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FINITE-SIZE CORRECTIONS FOR NESTED BETHE ANSATZ MODELS
AND CONFORMAL INVARIANCE

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A B S T R A C T

The method presented here permits to systematically compute physical magnitudes for large but finite size in theories solvable by nested Bethe ansatz. The value of the central charge c and the surface tension are explicitly calculable in this way. For the fundamental vertex and spin models associated to simply-laced Lie algebras, c turns out to be equal to the rank of the algebra.

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1. - INTRODUCTION

Integrable theories are those possessing as many commuting and conserved physical quantities as degrees of freedom. So there is an infinite number in the thermodynamical limit. The explicit solution of these models is to a large extent possible through the use of the Bethe ansatz and its generalizations¹⁾.

The Bethe ansatz for the eigenvectors lead a set of algebraic equations whose roots determine the common eigenvalues of the commuting operators of the theory including the Hamiltonian. In the thermodynamical limit the number of equations and roots tend to infinity and the Bethe ansatz equation (BAE) usually yields a set of linear integral equations for the densities of roots. Fourier transforms solve the equation in this limit. However, the explicit resolution of the BAE for a finite size N is a formidable problem. Besides its interest from the point of view of integrable theories the finite size resolution of these models provide deep insights on their conformal content when they are gapless. For models with non-zero gap it gives information on the surface tension.

A method to systematically compute large but finite-size corrections in integrable theories was presented in Ref. 2) for the six- and eight-vertex³⁾ models and the associated magnetic chains. These methods are elaborated and developed in Refs. 4)-7).

The aim of this paper is to generalize the method of Ref. 2) to theories solvable by nested Bethe ansatz (NBA). That is, theories with internal structure: vertex models where the links can be in q different states (with $q > 2$) or q -component spin models. The field theoretical models solvable by NBA usually exhibit non-Abelian internal symmetries as the chiral fermionic models of Refs. 8)-10), the Gross-Neveu model and the sigma models connected with multiflavour fermionic models¹¹⁾.

The calculation method presented here is based on a new integral representation for the physical quantities (like the energy) which are exact for all N (number of sites): These integral representations generalize those of Refs. 2) and 3) and they contain functions $\sigma_N^{(l)}(\lambda)$ ($1 \leq l \leq r$, r = number of steps in the NBA) that can be thought of as finite N generalizations of the densities of roots. We find that the contribution of the different steps of the NBA simply adds. The expressions obtained appear as infinite series [Eqs. (2.31)- (2.33)] that can be summed up as geometric series. Then the large N behaviour

follows by the saddle point method. In massive models (non-zero gap) the relevant saddle points are complex roots of

$$\sigma_{\infty}^{(\ell)}(\lambda) = 0, \quad 1 \leq \ell \leq r. \quad (1.1)$$

They dominate the large N behaviour of all physical quantities. For gapless theories end-points of integration ($\lambda=\pm\infty$) govern the large N regime. In the first case the finite N corrections are exponentially small in N and in the second case they are power-type. In both cases closed-form expressions can be explicitly derived. In the transition between the two regimes, logarithmic terms appear also⁶⁾ (already with a one-step Bethe ansatz).

The method presented in Section 2 is valid for any theory solvable by Bethe ansatz where the ground state is formed by real roots.

We apply this method to a large class of fundamental vertex models associated to simple Lie algebras. The NBAE for all these models can be written systematically in Lie algebraic terms¹²⁾. For the models associated to simply laced Lie algebras (A_n, D_n, E_6, E_7, E_8) we find a simple expression for the dominant finite size corrections to the free energy

$$f_N - f_{\infty} = - \frac{\pi}{6N^2} r \quad (1.2)$$

for periodic boundary conditions. Here r stands for the rank of the algebra. Comparison with Ref. 13) implies that the central conformal charge equals r for all these models. Equation (1.2) is also true for the $U(1)^r$ symmetric model of Refs. 1e) and 14) in its gapless regime. This last model possesses a massive regime. We compute for this regime the interfacial tension S in Section 4. It turns out that S follows from the zeros of an elliptic function [Eq. (4.19)]. Explicit expressions of S for large and small anisotropies (γ) are derived in Section 4 [Eqs. (4.22), (4.23) and (4.26)]. The results presented generalize for this $q(2q-1)$ vertex model the formulas of Baxter for the six-vertex model ($q=2$)^{1c)}.

2. - FINITE-SIZE SOLUTION OF NESTED BETHE ANSATZ MODELS

A method to compute finite-size corrections for integrable theories solvable by nested Bethe ansatz (NBA) is presented in this section. Specific applications are exposed in Section 4.

