quad (52)$$

where Q is determined by the normalization condition (4). Equation (52) reduces to (43) when $K(q) = \omega q$, as it should.

Another remarkable fact is that the argument in the exponent is basically the Hamiltonian of the system in the absence of damping and noise, with the integral corresponding to the potential well term. The stationary distribution of a Fokker-Planck system in a nonlinear potential well is therefore given by the exponential of the unperturbed Hamiltonian of the system.

One practical application of Eq.(52) is to give the stationary longitudinal distribution of an electron beam in a nonlinear rf bucket. The potential well provided by the sinusoidal rf voltage gives rise to the stationary distribution

$$\psi_0 = Q \exp \left(\frac{-2\alpha H}{D\omega} \right) \quad (53)$$

where H is given by Eq.(21).

Another practical case of potential well distortion occurs as a result of wake fields. The results obtained for the Vlasov equation still apply here for the Fokker-Planck equation except that the gaussian form (24) is no longer arbitrary but is compulsory, and that the rms value σ_p is no longer arbitrary but has to be equal to $(D/2\alpha)^{1/2}$.

3.3 Linear Coupling

The description so far is for a one-dimensional system. Fokker-Planck equation can be extended to 2- or 3-dimensional systems following similar analyses. This is especially useful if two or more of the dimensions are coupled. The expectation is that, if the coupling is linear, the stationary solution to the coupled Fokker-Planck equation remains gaussian in the multi-dimensional phase space.

In an electron storage ring, linear coupling can result from a skew quadrupole element, a solenoid or an rf cavity located at a location with finite dispersion. In the following, we will assume the linearly coupled equations of motion are known and can be written as

$$\frac{dX}{dt} = CX + \sum_k E_k \delta(t-t_k) \quad (54)$$

For example, in the case of coupled x- and y-motions, X is the column matrix with elements (x, p_x, y, p_y) ; C is the coupling matrix; E_k is the vector describing the response in the four phase space coordinates to a single noise source ϵ_k which occurs at times t_k ,

$$E_k = \epsilon_k \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} \quad (55)$$

The quantities η_1 to η_4 specify the sensitivity of x, p_x, y, p_y to the single noise perturbation ϵ_k .

Near the coupling resonance with $\omega_x \approx \omega_y$, C is approximately parametrized by¹¹

$$C = \begin{bmatrix} 0 & \omega_x & -k_2 & 0 \\ -\omega_x & -2\alpha_x & -2k_1 & -k_2 \\ k_2 & 0 & 0 & \omega_y \\ -2k_1 & k_2 & -\omega_y & -2\alpha_y \end{bmatrix} \quad (56)$$

where $k_{1,2}$ are coupling constants related to the skew quadrupole or solenoid strengths. Parameters $\omega_{x,y}$ and $\alpha_{x,y}$ in C specify the damped simple harmonic behavior of the system.

The diffusion effects are described by another symmetric matrix D whose elements are

$$D_{ij} = \frac{1}{2} \langle \dot{N} \epsilon^2 \rangle \eta_i \eta_j \quad (57)$$

In an ideal planar electron storage ring, the only nonvanishing elements of the D matrix are D_{22} and D_{44} , whose values are given in Table 1. In general, however, a photon emission event simultaneously perturbs all four coordinates in a correlated manner. Such correlations can be included by off-diagonal elements of the D matrix. In the remaining of this section, the general case is considered.

The Fokker-Planck equation for this case can be derived similarly to that leading to Eq.(42).

The resulting expression is

$$\frac{\partial \psi}{\partial t} + \sum_i \frac{dx_i}{dt} \frac{\partial \psi}{\partial x_i} = 2(\alpha_x + \alpha_y) \psi + \sum_i \sum_j D_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j}$$

or equivalently

$$\frac{\partial \psi}{\partial t} + \sum_i \sum_j C_{ij} x_j \frac{\partial \psi}{\partial x_i} = - \sum_i C_{ii} \psi + \sum_i \sum_j D_{ij} \frac{\partial^2 \psi}{\partial x_i \partial x_j} \quad (58)$$

Knowing C and D matrices, we now will look for the stationary solution to Eq.(58). By declaring the solution shall be gaussian, we have the ansatz

$$\psi = Q \exp(-X^t A X) \quad (59)$$

where X^t is the transpose of X, A is a symmetric matrix yet to be solved and Q is a normalization constant not important in the following discussions. Substituting Eq.(59) into (58) and equating coefficients for terms containing the same $x_i x_j$, we obtain

$$(AC)^t + AC = -4 ADA \quad (60)$$

and

$$\text{trace}(C) = -2 \text{trace}(DA) \quad (61)$$

There are $m(2m+1)$ conditions and the same number of unknowns in Eq.(60) where m is the degree of freedom considered ($m=2$ for the case of coupled x-y motion). The trace equation (61) is redundant since it follows simply by rewriting Eq.(60) as $A^{-1}C^t A + C = -4DA$ and then taking trace of this equation.

Before proceeding, we first note that the matrix A is not as interesting as A^{-1} . This is because the second moments of the gaussian distribution, defined by the symmetric matrix with elements

$\Sigma_{ij} = \langle x_i x_j \rangle$, is related to A^{-1} by

$$\Sigma = \frac{1}{2} A^{-1} \quad (62)$$

For example, the elementary special case of $\exp(-x^2/2\sigma^2)$ has a 1x1 matrix $A=1/2\sigma^2$. Applying Eq.(62) gives $\langle x^2 \rangle = \sigma^2$ as it should. The general case is left for the reader to prove.

The second moments uniquely determine the rms sizes and orientation of the gaussian distribution in the phase space. The tilt angle, for example, of the distribution projected onto the x_i - x_j plane is given by

$$\tan 2\theta_{ij} = \frac{2 \langle x_i x_j \rangle}{\langle x_i^2 \rangle - \langle x_j^2 \rangle} \quad (63)$$

Using expression (62), Eq.(60) can be rewritten as

$$\Sigma C^t + C \Sigma = -2D. \quad (64)$$

Although in principle it is straightforward to solve Eq.(64) algebraically for Σ_{ij} , it is useful and perhaps instructive to have a formal solution of it. To do so, we diagonalize the matrix C by

$$C = V \lambda V^{-1} \quad \text{or} \quad C^t = (V^t)^{-1} \lambda V^t \quad (65)$$

where λ is the diagonal matrix whose elements are the eigenvalues of the matrix C and V is composed of the corresponding eigenvectors, i.e. the i -th column of V is given by the eigenvector corresponding to eigenvalue λ_i . The real part of all eigenvalues $\text{Re}(\lambda_i)$ must be negative for single particle motion to be damped.

Having established λ and V , the solution to Eq.(64) can be written as

$$\Sigma = V B V^t \quad (66)$$

where the matrix B is the second moment matrix viewed in the eigenvector space with elements

$$B_{ij} = \frac{-2}{\lambda_i + \lambda_j} [V^{-1} D (V^t)^{-1}]_{ij} \quad (67)$$

Example 9 As a first application of this formalism, consider the 1-dimensional case treated before.

We have

$$C = \begin{bmatrix} 0 & \omega \\ -\omega & -2\alpha \end{bmatrix} \quad (68)$$

and

$$D = \begin{bmatrix} 0 & 0 \\ 0 & D_x \end{bmatrix} \quad (69)$$

We find then

$$\lambda_{\pm} = -\alpha \pm i(\omega^2 - \alpha^2)^{1/2} \quad (70)$$

$$V = \begin{bmatrix} 1 & 1 \\ \frac{\lambda_+}{\omega} & \frac{\lambda_-}{\omega} \end{bmatrix} \quad (71)$$

$$B = \frac{D_x}{4} \frac{\omega^2}{\omega^2 - \lambda^2} \begin{bmatrix} \frac{1}{\lambda_+} & \frac{1}{\alpha} \\ \frac{1}{\alpha} & \frac{1}{\lambda_-} \end{bmatrix} \quad (72)$$

and the expected result

$$\Sigma = \frac{D_x}{2\alpha} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (73)$$

The same result can of course be obtained by algebraically solving the three simultaneous equations imbedded in Eq.(64).

Example 10 A nontrivial application is for the 2-dimensional case of x-y coupled motion when the horizontal and vertical betatron frequencies are close to each other. The matrix C is given in Eq.(56). If we assume that the noise affects only p_x and p_y and not the other coordinates, matrix D is

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & D_x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_y \end{bmatrix} \quad (74)$$

Straightforwardly solving the 10 simultaneous equations imbedded in Eq.(64) or using the formal approach in Eqs.(65) to (67) gives¹¹

$$\begin{aligned}
\langle x^2 \rangle &= \frac{\langle x^2 \rangle_0}{1+\kappa_y} + \frac{\alpha_y}{\alpha_x} \frac{\kappa_x \langle y^2 \rangle_0}{1+\kappa_x} \\
\langle y^2 \rangle &= \frac{\alpha_x}{\alpha_y} \frac{\kappa_y \langle x^2 \rangle_0}{1+\kappa_y} + \frac{\langle y^2 \rangle_0}{1+\kappa_x} \\
\langle xy \rangle &= \kappa' (\langle x^2 \rangle_0 - \langle y^2 \rangle_0)
\end{aligned} \tag{75}$$

where $\langle x^2 \rangle_0 = D_x/2\alpha_x$ and $\langle y^2 \rangle_0 = D_y/2\alpha_y$ give the horizontal and vertical beam sizes in the absence of coupling; $\kappa_{x,y}$ and κ' are coupling constants given by

$$\begin{aligned}
\kappa_y &= \frac{\frac{\alpha_y}{\alpha_x} (\alpha_x + \alpha_y) k^2}{(\alpha_x + \alpha_y) k^2 + \alpha_y \Delta\omega^2} \\
\kappa_x &= \frac{\frac{\alpha_x}{\alpha_y} (\alpha_x + \alpha_y) k^2}{(\alpha_x + \alpha_y) k^2 + \alpha_x \Delta\omega^2} \\
\kappa' &= \frac{\alpha_x \alpha_y \Delta\omega k_1}{(\alpha_x + \alpha_y)^2 k^2 + \alpha_x \alpha_y \Delta\omega^2}
\end{aligned} \tag{76}$$

where $k^2 = k_1^2 + k_2^2$ and $\Delta\omega = \omega_x - \omega_y$ is the small difference between the horizontal and vertical betatron frequencies. Expressions for the remaining seven second moments are omitted. Coupling constant κ_y specifies the amount of beam emittance that is transferred from x- to y-dimension. Similarly, κ_x specifies the emittance transfer from y- to x-dimension. The constant κ' determines $\langle xy \rangle$ which is associated with the tilt angle of the beam distribution in the x-y plane. From Eqs.(75) and (76) one observes that, if the uncoupled beam has equal x- and y-emittances, the beam remains round and it does not tilt in the x-y plane due to linear coupling.

Equation (75) can be combined to give an interesting invariant condition,

$$\alpha_x \langle x^2 \rangle + \alpha_y \langle y^2 \rangle = \alpha_x \langle x^2 \rangle_0 + \alpha_y \langle y^2 \rangle_0 \tag{77}$$

The quantity on the left hand side of Eq.(77) is thus an invariant, independent of the linear coupling that exists in the storage ring.

Equations (75) and (76) allow the calculation of beam moments once the linearly coupled equations of motion are known. One special case of particular interest is when $\alpha_x = \alpha_y$ and $\langle y^2 \rangle_0 = 0$, which are often approximately valid in practice. We then have⁹

$$\begin{aligned}\langle x^2 \rangle &= \frac{\langle x^2 \rangle_0}{1 + \kappa_y} \\ \langle y^2 \rangle &= \frac{\kappa_y \langle x^2 \rangle_0}{1 + \kappa_y} \\ \langle xy \rangle &= \kappa' \langle x^2 \rangle_0\end{aligned}\tag{78}$$

with the coupling parameters

$$\begin{aligned}\kappa_y &= \frac{2k^2}{2k^2 + \Delta\omega^2} \\ \kappa' &= \frac{\Delta\omega k_1}{4k^2 + \Delta\omega^2}\end{aligned}\tag{79}$$

Equation (78) is a familiar result where the coupling constants are often regarded as phenomenological parameters. Application of the Fokker-Planck technique provides a way of obtaining explicit expressions of these coupling constants. The invariant condition (77) becomes

$$\langle x^2 \rangle + \langle y^2 \rangle = \langle x^2 \rangle_0\tag{80}$$

and the tilt angle in the x-y plane, using Eq.(63), satisfies

$$\tan 2\theta = \frac{2k_1}{\Delta\omega}\tag{81}$$

The beam tilts by 45° in the strong coupling limit $k_1 \gg \Delta\omega$.

3.4 Transient Distributions

Another application of the Fokker-Planck equation is to find the transient beam behavior. Consider an electron beam injected into a storage ring with a distribution ψ_i that is different from the equilibrium gaussian distribution ψ_0 . As time goes on, it evolves towards the equilibrium form ψ_0 . This transient evolution of beam distribution is discussed in this section.

Consider the general Fokker-Planck equation, Eq.(58), for a linearly coupled system. We have shown that the equilibrium beam distribution of this system is a specific gaussian form given

by Eq.(59). The matrix A -- or rather, the matrix Σ -- is solved by Eq.(64), or Eqs.(66) and (67).

A similar technique will be used to describe the transient effect.

At $t=0$, we assume that the initial distribution is a displaced gaussian,

$$\psi_i = Q(0) \exp\{ - [X-X_0(0)]^t A(0) [X-X_0(0)] \} \quad (82)$$

where $Q(0)$ is the normalization constant at time $t=0$ -- we are anticipating time dependence on the normalization -- $X_0(0)$ is the column vector specifying the beam center at $t=0$ and $A(0)$ is the symmetric matrix that specifies the initial beam distribution moments. We assume that $Q(0)$, $A(0)$ and $X_0(0)$, as well as the matrices C and D in Eq.(58), are known.