Generally speaking, the free energy for such models writes for a large but finite number of sites N

$$f_N(\theta) = -\frac{1}{N} \log \Lambda_{\max}(\theta) + O(e^{-\delta N}) \quad (2.1)$$

Here $\Lambda_{\max}$ is the maximum eigenvalue of the transfer matrix $\tau(\theta)$, θ the spectral parameter and $\delta > 0$ is related to the next-to-leading eigenvalue of $\tau(\theta)$. Then,

$$\log \Lambda_{\max} = i \sum_{k=1}^r \sum_{j_k=1}^{p_k} \phi(\lambda_{j_k}^{(k)} + i\theta, \omega_k) \quad (2.2)$$

where the functions $\phi(z, \alpha)$ are given in Section 3, k labels the steps of the NBA ($1 \leq k \leq r$), and the $\lambda_{j_k}^{(k)}$, $j_k = 1, \dots, p_k$ are the roots of the NBA equations

$$\sum_{k=1}^r \sum_{j_k=1}^{p_k} \phi(\lambda_{j_e}^{(e)} - \lambda_{j_k}^{(k)}, \omega_{ek}) = N \phi(\lambda_{j_e}^{(e)}, \omega_e) - 2\pi I_{j_e}^{(e)} \quad (2.3)$$

The parameters $\omega_{ek} = (\alpha_e, \alpha_k)$ and $\omega_k = (\omega, \alpha_k)$ are given and they are related to roots and weights of the underlying Lie algebra (see Section 3). The $I_{j_k}^{(k)}$ are half-odd-integers.

It is convenient to introduce the functions

$$t_N^{(e)}(\lambda) = \phi(\lambda, \omega_e) - \frac{1}{N} \sum_{k=1}^r \sum_{j_k=1}^{p_k} \phi(\lambda - \lambda_{j_k}^{(k)}, \omega_{ek}) \quad (2.4)$$

They are continuous functions of λ for real λ . At the roots of the NBAE (2.3)

$$t_N^{(e)}(\lambda_{j_e}^{(e)}) = \frac{2\pi}{N} I_{j_e}^{(e)} \quad (2.5)$$

Define now

$$\sigma_N^{(k)}(\lambda) = \frac{1}{2\pi} \frac{d t_N^{(k)}}{d \lambda} \quad (2.6)$$

The half-odd integers $I_{j_k}^{(k)}$ form for fixed k a monotonic sequence for the anti-ferromagnetic vacuum

$$I_{j_{k+1}}^{(k)} - I_{j_k}^{(k)} = + 1, \quad 1 \leq k \leq r, \quad 1 \leq j_k \leq p_k \quad (2.7)$$

For excited states these sequences exhibit jumps for some values of j_k

$$I_{j_{k+1}}^{(k)} - I_{j_k}^{(k)} = 1 + \sum_{h=1}^{N_h^{(k)}} \delta_{j_k, i_h^{(k)}} \quad (2.8)$$

The values of $\lambda_{j_k}^{(k)}$ associated to these missing half-integers are called holes and denoted by $\theta_h^{(k)}$

$$t_N^{(k)}(\theta_h^{(k)}) = \frac{2\pi}{N} (1 + I_{i_h^{(k)}}^{(k)}) \quad (2.9)$$

When N goes to infinity, the p_k also tend to infinity for antiferromagnetic states and the $\lambda_{j_k}^{(k)}$ tend to have a continuous distribution for each value of k with density

$$\rho^{(k)}(\lambda_{j_k}^{(k)}) = \lim \frac{1}{N(\lambda_{j_{k+1}}^{(k)} - \lambda_{j_k}^{(k)})} \quad (2.10)$$

One finds from Eqs. (2.5) and (2.6) for large N

$$\sigma_\infty^{(k)}(\lambda_{j_k}^{(k)}) = \lim_{N \rightarrow \infty} \frac{I_{j_{k+1}}^{(k)} - I_{j_k}^{(k)}}{N(\lambda_{j_{k+1}}^{(k)} - \lambda_{j_k}^{(k)})} \quad (2.11)$$

Then, Eqs. (2.8)-(2.11) yield for large N

$$\sigma_\infty^{(k)}(\lambda) = \rho^{(k)}(\lambda) + \frac{1}{N} \sum_{h=1}^{N_h^{(k)}} \delta(\lambda - \theta_h^{(k)}) \quad (2.12)$$

The NBA Eq. (2.3) admit both real and complex roots. We discuss in the present paper effects related to real roots. Taking the difference in that case between Eq. (2.3) for $j_n = \lambda_{n+1}$ and $j_n = \lambda_n$ yields in the $N = \infty$ limit

$$\rho^{(j)}(\lambda) - \sum_{e=1}^r \int_{-a_e}^{a_e} d\mu K_{je}(\lambda-\mu) \rho^{(e)}(\mu) = \frac{\phi'(\lambda, \omega_j)}{2\pi} - \frac{1}{N} \sum_{h=1}^{N_h^{(e)}} \delta(\lambda - \Theta_h^{(e)}) \quad (2.13)$$

or

$$\sigma_\infty^{(j)}(\lambda) - \sum_{e=1}^r \int_{-a_e}^{a_e} d\mu K_{je}(\lambda-\mu) \rho^{(e)}(\mu) = \frac{\phi'(\lambda, \omega_j)}{2\pi} - \frac{1}{N} \left[\sum K_{je}(\lambda - \Theta_h^{(e)}) \right] \quad (2.14a)$$

where

$$K_{je}(\lambda) = 2\pi \phi'(\lambda, \omega_{je}) \quad (2.14b)$$

The following relation was used to derive Eqs. (2.13) and (2.14a)

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^{p_k} f(\lambda_{\lambda_k}^{(k)}) = \int_{-a_e}^{a_e} d\lambda f(\lambda) \rho^{(k)}(\lambda) \quad (2.15)$$