It turns out that if the initial beam distribution is gaussian, the beam will remain gaussian throughout the transient period. The distribution at time t can be written as

$$\psi = Q(t) \exp\{ - [X-X_0(t)]^t A(t) [X-X_0(t)] \} \quad (83)$$

Substituting Eq.(83) into Eq.(58) and equating the coefficients for terms containing the same factors, we obtain¹¹

$$-\frac{dA}{dt} = (AC)^t + AC + 4 ADA \quad (84)$$

$$\frac{dX_0}{dt} = CX_0 \quad (85)$$

$$\frac{1}{Q} \frac{dQ}{dt} = \text{trace}(C) + 2 \text{trace}(DA) \quad (86)$$

Equations (84) and (86) can be compared with the corresponding equations (60) and (61) for the equilibrium distribution.

It is interesting to note that Eq. (85), which describes the motion of the beam distribution center, acts as if the beam is regarded as a single particle ignoring diffusion but take into account of the damping effect. The beam center thus executes a damped, linearly coupled motion, starting with the initial value $X_0(0)$.

Like Eq.(60), there are $m(2m+1)$ conditions and unknowns in Eq. (84). Equation (85) has $2m$ conditions and $2m$ unknowns for X_0 . The two equations (84) and (85) for A and X_0 are decoupled from each other. Equation (86) can be shown to be redundant; the proof is omitted here.

Equation (84) can be rewritten in terms of matrix Σ using expression (62), yielding

$$\frac{d\Sigma}{dt} = \Sigma C^t + C\Sigma + 2D \quad (87)$$

We then use the diagonal form (65) for matrix C to obtain

$$\frac{dB}{dt} = B\lambda + \lambda B + 2 V^{-1} D (V^t)^{-1} \quad (88)$$

where B is related to Σ by Eq.(66). Equation (88) is to be solved with the initial condition

$$B(0) = V^{-1} \Sigma(0) (V^t)^{-1} \quad (89)$$

The solution is

$$B_{ij}(t) = B_{ij}(\infty) + [B_{ij}(0) - B_{ij}(\infty)] \exp[(\lambda_i + \lambda_j)t] \quad (90)$$

with the asymptotic values $B_{ij}(\infty)$ given in Eq.(67). At time $t=0$, Σ is equal to the initial value $\Sigma(0)$. As $t \rightarrow \infty$, the exponential term of (90) diminishes and the beam distribution approaches that of the equilibrium.

The solution of Eq.(85) for the beam center can be written as

$$X_0(t) = V U(t) \quad (91)$$

with

$$u_i(t) = u_i(0) \exp(\lambda_i t)$$

$$U(0) = V^{-1} X_0(0).$$

The initial displacement is $X_0(0)$ and the asymptotic displacement vanishes, as expected.

Example 11 The transient problem is now formally solved. As an example of its application, consider the 1-dimensional case discussed in Example 9. Let the injected beam have

$$\psi_i(q,p) = \delta(q-q_i) \delta(p-p_i) \quad (92)$$

i.e., the beam with vanishing size is injected off-center by (q_i, p_i) . The transient behavior found below for this initial distribution is in fact the Green's function of the system. Knowing the Green's function, the transient behavior of arbitrary initial distributions can be obtained by superposition.

Distribution (92) can be described as a gaussian with

$$X_0 = \begin{bmatrix} q_i \\ p_i \end{bmatrix}$$

$$\Sigma(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad B(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

With C, D, V, λ and $B(\infty)$ given by Eqs.(68) to (72) and following the prescription discussed above, the time dependent solution is found to be

$$X_0(t) = \begin{bmatrix} q_0(t) \\ p_0(t) \end{bmatrix} = e^{-\alpha t} \begin{bmatrix} (\alpha q_i + \omega p_i) \frac{\sin \Omega t}{\Omega} + q_i \cos \Omega t \\ -(\alpha p_i + \omega q_i) \frac{\sin \Omega t}{\Omega} + p_i \cos \Omega t \end{bmatrix} \quad (93)$$

and

$$B(t) = \frac{\omega^2 D}{4\Omega^2} \begin{bmatrix} \frac{1 - \exp(2\lambda_+ t)}{\lambda_+} & \frac{1 - \exp(\lambda_+ t + \lambda_- t)}{\alpha} \\ \frac{1 - \exp(\lambda_+ t + \lambda_- t)}{\alpha} & \frac{1 - \exp(2\lambda_- t)}{\lambda_-} \end{bmatrix} \quad (94)$$

where $\Omega = (\omega^2 - \alpha^2)^{1/2}$. The distribution moments are given by $\Sigma(t) = VB(t)V^t$, or explicitly,

$$\begin{aligned} \langle (q(t) - q_0(t))^2 \rangle &= \Sigma_{11}(t) \\ &= \frac{\omega^2 D}{2\alpha\Omega^2} (1 - e^{-2\alpha t}) - \frac{\alpha D}{2\Omega^2} (1 - e^{-2\alpha t} \cos 2\Omega t) - \frac{D}{2\Omega} e^{-2\alpha t} \sin 2\Omega t \\ \langle (q(t) - q_0(t)) (p(t) - p_0(t)) \rangle &= \Sigma_{12}(t) = \Sigma_{21}(t) \\ &= \frac{\omega D}{2\Omega^2} (1 - \cos 2\Omega t) e^{-2\alpha t} \\ \langle (p(t) - p_0(t))^2 \rangle &= \Sigma_{22}(t) \\ &= \frac{\omega^2 D}{2\alpha\Omega^2} (1 - e^{-2\alpha t}) - \frac{\alpha D}{2\Omega^2} (1 - e^{-2\alpha t} \cos 2\Omega t) + \frac{D}{2\Omega} e^{-2\alpha t} \sin 2\Omega t \end{aligned} \quad (95)$$

As a check, note that $\Sigma_{11}(0) = \Sigma_{12}(0) = \Sigma_{21}(0) = \Sigma_{22}(0) = 0$ and $\Sigma_{11}(\infty) = \Sigma_{22}(\infty) = D/2\alpha$ and $\Sigma_{12}(\infty) = \Sigma_{21}(\infty) = 0$.

For the case of weak damping, as is the case for most applications, we may keep only terms proportional to $1/\alpha$. Equations (93) and (95) then acquire simpler forms,

$$X_0(t) \approx e^{-\alpha t} \begin{bmatrix} p_i \sin \omega t + q_i \cos \omega t \\ -q_i \sin \omega t + p_i \cos \omega t \end{bmatrix} \quad (96)$$

and

$$\Sigma(t) \approx \frac{D}{2\alpha} \begin{bmatrix} 1-e^{-2\alpha t} & 0 \\ 0 & 1-e^{-2\alpha t} \end{bmatrix} \quad (97)$$

The beam moments thus simply approach exponentially from the initial values to the equilibrium values and the beam basically remains upright during the transient. Figure 4 illustrates the transient behavior of the injected beam. The beam has vanishing size and its center is displaced at time $t=0$. The beam size diffuses towards its equilibrium value with rate 2α as the beam center damps out with rate α . In the more general case when the injected beam has finite size, the right hand side of Eq.(97) acquires additional terms $\Sigma_{ij}(0)e^{-2\alpha t}$ which indicate the initial knowledge of beam size is exponentially damped out.

FIG. 4. Transient behavior of a delta-function electron beam injected off-centered into a storage ring. The shaded area represents the rms beam size.

3.5 Quantum Lifetimes

We have shown the Fokker-Planck equation (42) has a bi-gaussian equilibrium solution (43). However, this solution assumes that there is no physical limit on the value of q and p , which in reality means ignoring objects such as the vacuum chamber wall in the storage ring. Although the distribution is stationary, actual particles are not; they rapidly circulate around the phase angle ϕ and slowly change their amplitude due to diffusion and damping. In the presence of a vacuum

chamber wall, the change in a makes it possible for particles to hit the wall and consequently be removed from the beam. The beam distribution will no longer be stationary; instead, it will decay with time. The corresponding beam lifetime is called the quantum lifetime.⁹

In terms of the Fokker-Planck equation, the new ingredient is that we have to impose the extra condition that the distribution vanishes at the wall boundary. In the following, we will calculate the quantum lifetimes for the three dimensions of motion. The simplest case is that for the vertical y -motion; it will be discussed first. The more complicated cases of longitudinal and horizontal quantum lifetimes will be discussed subsequently. No x - y coupling is included in these discussions.

Vertical. The rapid rotation in phase angle in particle motion provides the simplification of the problem. Figure 5(a) compares the single particle motion with and without the presence of a wall boundary. Physically the wall located at coordinate $q=A$ restricts particles to the region $q < A$ without any restriction on the momentum p . However, the rapid phase motion means that the effective wall is specified by $a = (p^2 + q^2)^{1/2} < A$. Figures 5(b) and (c) illustrate the effect of the wall on beam distribution for various modes (see Example 12 and discussions later).

FIG. 5 (a) Particle motion in phase space with and without a wall boundary. A wall imposes an effective boundary condition that no particles exist in the region $a > A$ although it physically only removes particles with $q > A$. (b) Normal modes of beam distribution in the absence of wall; n is the mode index. Damping times of the various modes are indicated. (c) Same as (b) with wall. In case the wall is sufficiently far away from the beam, the damping times are only slightly perturbed from their values indicated in (b). However, the slight perturbation in damping time for the $n=0$ mode is of particular importance; it is identified as the quantum lifetime of the system and is the quantity to be calculated in this section.

We first transform the problem from the Cartesian coordinates (q,p) to the polar coordinates (a,ϕ) according to Eq.(11). Expecting a quantum lifetime much longer than the rotation period, the beam distribution is accurately given by a function of a and t only, i.e. $\psi=\psi(a,t)$, independent of phase angle ϕ . Substituting (11) into the Fokker-Planck equation (42) and averaging the resulting equation over ϕ give

$$\frac{\partial \psi}{\partial t} = 2\alpha\psi + \alpha a \frac{\partial \psi}{\partial a} + \frac{D}{2a} \frac{\partial}{\partial a} \left[a \frac{\partial \psi}{\partial a} \right] \quad (98)$$

The average over phase angle is valid only if we are interested in slow effects such as the effect of quantum lifetime. As will be seen later when we discuss the normal modes in Sec. 3.6, this simplification does not apply in general.

The boundary condition imposed by the wall is

$$\psi(a=A, t) = 0 \quad (99)$$

for all time t . Equation (98) is to be solved together with Eq.(99).

To find the quantum lifetime, we need to find the normal modes of the system. In this context, a normal mode is a particular pattern in phase space, each characterized by a particular value of lifetime. A beam injected into the storage ring at time $t=0$ can be decomposed into a superposition of these normal modes. As time proceeds after injection, only the component in the mode with the longest lifetime remains. Quantum lifetime is thus identified to be the lifetime for the longest living mode, i.e. the $n=0$ mode sketched in Fig.5(c).

Consider one normal mode with mode number n and lifetime τ_n . Let the normal mode be written as

$$\psi^{(n)}(a,t) = \psi_n(a) \exp\left(\frac{-t}{\tau_n}\right) \quad (100)$$

i.e. the mode has a fixed overall shape in phase space given by $\psi_n(a)$, but its magnitude diminishes with a lifetime τ_n . Substituting Eq.(100) into (98) gives

$$\psi_n'' + \left(\frac{1}{a} + \frac{2\alpha a}{D} \right) \psi_n' + \frac{2}{D} \left(2\alpha + \frac{1}{\tau_n} \right) \psi_n = 0 \quad (101)$$

To solve Eq.(101) together with the boundary condition $\psi_n(A)=0$, we first let

$$\psi_n(a) = \exp\left(\frac{-a^2}{2\sigma^2}\right) g_n(a) \quad (102)$$

where $\sigma = (D/2\alpha)^{1/2}$ is the natural rms beam size in the absence of wall. Substituting Eq.(102) into (101) leads to

$$g_n'' + \left(\frac{1}{a} - \frac{a}{\sigma^2}\right) g_n' + \frac{1}{\tau_n \alpha \sigma^2} g_n = 0 \quad (103)$$

We further let

$$g_n(a) = 1 + \sum_{k=1}^{\infty} c_k \left(\frac{a^2}{2\sigma^2}\right)^k \quad (104)$$

The coefficients c_k 's can be obtained by substituting (104) into (103) and identifying terms with equal powers in a . This leads to the solution

$$c_k = \frac{1}{(k!)^2} \left[(k-1) - \frac{1}{2\alpha\tau_n} \right] \left[(k-2) - \frac{1}{2\alpha\tau_n} \right] \dots \left[\frac{-1}{2\alpha\tau_n} \right] \quad (105)$$

The eigenvalue τ_n is determined by the $(n+1)$ -th solution for $g_n(A)=0$. The lowest mode has $n=0$. The problem is therefore formally solved.