Here a_j may be finite or infinite. Equations (2.13) and (2.14a) can be solved by the Fourier expansion

$$\sigma_\infty^{(j)}(\mu) = \int_{-\infty}^{+\infty} \frac{dx}{2\pi} e^{i\mu x} \hat{\sigma}^{(j)}(x) \quad (2.16)$$

when $a_e = \infty$ or

$$\sigma_\infty^{(j)}(\mu) = \sum_{n \in \mathbb{Z}} e^{2in\mu} \hat{\sigma}_n^{(j)} \quad \text{for } a_e = \pi/2.$$

It is convenient to express the solutions of Eqs. (2.14a) and (2.14b) in terms of the resolvent $R_{jk}(\lambda)$. That is the solution of the equation

$$R_{ek}(\lambda) - \sum_{j=1}^r \int_{-a_j}^{a_j} d\mu K_{ej}(\lambda-\mu) R_{jk}(\mu) = \delta_{ek} \delta(\lambda) \quad (2.17)$$

$; 1 \leq k, e \leq r$

One finds in Fourier space

$$\hat{R}_{ek}(x) = [(1 - \hat{K}(x))^{-1}]_{ek} ; 1 \leq k, e \leq r. \quad (2.18)$$

Explicit expressions for this inverse matrix are available for each specific model (Section 4). Then the densities $\sigma^{(l)}(\lambda)$ write

$$\sigma_{\infty}^{(l)}(\lambda) = \sigma_v^{(l)}(\lambda) + \frac{1}{N} \sum_{j=1}^r \sum_h^{N_h^{(l)}} \sigma_{ej}(\lambda - \theta_h^{(j)}) \quad (2.19)$$

where

$$\sigma_v^{(l)}(\lambda) = \sum_{k=1}^r \int_{-a_k}^{a_k} \frac{d\mu}{2\pi} \phi'(\mu, \omega_k) R_{ek}(\lambda - \mu) \quad (2.20)$$

and

$$\sigma_{ej}(\lambda) = \delta_{ej} \delta(\lambda) - R_{ej}(\lambda) \quad (2.21)$$

Here $\sigma_v^{(l)}(\lambda)$ describes the root density for the antiferromagnetic vacuum at $N = \infty$, that is in the absence of holes. In the same limit the free energy per site reads from Eqs. (2.1), (2.2) and (2.15)

$$f_{\infty}(\theta) = -i \sum_{k=1}^r \int_{-a_k}^{a_k} d\lambda \sigma_{\infty}^{(k)}(\lambda) \phi(\lambda + i\theta, \omega_k) \quad (2.22)$$

Now, the second term in Eq. (2.19) describes the hole contribution to the density of roots. The contribution to the free energy from a single hole at $\phi = \theta_h^{(l)}$ reads

$$f_e(\varphi) = i \sum_{k=1}^r \int_{-a_k}^{a_k} R_{ek}(\lambda - \varphi) \phi(\lambda + i\theta, \omega_k) d\lambda = i [t_{\infty}^{(l)'}(\varphi + i\theta) + K_e] \quad (2.23)$$

The last equality follows from Eqs. (2.6) and (2.20). K_e is an integration constant (see Section 4).

Let us now consider the finite size corrections to $f_{\infty}(\theta)$, that is

$$L_N(\theta) \equiv f_N(\theta) - f_{\infty}(\theta) \quad (2.24)$$

One finds, with the help of Eqs. (2.21), (2.2), (2.15) and (2.22)

$$L_N(\theta) = -i \sum_{\ell=1}^r \int_{-a_\ell}^{a_\ell} d\mu \phi(\mu + i0, \omega_\ell) \left\{ \frac{1}{N} \sum_{j=1}^{p_\ell} \delta(\mu - \lambda_{j\ell}^{(\ell)}) - \sigma_N^{(\ell)}(\mu) \right\} \\ - i \sum_{k=1}^r \int_{-a_k}^{a_k} d\lambda \phi(\lambda + i0, \omega_k) \left[\sigma_N^{(k)}(\lambda) - \sigma_\infty^{(k)}(\lambda) \right]. \quad (2.25)$$

We now study the difference $\sigma_N^{(k)}(\lambda) - \sigma_\infty^{(k)}(\lambda)$. It follows, using Eqs. (2.4), (2.6), (2.14a) and the resolvent (2.18), that

$$\sigma_N^{(k)}(\lambda) - \sigma_\infty^{(k)}(\lambda) = - \sum_{\ell=1}^r \int_{-a_\ell}^{a_\ell} d\mu \left[\delta_{k\ell} \delta(\lambda - \mu) - R_{k\ell}(\lambda - \mu) \right] \\ \left\{ \frac{1}{N} \sum_{j=1}^{p_\ell} \delta(\mu - \lambda_{j\ell}^{(\ell)}) - \sigma_N^{(\ell)}(\mu) \right\}. \quad (2.26)$$

Equations (2.25) and (2.26) hold for the antiferromagnetic vacuum. In the presence of holes, one finds instead inside the brackets in Eqs. (2.25) and (2.26)

$$\frac{1}{N} \sum_{j=1}^{p_\ell} \delta(\mu - \lambda_{j\ell}^{(\ell)}) + \frac{1}{N} \sum_{k=1}^{N_h^{(\ell)}} \delta(\mu - \theta_k^{(\ell)}) - \sigma_N^{(\ell)}(\mu) \quad (2.27)$$