Example 12 In the absence of wall, Eq.(104) gives a spectrum of normal modes that have terminated power series. The first few are given below:

$$\begin{aligned} n=0, \tau_0 &= \infty, \quad g_0(a) = 1 \\ n=2, \tau_2 &= \frac{1}{2\alpha}, \quad g_2(a) = 1 - \frac{a^2}{2\sigma^2} \\ n=4, \tau_4 &= \frac{1}{4\alpha}, \quad g_4(a) = 1 - \frac{a^2}{\sigma^2} + \frac{1}{2} \left(\frac{a^2}{2\sigma^2}\right)^2 \\ n=6, \tau_6 &= \frac{1}{6\alpha}, \quad g_6(a) = 1 - \frac{3a^2}{2\sigma^2} + \frac{3}{2} \left(\frac{a^2}{2\sigma^2}\right)^2 - \frac{1}{6} \left(\frac{a^2}{2\sigma^2}\right)^3 \end{aligned} \quad (106)$$

In general, $\tau_{2n}=1/2n\alpha$ and $g_{2n}(a)$ is an $2n$ -th power polynomial of a , which satisfies the normalization condition $\int_0^\infty da \psi_{2n}(a) = 0$ except for $n=0$ where ψ_{2n} is related to g_{2n} by Eq. (102).

The lifetimes for $n \neq 0$ modes are due to the damping motion and not due to any wall boundary. The quantum lifetime, i.e., the lifetime for the $n=0$ mode, is $\tau_0 = \infty$. A few of these modes are illustrated in Fig. 5(b). Note that the mode sequence (106) is an even sequence. There exists another sequence of odd modes which can not be discussed by the phase-averaged equation (98). Some of this discussion can be found in Sec. 3.6.

In most practical cases, the wall is far from the beam, meaning $A \gg \sigma$. Under this condition, the quantum lifetime is much longer than the damping time $1/\alpha$, i.e. $\alpha\tau_0 \gg 1$. Equation (105) becomes

$$c_k \approx \frac{(k-1)!}{(k!)^2} \left[\frac{-1}{2\alpha\tau_0} \right] \quad (107)$$

which gives

$$g_0(a) \approx 1 - \frac{1}{2\alpha\tau_0} h\left(\frac{a^2}{2\sigma^2}\right) \quad (108)$$

where we have defined a function

$$h(x) = \sum_{k=1}^{\infty} \frac{x^k}{k(k!)} \quad (109)$$

The function $h(x)$ can also be written as

$$h(x) = \int_0^x dy \frac{e^y - 1}{y} \quad (110)$$

For $x \gg 1$, it can be shown that $h(x) \approx e^x/x$.

The boundary condition $g_0(A)=0$ gives, using Eq.(108), the familiar expression of quantum lifetime⁹

$$\begin{aligned} \tau_0 &\approx \frac{1}{2\alpha} h\left(\frac{A^2}{2\sigma^2}\right) \\ &\approx \frac{\sigma^2}{\alpha A^2} \exp\left(\frac{A^2}{2\sigma^2}\right) \end{aligned} \quad (111)$$

The validity criterion that $\alpha\tau_0 \gg 1$ is assured if $A \gg \sigma$. For example, if $A=6\sigma$, the quantum lifetime is 1.8×10^6 times the damping time $1/\alpha$.

From Eqs.(100), (102), (108) and (111), we find the normal mode distribution for $n=0$,¹²

$$\psi^{(0)}(a,t) \approx \exp\left(-\frac{t}{\tau_0} - \frac{a^2}{2\sigma^2}\right) \left[1 - \frac{h(a^2/2\sigma^2)}{h(A^2/2\sigma^2)} \right] \quad (112)$$

The quantity in the square bracket of Eq.(112), for several values of A/σ , is shown in Fig.6. Note that the distribution is accurately given by the gaussian form until a comes very close to the wall A .

XBL 872-8928

FIG. 6. Distribution of the $n=0$ mode in the presence of a wall boundary for several values of A/σ .

Example 13 Equation (111) is valid when $A \gg \sigma$ and $\tau_0 \gg 1/\alpha$. This is not too far wrong when A approaches σ but stays above it. On the other hand, it predicts that the quantum lifetime reaches a minimum at $A = \sqrt{2}\sigma$ with $\tau_0 = e/2\alpha$, which cannot be correct because smaller A is bound to give an even shorter lifetime.

For the other extreme with $A \ll \sigma$, an approximate expression of quantum lifetime can be obtained by noting $\tau_0 \ll 1/\alpha$, which implies

$$c_k \approx \frac{1}{(k!)^2} \left(\frac{-1}{2\alpha\tau_0} \right)^k \quad (113)$$

One notices that these coefficients are those appearing in the Taylor expansion of the Bessel

function J_0 . Indeed, substituting (113) into (104) gives

$$g_0(a) \approx J_0\left(\frac{a}{\sigma\sqrt{\alpha\tau_0}}\right) \quad (114)$$

The boundary condition $g_0(A)=0$ then gives

$$\tau_0 = \frac{1}{\alpha} \left(\frac{A}{\sigma\lambda_1}\right)^2 \quad (115)$$

where $\lambda_1=2.405$ is the first root of the $J_0(x)$. The distribution (114) can also be written as

$$g_0(a) \approx J_0(\lambda_1 a/A).$$

One can also easily obtain exact values of quantum lifetime using Eq.(105) for a few special cases. For example, $\tau_0=1/2\alpha$ gives $c_1=-1$ and $c_k=0$ for $k \geq 2$. This gives a terminated polynomial $g_0(a)=1-(a^2/2\sigma^2)$. The wall condition $g_0(A)=0$ then gives $A=\sqrt{2}\sigma$. Similarly, $\tau_0=1/4\alpha$ means $g_0(a)=1-(a^2/\sigma^2)+(a^4/8\sigma^4)$ which has a boundary at $A=(4-2\sqrt{2})^{1/2}\sigma=1.082\sigma$. Another special case is for $\tau_0=1/6\alpha$. The corresponding boundary occurs at $A=[6-4\sqrt{3}\cos(4\pi/3-\phi/3)]^{1/2}\sigma=0.912\sigma$ where $\phi=\tan^{-1}\sqrt{2}$. These special cases, in fact, are just those given in Eq. (106) when the wall is located at the first node of the eigen-distribution. Figure 7 is a comparison among the two asymptotic expressions of quantum lifetime, Eq.(111) and Eq.(115), and that of the exact calculation.

A Shorter Derivation

Before leaving the subject of vertical quantum lifetime, we will offer another shorter derivation of Eq. (111), provided one is not interested in detailed information such as the actual eigenmode distribution. The technique developed will be useful later in the more complex calculations of the horizontal and longitudinal quantum lifetimes. To do so, we substitute Eq.(100) into (98) and integrate the resulting equation over $2\pi a da$ from 0 to A. An integration by parts gives¹²

$$\frac{-1}{\tau_n} = \pi D A \frac{d\psi_n(A)}{dA} \quad (116)$$

Equation (116) relates the lifetime τ_n to the derivative of ψ_n at the wall boundary. Without knowing ψ_n , Eq. (116) is not very useful. However, for the lowest mode, it is possible to obtain an approximate expression of τ_0 by replacing ψ_0 on the right hand side of (116) by the natural

XBL 872-8929

FIG. 7. The solid curve gives the exact quantum lifetime as a function of A/σ . The dashed curve is that predicted by the approximate expression (111), which is valid for $A \gg \sigma$. The dotted curve is that of Eq.(115), which applies when $A \ll \sigma$.

gaussian $(1/2\pi\sigma^2)\exp(-a^2/2\sigma^2)$. This is because the diffusion flux outward across the wall boundary is approximately unperturbed by the presence of the wall. Indeed, substituting ψ_0 by the natural gaussian in Eq.(116) leads to the familiar result Eq.(111).