Now, inserting Eq. (2.26) into (2.25) yields

$$L_N(\theta) = -i \sum_{\ell=1}^r \int_{-a_\ell}^{a_\ell} d\mu \left[t_\infty^{(\ell)}(\mu + i0) + \kappa_\ell \right] \left\{ \frac{1}{N} \sum_{j=1}^{p_\ell} \delta(\mu - \lambda_{j\ell}^{(\ell)}) - \sigma_N^{(\ell)}(\mu) \right\} \quad (2.28)$$

where Eq. (2.23) was used. This formula (2.28) has the right structure in order to evaluate its large N behaviour. It must be noticed that it is the sum of r terms, so one can apply essentially the same method as in Refs. 2) and 3) to each term separately. It can be noticed that a formula analogous to (2.28) holds for excited states: the curly bracket in the right-hand side must be replaced by Eq. (2.27).

Generally speaking, it is possible to derive the large N behaviour of any expression with the form

$$I_N = \sum_{e=1}^r \int_{-ae}^{ae} d\mu_e f_e(\mu_e) \left\{ \frac{1}{N} \sum_{j=1}^{p_e} \delta(\mu_e - \lambda_{j_e}^{(e)}) + \frac{1}{N} \sum_{h=1}^{N_h^{(e)}} \delta(\mu_e - \Theta_h^{(e)}) - \sigma_N^{(e)}(\mu_e) \right\} \quad (2.29)$$

where the function $f_e(\mu)$ is given. Changing the integration variables in Eq. (2.29) to $t_e = t_N^{(e)}(\mu_e)$ as defined by Eq. (2.4) yields

$$I_N = \sum_{e=1}^r \int_0^{Q_e} \frac{dt_e}{2\pi} f_e(\mu_e(t_e)) \left\{ \frac{1}{N} \sum_{k=1}^{p_e + N_h^{(e)}} \delta(t_e - t_k^{(e)}) - 1 \right\} \quad (2.30)$$

where

$$t_k^{(e)} = \frac{2\pi}{N} (k + 1/2), \quad k = 1, 2, \dots, p_e + N_h^{(e)} \text{ and } Q_e = \int_{-ae}^{ae} d\lambda \sigma_N^{(e)}(\lambda)$$

Fourier expanding the $\delta(z)$ gives

$$I_N = \sum_{e=1}^r \sum_{\substack{\alpha e = -\infty \\ \alpha e \neq 0}}^{+\infty} (-1)^{\alpha e} T_{N\alpha e}^e \quad (2.31)$$

where

$$T_m^e = \int_{-ae}^{ae} d\lambda f_e(\lambda) \sigma_N^{(e)}(\lambda) e^{im t_N^{(e)}(\lambda)} \quad (2.32)$$

Now, in order to find the dominant large N behaviour, one can replace

$$T_m^e \rightarrow T_m^{e,as} \equiv \int_{-ae}^{ae} d\lambda f_e(\lambda) \sigma_\infty^{(e)}(\lambda) e^{im t_\infty^{(e)}(\lambda)} \quad (2.33)$$

The right-hand side of Eq. (2.33) is exactly known from Eqs. (2.18)-(2.21). One must take

$$f_e(\lambda) = -i \left[t_\infty^{(e)}(\lambda + i0) + \kappa_e \right] \quad (2.34)$$

to compute finite-size corrections to the free energy.

For large n , $T_n^{l,as}$ (and T_n^l) are dominated by the stationary points where

$$\frac{dt_\infty(\lambda_0)}{d\lambda} = 2\pi \sigma_\infty^{(e)}(\lambda_0) = 0 \quad (2.35)$$

These equations have finite complex solutions when the mass gap is non-zero [see Refs. 2) and 3) and Section 4]. When the model is gapless, $\lambda_0 = \infty$ [Ref. 4) and Section 4] and the integral (2.33) is dominated for large n by its endpoints. In the former case ($\lambda_0 < \infty$), T_n^l is exponentially small in n and so is I_N in N ; that is

$$I_N \sim |e^{it_\infty(\lambda_0)}|^N \quad (2.36)$$

(plus power corrections). On the contrary, one finds power-like behaviour plus possible logarithmic corrections in the gapless regime.

The method exposed here holds for any NBA state formed with real roots and eventually a finite number of complex roots.

3. - INTEGRABLE THEORIES AND SEMI-SIMPLE LIE ALGEBRAS

We give in this section a non-exhaustive description of integrable models where the methods of Section 2 apply.

A deep connection exists between integrable theories and simple Lie algebras^{1),12)}. It is possible to associate an integrable vertex model to each representation of a simple Lie algebra. These models are invariant under the corresponding Lie group G , since this R -matrix obey

$$[R(\theta), g \otimes g] = 0 \quad (3.1)$$

Moreover, the structure of their BAE looks like the one of their respective Dynkin diagram. It must be noticed that the BAE are known to lead to the eigenvectors and eigenvalues of the transfer matrix only for a subset of models: those associated to $U(N)$ ^{1c),14),15)}, $Sp(2N)$ ¹⁶⁾ and $SO(2N)$ ¹⁷⁾ and some others. However, it is plausible to assume the validity of these statements for all the semi-simple Lie algebras. Moreover, the Dynkin diagram structure of the BAE is not restricted to non-Abelian invariant models fulfilling Eq. (3.1). In the case

of the model of Refs. 1e) and 14), $G = U(1)^{q-1} \times Z_q$ and the NBAE have the structure of the A_{q-1} Dynkin diagram, that is, the same structure as in the isotropic limit where $G = SU(q)$.