One may doubt whether the derivative $d\psi(A)/dA$ at the wall boundary is indeed approximately equal to the derivative of the natural gaussian at $a=A$ since the derivative appears to be very large at the wall according to Fig.6 while the natural gaussian has a gentle slope at $a=A$. A more careful analysis very close to $a=A$ of the distribution (112), however, removes the doubt. The takeoff of ψ_0 very near the wall boundary is in fact quite gentle; the seemingly sharp rise is the result of an exponential factor.

Equation (116) gives the direct connection between beam tail distribution and the quantum lifetime. In the analysis so far, we have considered a damped simple harmonic system whose natural distribution is gaussian. In a practical storage ring, nonlinearities may cause the beam tail to be more populated than that expected of a gaussian. The quantum lifetime predicted by Eq.(111) can easily be too optimistic by one or two orders of magnitude due to the sensitivity of quantum lifetime to details of the tail distribution. Conversely, measuring quantum lifetime is a sensitive method to obtain experimental information on the beam tail distribution using Eq.(111).

Example 14 Equation (101) can be solved in the case of no damping, i.e. $\alpha=0$. The solution is that ψ_n is given by the Bessel function

$$\psi_n(a) = J_0\left(\frac{\lambda_n a}{A}\right) \quad (117)$$

with lifetime

$$\tau_n = \frac{2A^2}{D\lambda_n^2} \quad (118)$$

where λ_n is the n-th root of $J_0(x)$. The quantum lifetime corresponds to $n=1$ with $\lambda_1 = 2.405$. Expressions (117) and (118) were discussed before in Eqs.(114) and (115), in a slightly different, but very much related, context. This example is one relatively rare case when solutions to the Fokker-Planck equation does not relate directly to a gaussian form. Translated to the current discussion, the example applies when the wall boundary cuts deeply into the beam, i.e., $A \ll \sigma$.

Example 15 Following a calculation similar to that leading to Eq.(111) and (112), one can also have an accurate estimate of the lifetime for mode $n=2$ in the presence of wall. From Example 12, we know that the lifetime τ_2 is going to be very close to $1/2\alpha$. Let $\tau_2=(1+\Delta)/2\alpha$ with $|\Delta| \ll 1$, we find

$$\begin{aligned} c_1 &\approx -1 \\ c_k &\approx \frac{-\Delta}{k(k-1)(k!)} \quad \text{for } k \geq 2 \end{aligned} \quad (119)$$

Therefore,

$$g_2 \approx 1 - \frac{a^2}{2\sigma^2} - \Delta h_2 \left(\frac{a^2}{2\sigma^2} \right) \quad (120)$$

where

$$h_2(x) = \sum_{k=2}^{\infty} \frac{x^k}{k(k-1)(k!)} \quad (121)$$

Asymptotically we have $h_2(x) \approx e^x/x^2$ for large x . It follows from the boundary condition $g_2(A)=0$ that the $n=2$ lifetime is approximately given by

$$\tau_2 \approx \frac{1}{2\alpha} \left[1 - \left(\frac{A^2}{2\sigma^2} \right)^3 \exp \left(\frac{-A^2}{2\sigma^2} \right) \right] \quad (122)$$

and the eigenmode distribution is

$$\psi^{(2)}(a,t) \approx \exp \left(\frac{-t}{\tau_2} - \frac{a^2}{2\sigma^2} \right) \left[1 - \frac{a^2}{2\sigma^2} - \frac{\left(1 - \frac{A^2}{2\sigma^2} \right) h_2 \left(\frac{a^2}{2\sigma^2} \right)}{h_2 \left(\frac{A^2}{2\sigma^2} \right)} \right] \quad (123)$$

Longitudinal The vertical quantum lifetime calculation discussed above applies to a damped linear system. In the longitudinal dimension, this no longer applies because the rf voltage is in general not linear but sinusoidal in the longitudinal coordinate. The resulting stationary distribution is given by Eq.(53) with Hamiltonian H given by Eq.(21).

The value of the Hamiltonian is zero at origin $p=q=0$, and increases as p and q increase. At the separatrix, the Hamiltonian has the value

$$\hat{H} = \frac{2\omega_s c^2}{\omega_{rf}^2} \quad (124)$$

A particle with Hamiltonian larger than $\hat{H}$ is rapidly removed from the beam, as illustrated in Fig.2(b), because it no longer executes bounded synchrotron oscillation within the rf bucket. This means there is equivalently a wall boundary specified by the condition $H=\hat{H}$. Associated with this particle loss mechanism is the longitudinal quantum lifetime.

To calculate the quantum lifetime for this case, we follow a procedure similar to the previous case except that the distribution is no longer a function simply of amplitude a but of Hamiltonian H . Letting

$$\psi = e^{-t/\tau} g(H), \quad (125)$$

the Fokker-Planck equation becomes

$$-\frac{1}{\tau} g(H) = 2\alpha g(H) + 2\alpha H g'(H) + \omega_s D [H g''(H) + g'(H)] \quad (126)$$

where we have replaced $\omega_s p^2$ by H , which is the result of averaging over phase angle. Integrating Eq.(126) over dH from 0 to $\hat{H}$ and performing integrations by parts as before give

$$\frac{-1}{\tau} = \omega_s D \hat{H} g'(\hat{H}) \quad (127)$$

which is a generalization of Eq.(116).

Equation (53) with proper normalization gives the natural distribution $(2\alpha/D\omega_s)\exp(-2\alpha H/D\omega_s)$. Approximating $g(H)$ on the right hand side of Eq.(127) by the natural distribution then gives

$$\tau = \frac{\omega_s D}{4\alpha^2 \hat{H}} \exp\left(\frac{2\alpha \hat{H}}{D\omega_s}\right) \quad (128)$$

Substituting $\hat{H}$ from Eq.(124) into Eq.(128) then gives the longitudinal quantum lifetime. The same result (128) can be applied when the potential well distortion comes not from the rf bucket but from the wake fields.

Horizontal The quantum lifetime calculations discussed above applies to 1-dimensional systems, either in the vertical betatron or the synchrotron dimension. A complication results if the aperture limit is in the horizontal dimension at a location with energy dispersion η . This is because the horizontal beam size at such a location contains both horizontal betatron and synchrotron contributions and the aperture limitation is no longer 1-dimensional.

Let A be the amplitude of the wall boundary, the aperture limitation can be written as

$$a_x + \eta a_\delta < A \quad (129)$$

where a_x and a_δ are the oscillation amplitudes in the horizontal betatron coordinate x and synchrotron energy deviation $\delta = \Delta E/E$ respectively. We assume η is positive; otherwise it is to be replaced by $|\eta|$.