Let us briefly describe the BAE for the solutions of the non-Abelian symmetric models. We refer for the derivation of these equations (when they exist already) to Refs. 1), 14)-17).

The eigenvalues of the transfer matrix write as a sum of terms. The dominant one in the infinite volume limit ($N \rightarrow \infty$) is

$$\lambda_\omega(\theta, \{\lambda^{(j)}\}) = \prod_{a=1}^N \prod_{k=1}^r \prod_{j_k=1}^{p_k} \frac{i(\theta - \theta_a) + \lambda_{j_k}^{(k)} - i(\omega_a, \alpha_k)}{i(\theta - \theta_a) + \lambda_{j_k}^{(k)} + i(\omega_a, \alpha_k)} \quad (3.2)$$

for θ in the vicinity of $\theta = 0$, $|\theta| < \theta_0$. Here $\theta_1, \dots, \theta_N$ are given numbers describing inhomogeneities in the lattice^{1c), 18)}. The ω_a are fundamental weights and the α_k are the simple roots of the associated Lie algebra, whose rank is r (α, β) stands for the usual inner product in root space. The $\lambda_{j_k}^{(k)}$ ($1 \leq j_k \leq p_j$, $1 \leq k \leq r$) are solutions of the NBAE. They admit the Lie algebraic expression

$$\prod_{k=1}^r \prod_{j_k=1}^{p_k} \frac{\lambda_{j_i}^{(i)} - \lambda_{j_k}^{(k)} - i(\alpha_i, \alpha_k)}{\lambda_{j_i}^{(i)} - \lambda_{j_k}^{(k)} + i(\alpha_i, \alpha_k)} = \prod_{a=1}^N \frac{\lambda_{j_i}^{(i)} - \theta_a - i(\omega_a, \alpha_i)}{\lambda_{j_i}^{(i)} - \theta_a + i(\omega_a, \alpha_i)} \quad (3.3)$$

Here the upper indices (i) label the steps in the nested Bethe ansatz. Each step is associated to a simple root α_i . The structure of Eqs. (3.3) coincides with the respective Dynkin diagram: when two roots, say α_ℓ and α_i , are orthogonal, their associated parameters $\lambda_{j_\ell}^{(\ell)}$ and $\lambda_{j_i}^{(i)}$ ($1 \leq j_\ell \leq p_\ell$, $1 \leq j_i \leq p_i$) are not directly coupled through Eq. (3.3) since $(\alpha_i, \alpha_\ell) = 0$. It must be noticed that due to the orthogonality of fundamental weights and simple roots¹⁹⁾

$$(\omega_a, \alpha_i) = \frac{\delta_{ai}}{2} (\alpha_i, \alpha_i) \quad (3.4)$$

The normalization of the simple roots can be absorbed as a multiplicative factor on the $\lambda_{j_i}^{(i)}$.

In Section 2 we dealt with the homogeneous models where $\theta_a = 0$ and $\omega_a = \omega$ for $1 \leq a \leq N$. Then

$$\phi(z, \alpha) = i \log \frac{z + i\alpha}{z - i\alpha} \quad (3.5)$$

in the equations of Section 2. We have, for $|\text{Im}z| < \alpha$

$$\phi(z, \alpha) = 2 \int_0^\infty dk \frac{\sin kz}{k} e^{-k\alpha} + \pi \quad (3.6)$$

Here we take the cut of the logarithm in Eq. (3.5) such that $\phi(x, \alpha)$ is a continuous function for real x and $\phi(0, \alpha) = \pi$. It must be noticed that

$$\left. \frac{\partial \phi}{\partial z}(z, \alpha) \right|_{\alpha=0} = 0$$

Then Eqs. (2.14a)-(2.16) and (3.6) yield for the kernel of the NBAE (2.14)

$$\hat{K}_{je}(x) = -\text{sign}[(\alpha_j, \alpha_e)] e^{-|(\alpha_j, \alpha_e)x|} \quad (3.7)$$

with $\text{sign}(0) \equiv 0$. Therefore, one gets for the resolvent from Eq. (2.18)

$$[\hat{R}^{-1}(x)]_{je} = 1 + \text{sign}[(\alpha_j, \alpha_e)] e^{-|(\alpha_j, \alpha_e)x|} \quad (3.8)$$

Hence, one has to invert this $r \times r$ matrix. This is not a formidable problem since it is a sparse matrix²⁰⁾ whose characteristic diagram is precisely the Dynkin diagram of the corresponding Lie algebra. Explicit formulae for $R_{jl}(x)$ can be derived for each Lie algebra. One has for A_n

$$\hat{R}_{je}(x) = e^{-|x|} \frac{\text{sh } x(n+1-\ell_j) \text{sh } (x\ell_e)}{\text{sh } x(n+1) \text{sh } x} \quad (3.9)$$

For D_n , see Refs. 10) and 17). For E_6 , E_7 and E_8 , no general formula is known but $R_{jl}(x)$ can be worked explicitly.