Let the distribution be

$$\psi(a_x, a_\delta, t) = e^{-\frac{t}{\tau}} g(a_x, a_\delta) \quad (130)$$

If we denote the allowed region in the (a_x, a_δ) space by R and the boundary of R by C , we require the normalization and boundary conditions

$$\iint_R 4\pi^2 a_x da_x a_\delta da_\delta g(a_x, a_\delta) = 1 \quad (131)$$

$$[g(a_x, a_\delta)]_C = 0 \quad (132)$$

where $[]_C$ means evaluating the quantity at the boundary C .

The Fokker-Planck equation in the 2-dimensional case is [compare with Eq.(98)]

$$\frac{\partial \psi}{\partial t} = \sum_{i=x,\delta} \left[2\alpha_i \psi + \alpha_i a_i \frac{\partial \psi}{\partial a_i} + \frac{D_i}{\partial a_i} \frac{\partial}{\partial a_i} \left(a_i \frac{\partial \psi}{\partial a_i} \right) \right] \quad (133)$$

Substituting Eq.(130) into (133), integrating over the phase space region R and applying integration by parts yield

$$\frac{1}{\tau} = \alpha_x I_x + \alpha_\delta I_\delta \quad (134)$$

where

$$I_x = -4\pi^2 \sigma_x^2 \int_R a_\delta da_\delta \left[a_x \frac{\partial g}{\partial a_x} \right]_C$$

$$I_\delta = -4\pi^2 \sigma_\delta^2 \int_R a_x da_x \left[a_\delta \frac{\partial g}{\partial a_\delta} \right]_C \quad (135)$$

Equations (134) and (135) are the 2-dimensional counterpart of Eq.(116) or (127). Following the technique developed in the previous section, we approximate g by its natural form

$$g(a_x, a_\delta) = \frac{1}{4\pi^2 \sigma_x^2 \sigma_\delta^2} \exp \left(-\frac{a_x^2}{2\sigma_x^2} - \frac{a_\delta^2}{2\sigma_\delta^2} \right) \quad (136)$$

where $\sigma_i = (D_i/2\alpha_i)^{1/2}$, $i=x, \delta$. This leads to the expressions

$$I_x = \frac{1}{\sigma_x^2 \sigma_\delta^2} \int_0^{A/\eta} a_\delta da_\delta (A-\eta a_\delta)^2 \exp \left[-\frac{(A-\eta a_\delta)^2}{2\sigma_x^2} - \frac{a_\delta^2}{2\sigma_\delta^2} \right] \quad (137)$$

$$I_\delta = \frac{1}{\eta^2 \sigma_x^2 \sigma_\delta^2} \int_0^A a_x da_x (A-a_x)^2 \exp \left[-\frac{a_x^2}{2\sigma_x^2} - \frac{(A-a_x)^2}{2\eta^2 \sigma_\delta^2} \right]$$

By changing variables and defining $\sigma_T^2 = \sigma_x^2 + \eta^2 \sigma_\delta^2$, $r = \eta^2 \sigma_\delta^2 / \sigma_T^2$, Eq.(137) becomes¹²

$$I_x = \frac{A^4}{\eta^2 \sigma_x^2 \sigma_\delta^2} \exp \left(\frac{-A^2}{2\sigma_T^2} \right) \int_{-r}^{1-r} (y+r) (1-r-y)^2 dy \exp \left(-\frac{A^2 y^2}{2r\sigma_x^2} \right) \quad (138)$$

$$I_\delta = \frac{A^4}{\eta^2 \sigma_x^2 \sigma_\delta^2} \exp \left(\frac{-A^2}{2\sigma_T^2} \right) \int_{r-1}^r (y+4-r) (r-y)^2 dy \exp \left(-\frac{A^2 y^2}{2r\sigma_x^2} \right)$$

The quantum lifetime is thus obtained by substituting Eq.(138) into Eq.(134).

Example 16 In case the aperture limit occurs at a nondispersive point with $\eta=0$, we have from Eq.(138),

$$I_x = \frac{A^2}{\sigma_x^2} \exp \left(\frac{-A^2}{2\sigma_x^2} \right) \quad (139)$$

$$I_\delta = 0$$

which means the problem reduces to 1-dimensional and the lifetime is that already found in Eq.(111), as it should.

Example 17 If we assume σ_x and $\eta\sigma_\delta$ are comparable and are both much smaller than A , or more precisely, if both r and $1-r$ are much larger than $\sigma_x \eta \sigma_\delta / A \sigma_T$, Eq.(138) can be simplified. This is

because the gaussian factor in the integrand limits significant contributions to the region $y^2 < r\sigma_x^2/A^2$. As a result, the algebraic part of the integrand can be approximated by setting $y=0$ and the limits of the integral can be replaced by $\pm\infty$. After this process, we find

$$I_x \approx \sqrt{2\pi} \frac{A^3 \sigma_x^3 \eta \sigma_\delta}{\sigma_T^7} \exp\left(\frac{-A^2}{2\sigma_T^2}\right)$$

and

(140)

$$I_\delta \approx \sqrt{2\pi} \frac{A^3 \sigma_x \eta^3 \sigma_\delta^3}{\sigma_T^7} \exp\left(\frac{-A^2}{2\sigma_T^2}\right)$$

Substituting (140) into (134) and setting $\alpha_\delta = 2\alpha_x$ for an ideal storage ring (see Table 1), we obtain the final expression of quantum lifetime.

$$\tau = \frac{\exp(\frac{n^2}{2})}{\sqrt{2\pi} \alpha_x n^3} \frac{1}{(1+r) \sqrt{r(1-r)}} \quad (141)$$

where we have defined $n=A/\sigma_T$. For given σ_T , the shortest lifetime occurs when $r=(1+\sqrt{17})/8=0.640$, i.e., $\eta\sigma_\delta = 0.80 \sigma_T$ and $\sigma_x = 0.60 \sigma_T$, which gives

$$\tau_{\min} = 0.507 \frac{\exp(\frac{n^2}{2})}{\alpha_x n^3} \quad (142)$$

A comparison with Eq.(111) indicates that the quantum lifetime in this 2-dimensional case is shorter than the quantum lifetime of a 1-dimensional case because, in addition to a numerical factor, the denominator now contains n^3 instead of n^2 , where $n \gg 1$.

3.6 Normal Modes

In the description of a multi-particle system using the Vlasov and Fokker-Planck equations, the collective beam motion is often analysed in terms of normal modes. In the presence of wake fields, for example, the complex frequencies of these normal modes provide the criterion for their stability. The system is unstable if any one of its normal modes is found unstable.