For non-simply laced Lie algebras, the ground state is formed by complex roots and hence the treatment of Section 2 needs to be generalized.

In the $U(1)^q$ ($q \geq 2$) symmetric model of Refs. 1e) and 14), one finds an A_{q-1} structure for the NBAE although the model is not $SU(q)$ invariant. One must take for $\phi(z, \alpha)$

$$\phi(z, \alpha) = i \operatorname{Log} \frac{\sin(z + i\alpha)}{\sin(z - i\alpha)} \quad (3.10)$$

or

$$\phi(z, \alpha) = i \operatorname{Log} \frac{\operatorname{sh}(z + i\alpha)}{\operatorname{sh}(z - i\alpha)} \quad (3.11)$$

depending on the anisotropy parameter, instead of Eq. (3.5). That is, the NBAE are given by Eqs. (2.2) and (2.3) with (3.2) given by (3.10) or (3.11) and the Lie algebra parameters ω_k, ω_{jk} of A_{q-1} .

Let us conclude this section by deriving a few formulas that will be needed in the finite-size calculations of the next section. We start by the asymptotic behaviour of the vacuum density of roots $\sigma_{\infty}^{(\ell)}(\lambda)$. One finds from the Fourier representations (2.18)-(2.20)

$$\sigma_{\infty}^{(\ell)}(\lambda) = \sum_{k=1}^{\ell} \int_{-\infty}^{\infty} \frac{dx}{2\pi} A(x, \omega_k) [1 - \hat{K}(x)]^{-1} e^{i\lambda x} \quad (3.12)$$

Here $A(x, \alpha)$ is the Fourier transform of $\phi'(x, \alpha)$. So

$$A(x, \alpha) = e^{-|x\alpha|} \quad (3.13)$$

when Eq. (3.6) is used and

$$A(x, \alpha) = \frac{\operatorname{sh} x(\frac{\pi}{2} - \alpha)}{\operatorname{sh} \frac{x\pi}{2}} \quad (3.14)$$

when (3.11) holds. Equation (3.12) tells us that the large λ behaviour of $\sigma_{\infty}^{(\ell)}(\lambda)$ is determined by the zeros of $\det[1 - \hat{K}(x)]$ closer to the real axis. These values are clearly ℓ -independent. One finds from Eq. (3.12) by the residue method

$$\sigma_{\infty}^{(\ell)}(\lambda) = \frac{\kappa}{\pi} m_{\ell} e^{-\kappa|\lambda|} [1 + o(e^{-|\lambda|\delta})] \quad (3.15)$$

where $\delta > 0$. The parameters κ and m_{ℓ} are given in Table 2 for all models associated to simply-laced Lie algebras. κ is just 2π times the length square of

the shortest simple root in the normalization where¹⁹⁾

$$B(\epsilon_\alpha, \epsilon_{-\alpha}) = -1$$

and $B(x, y)$ is the Killing form. The coefficients may have the physical interpretation of particle masses in the context of the relativistic QFT with NBAE of the type (3.3). The chiral-invariant fermionic models⁸⁾ provide explicit realizations with such Lie algebraic structures. In addition, the mass spectrum has the invariance Δ of the corresponding Dynkin diagram, that is the automorphisms of the Lie algebra¹⁹⁾. Δ is trivial or isomorphic to Z_2 except for D_4 where it is the permutation group of three elements S_3 .

Although the detailed mass spectrum depends on the Lie algebra, it is true in all cases that m_λ increases monotonically going from the ends of the Dynkin diagrams to the middle.

In the case of the $U(1)^{r-1}$ symmetric model^{1e), 14)}, Eq. (3.14) holds with the m_λ and κ of the A_n symmetric theory renormalized by a factor $1/\gamma$.

4. - APPLICATIONS OF FINITE-SIZE COMPUTATIONS

A) Evaluation of the central charge for gapless models

Let us compute the leading finite-size correction to the free energy for a system solvable by NBA in the gapless regime. When the ground state is filled with real solutions of the NBAE, the formalism of Section 2 applies. As it is stated at the end of Section 2, the leading finite-size corrections are governed by the large λ behaviour of the integrand in Eq. (2.33) since we assume a zero gap. We know $\sigma_\infty^{(\lambda)}(\lambda)$ for large λ from Eq. (3.14) and we obtain for $t^{(\lambda)}(\lambda)$ [Eq. (2.6)]

$$t_\infty^{(\lambda)}(\lambda) = 2\pi q_e \Theta(\lambda) + \frac{m_e}{\pi} s_g(\lambda) e^{-\kappa|\lambda|} \quad (4.1)$$

$\lambda \rightarrow \pm\infty$

where

$$q_e = \int_{-\infty}^{+\infty} d\lambda \sigma_\infty^{(\lambda)}(\lambda) \quad (4.2)$$

is usually a rational number. This information is enough to compute the dominant large n behaviour of T_n^ℓ from Eq. (2.33). It is convenient to use the integration variable $t_\ell = t_\infty^{(\ell)}(\lambda_\ell)$ in Eq. (2.33). This yields for the free energy taking into account Eq. (2.28)

$$T_n^\ell(\theta) = \frac{\sin(x\theta)}{\pi n^2} + o\left(\frac{1}{n^3}\right) \quad (4.3)$$

where n is chosen such that q_n is an integer. Summation over α_ℓ and ℓ yield from Eqs. (2.28) and (2.31)

$$L_N(\theta) = - \frac{\pi r}{6 N^2} \sin(x\theta) \quad (4.4)$$

where we used $\zeta(2) = \pi^2/6$. Now, the free energy of a conformally invariant model with periodic boundary conditions in box of size L reads¹³⁾

$$f_L = f_\infty - \frac{\pi c}{6 L^2} + \text{smaller terms} \quad (4.5)$$

where c is the central charge value. However, one cannot blindly identify Eqs. (4.5) and (4.4). One must first verify the rotational invariance at least for long distances. Let us analyze the spectrum of elementary excitations for low momentum and energy. The eigenvalue of $\log \tau(\theta)$ reads, for a hole at $\phi \rightarrow -\infty$ in the ℓ^{th} branch ($1 \leq \ell \leq r$) from Eqs. (2.33) and (4.1)

$$f_\ell(\varphi) - f_\ell(-\infty) = - \frac{ime}{\pi} e^{x(\varphi+i\theta)} + o(e^{2x\varphi}) \quad (4.6)$$

Since the momentum operator is given here by

$$P = -i \log \tau(0) \quad (4.7)$$

one finds for this hole

$$p(\varphi) = - \frac{me}{\pi} e^{x\varphi} \quad (4.8)$$

[We have subtracted the constant K_ℓ such that $p(-\infty) = 0$]. In this context, the Hamiltonian can be identified with

$$H = - \text{Re} \log \tau(\theta) \quad (4.9)$$

So we find for the energy

$$\begin{aligned} \varepsilon &= - \lim_{\varphi \rightarrow -\infty} \frac{m\varphi}{\pi} e^{x\varphi} \sin(x\vartheta) + o(e^{2x\varphi}) \\ \varepsilon &= \lim_{p \rightarrow 0} p \sin(x\vartheta) + o(p^2) \end{aligned} \quad (4.10)$$

This indicates that we must renormalize the energy by a factor $1/\sin(x\vartheta)$ in order to have a relativistic dispersion law and hence rotational invariance for long distances. After this renormalization

$$L_N(\vartheta) \longrightarrow \tilde{L}_N(\vartheta) = \frac{1}{\sin x\vartheta} L_N(\vartheta) \quad (4.11)$$

Now, one finds from Eqs. (4.4) and (4.11)

$$c = r \quad (4.12)$$

that is, the value of c equals the number of steps in the NBA. Each gapless step contributes by one unity to the central charge. If there were some steps with a non-zero gap, they would give exponentially small corrections and no contribution to c . So the final result is that c equals the number of gapless steps of the NBA when the vacuum is formed by real roots.

B) Surface tension for a massive $q(2q-1)$ vertex model

Let us consider the finite-size corrections for the $q(2q-1)$ vertex model of Refs. 1e) and 14). In this model the links can be in q different states and the statistical weights are given in Fig. 1. For $\gamma \gtrsim 1$, $2\theta + \gamma$ fixed, the model has a long-range generalized ferroelectric order and the dominant configurations are formed mostly by vertices of type c_1 (for $\theta > 0$) and some c_{q-1} . There are q different predominant patterns following one from each other by shifting by one the state of all links. This generalizes the six-vertex ($q=2$) situation^{1c)}. The interfacial tension between domains of this type is then related by standard arguments^{1c)} to the ratio between the asymptotically degenerate largest eigenvalues of the transfer matrix. In this model we have q asymptotically degenerate eigenvalues²¹⁾ Λ^s ($0 \leq s \leq q-1$) and

$$\lim_{N \rightarrow \infty} \frac{\Lambda_s(\vartheta)}{|\Lambda_s(\vartheta)|} = e^{2\pi i s / q} \quad (4.13)$$

Now, one will expect

$$\frac{\Lambda_s(\theta)}{\Lambda_q(\theta)} = e^{2\pi i s/q} + O(e^{-NS'}) \quad (4.14)$$

where S is the interfacial tension divided by the absolute temperature.

Let us now apply the method of Section 2 to explicitly compute S . We start by looking at the stationary points of the integral (2.33) and (2.34), that is the zeros of the density of roots^{(1e), (14)}

$$\sigma_\infty^{(s)}(\lambda) = \frac{1}{\pi} \left\{ 1 - \frac{s}{q} + 2 \sum_{m=1}^{\infty} \frac{\cos(2m\lambda)}{\operatorname{sh}(m\gamma q)} \operatorname{sh}[m\gamma(q-s)] \right\}.$$