Due to the high degree of degeneracy of the Vlasov equation for a linear multi-particle system, described by Eq.(10) or (12), its normal modes are not very interesting. The n -th normal mode,

for example, can be written as

$$\psi^{(n)}(a, \phi, t) = \exp(-i\Omega^{(n)}t) A^{(n)}(a) \exp(im\phi) \quad (143)$$

where the normal mode frequency is $\Omega^{(n)} = n\omega$ and the radial eigen-function $A^{(n)}(a)$ could be a member of some orthogonal polynomials. Each mode n in general degenerates into a set of modes, each specified by a radial eigen-function.

In the following, we will describe the normal modes of the Fokker-Planck equation in the ideal 1-dimensional case without wall boundary. We will start with Eq.(42) and recall that we averaged over phase and assumed no phase dependence of ψ to obtain Eq.(98). To find the normal modes, however, the phase dependence must now be kept. Substituting Eq.(11) into (42), the complete Fokker-Planck equation in polar coordinates is found to be

$$\begin{aligned} \frac{\partial \psi}{\partial t} - \omega \frac{\partial \psi}{\partial \phi} - 2\alpha a \sin^2 \phi \frac{\partial \psi}{\partial a} - \alpha \sin 2\phi \frac{\partial \psi}{\partial \phi} \\ = 2\alpha \psi + D \sin^2 \phi \frac{\partial^2 \psi}{\partial a^2} + D \frac{\sin 2\phi}{a} \left[\frac{\partial^2 \psi}{\partial a \partial \phi} - \frac{1}{a} \frac{\partial \psi}{\partial \phi} \right] + D \frac{\cos^2 \phi}{a} \left[\frac{\partial \psi}{\partial a} + \frac{1}{a} \frac{\partial^2 \psi}{\partial \phi^2} \right] \end{aligned} \quad (144)$$

Let the n -th normal mode be written as

$$\psi^{(n)} = \exp\left(-i\Omega^{(n)}t - \frac{\alpha a^2}{D}\right) \sum_{m=-\infty}^{+\infty} A_m^{(n)}(a) \exp(im\phi) \quad (145)$$

where $\Omega^{(n)}$ is the normal mode frequency and functions $A_m^{(n)}(a)$ are yet to be determined.

Substituting (145) into (144), and identifying the proper Fourier components, we obtain the iteration equation

$$\begin{aligned} -i[\Omega^{(n)} + m\omega] A_m = \frac{D}{2} A_m'' + \left(\frac{D}{2a} - \alpha a\right) A_m' - \frac{Dm^2}{2a^2} A_m - \frac{D}{4} A_{m-2}'' \\ + \left(\frac{\alpha a}{2} + \frac{mD}{2a} - \frac{3D}{4a}\right) A_{m-2}' - (m-2) \left(\frac{\alpha}{2} + \frac{mD}{4a^2}\right) A_{m-2} - \frac{D}{4} A_{m+2}'' \\ + \left(\frac{\alpha a}{2} - \frac{mD}{2a} - \frac{3D}{4a}\right) A_{m+2}' + (m+2) \left(\frac{\alpha}{2} - \frac{mD}{4a^2}\right) A_{m+2} \end{aligned} \quad (146)$$

where we have dropped the (n) superscripts on the $A_m^{(n)}(a)$ functions. Primes denote taking derivatives with respect to a .

For the lowest mode, $n=0$, the solution is

$$\begin{aligned} A_0^{(0)}(a) &= 1, \quad A_m^{(0)} = 0 \text{ for all } m \neq 0 \\ \Omega^{(0)} &= 0 \end{aligned} \quad (147)$$

This is just the stationary solution of the Fokker-Planck equation.

There are two modes with $n=1$. They are given by

$$\begin{aligned} A_1^{(1)}(a) &= a, \quad A_{-1}^{(1)}(a) = fa, \quad A_m^{(1)} = 0 \text{ for all other } m \\ \Omega^{(1)} &= -i\alpha \pm (\omega^2 - \alpha^2)^{1/2} \end{aligned} \quad (148)$$

where

$$f = \frac{-i\omega}{\alpha} \pm \left(1 - \frac{\omega^2}{\alpha^2} \right)^{1/2}$$

The $\pm$ indices specify the two modes. These $n=1$ modes are the dipole modes which are damped with damping time $1/\alpha$. Their eigen-frequencies are just those given in Eq.(70). Note also that because of the $\exp(\pm i\phi)$ dependence, these modes do not survive phase averaging.

There are three $n=2$ modes given by

$$\begin{aligned} A_2^{(2)}(a) &= a^2, \quad A_0^{(2)}(a) = fa^2 + g, \quad A_{-2}^{(2)}(a) = ha^2, \quad A_m^{(2)} = 0 \text{ for all other } m \\ \Omega^{(2)} &= -2i\alpha, -2i\alpha \pm 2(\omega^2 - \alpha^2)^{1/2} \end{aligned} \quad (149)$$

where

$$f = 2 - \frac{i}{\alpha} (\Omega^{(2)} + 2\omega)$$

$$g = -1 + \left(1 - \frac{i\Omega^{(2)}}{2\alpha} \right) f$$

$$h = -\frac{2iD}{\Omega^{(2)}} (g + 1)$$

These are quadrupole modes with damping times $1/2\alpha$. If the phase angle information is averaged over, these modes correspond to the $n=2$ mode described by Eq.(106).

In general, there are $n+1$ modes with index n . If we designate them by another index k which assumes values $k = n, n-2, \dots, -n$, the eigen-frequencies are¹³

$$\Omega^{(n,k)} = -in\alpha + k (\omega^2 - \alpha^2)^{1/2} \quad (150)$$

These modes all have damping time $1/n\alpha$. Note that the mode frequencies are independent of the diffusion constant D . Note also that the higher modes have more complicated phase and amplitude structures, and they are damped faster.

REFERENCES

1. A.A. Vlasov, J. Phys. USSR 9, 25 (1945).
2. S. Chandrasekhar, Rev. Mod. Phys., 15, 1 (1943).
3. F. Sacherer, CERN/SI - BR/72-5 (1972), unpublished.
4. See, for example, K. Huang, "Statistical Mechanics," Wiley, New York, 1963.
5. E. D. Courant and H. S. Snyder, Ann. Phys., 3, 1 (1958)
6. C. Pellegrini and A.M. Sessler, Nuovo Cimento, 3A, 116 (1971).
7. J. Haissinski, Nuovo Cimento, 18 B, 72 (1973).
8. A. Hofmann, "Theoretical Aspects of the Behavior of Beams in Accelerators and Storage Rings," CERN 77-13 (1977), page 139. Also, A. Hofmann, lectures in this school.
9. M. Sands, "The Physics of Electron Storage Rings, An Introduction," SLAC-121 (1970).
10. Superconducting Super Collider Conceptual Design Report, SSC-SR-2020 (1986).
11. A.W. Chao and M.J. Lee, J. Appl. Phys. 47, 4453 (1976).
12. A.W. Chao, 1977 Part. Accel. Conf., Chicago, IEEE Trans. Nucl. Sci., NS-24, 1885.
13. A. Renieri, Frascati Lab report LNF-76/11 (R) (1976), unpublished.