This function can be expressed in terms of elliptic functions. One finds

$$\sigma_\infty^{(s)}(\lambda) = \frac{1}{2\pi} \frac{dt_\infty^{(s)}}{d\lambda} \quad (4.15)$$

$$t_\infty^{(s)}(\lambda) = \frac{1}{i} \log \frac{\vartheta_4\left(\frac{\lambda}{\pi} + \frac{i\gamma}{2\pi}[q-s] \mid \frac{i\gamma q}{\pi}\right)}{\vartheta_4\left(\frac{\lambda}{\pi} - \frac{i\gamma}{2\pi}[q-s] \mid \frac{i\gamma q}{\pi}\right)} + \left(1 - \frac{s}{q}\right)(2\lambda + \pi), \quad (4.16)$$

where $\vartheta_4(z/\tau)$ can be found in Ref. 22). From general properties of the ϑ functions, it follows that the solutions of

$$\sigma_\infty^{(s)}(\lambda) = 0; \quad s = 1, 2, \dots, q-1, \quad (4.17)$$

have the form

$$\frac{\lambda_s}{\pi} = -\frac{1}{2} + \frac{i\gamma q}{4\pi} + \frac{i\Delta_s(\gamma)}{\pi} \quad (4.18)$$

where $\Delta_s(\gamma) = -\Delta_{q-s}(\gamma)$. For $q = 2$ we have $\Delta_1(\gamma) = 0$ and we recover the result of Ref. 2). Δ_s fulfils the equation

$$\frac{\vartheta_2'}{\vartheta_2}\left(i\Delta_s + \frac{i\gamma}{4\pi}(q-2s)\right) - \frac{\vartheta_2'}{\vartheta_2}\left(i\Delta_s + \frac{i\gamma}{4\pi}(q+2s)\right) = \frac{2\pi i s}{q} \quad (4.19)$$

This equation can easily be solved for small γ and for large γ . One finds:

$$\Delta_s(\gamma) = - \frac{\gamma \frac{q}{2}}{\pi} e^{-\frac{\pi^2}{\gamma^2}} \left\{ 1 - 2\left(1 - \frac{c^2}{3}\right) e^{-\frac{2\pi^2}{\gamma^2}} + 6\left(1 + \frac{c^4}{5}\right) e^{-\frac{4\pi^2}{\gamma^2}} + O\left(e^{-\frac{6\pi^2}{\gamma^2}}\right) \right\} \quad (4.20)$$

$\gamma \rightarrow 0^+$

where $c \equiv \cos(\pi s/q)$

$$\Delta_s(\gamma) = \frac{2s-q}{4} \gamma + \frac{1}{2} \log\left(\frac{q}{s} - 1\right) + \frac{q^2}{2} \left[\frac{e^{-2\gamma(q-s)}}{s^2} - \frac{e^{-2\gamma s}}{(q-s)^2} \right] + \text{higher orders} \quad (4.21)$$

$\gamma \rightarrow +\infty$

Then, using Eq. (4.16)

$$M_s(\gamma) \equiv e^{i t_{\infty}^{(s)}(\gamma)} = 1 - 4T e^{-\frac{\pi^2}{\gamma^2}} + 8T^2 e^{-\frac{2\pi^2}{\gamma^2}} + 8T(1+T^2) e^{-\frac{3\pi^2}{\gamma^2}} + O\left(e^{-\frac{4\pi^2}{\gamma^2}}\right) \quad (4.22)$$

$\gamma \rightarrow 0^+$

where $T \equiv \sin \pi s/q$ and

$$M_s(\gamma) = e^{-\frac{\gamma^2(q-s)}{q}} \frac{q}{s} \left(\frac{q}{s} - 1\right)^{-1+s/q} \left\{ 1 - (3q-s) \frac{e^{-2\gamma(q-s)}}{s} + \frac{s^2 + 2qs - q^2}{(q-s)^2} e^{-2\gamma s} + \text{higher orders} \right\} \quad (4.23)$$

$\gamma \rightarrow +\infty$

Now the saddle-point method applied to Eqs. (2.33) and (2.34) yields

$$T_m^l(\theta) = \frac{(M_e)^m}{\sqrt{m}} \left[1 + O\left(\frac{1}{m}\right) \right] \quad (4.24)$$

$m \gg 1$

where D_ℓ is n independent. Then the finite-size corrections to the free energy (L_N) will be dominated by the larger M_ℓ , $1 \leq \ell \leq q$. Equations (4.22) and (4.23) indicate that $M_1 = M_{q-1} > M_\ell$ for $2 \leq \ell \leq q-2$. So, we have

$$L_N(\theta) = \frac{D_1}{\sqrt{N}} M_1(\gamma)^N \quad (4.25)$$

Comparison of Eqs. (4.14) and (4.25) yields for the interfacial tension

$$\sigma' = - \log M_1(\gamma) \quad (4.26)$$

The series (4.22)-(4.23) indicate that this quantity is always positive. It grows linearly with γ for large γ and vanishes as $\exp(-\pi\gamma/\delta)$ in the critical limit $\gamma \rightarrow 0^+$.

Table 1

Simply laced Lie algebras

Table 2

Lie algebra	χ	m_ℓ	Δ	q_ℓ
A_n	$\frac{2\pi}{n+1}$	$\sin \frac{\pi \ell}{n+1}, \quad 1 \leq \ell \leq n$	$\ell \leftrightarrow n + 1 - \ell$	$\frac{n+1-\ell}{n+1}$
D_n	$\frac{\pi}{(n-1)}$	$\sin \frac{\pi \ell}{2(n-1)}, \quad 1 \leq \ell \leq n-2$ $m_\pm = \frac{1}{2}$	$+\leftrightarrow -$ $n \neq 4$	$1, \quad 1 \leq \ell \leq n-2$ $\frac{1}{2}, \pm$
E_6	$\pi/6$	$m_1 = m_5 = \sqrt{3}/2, \quad m_2 = m_4 = \frac{3+\sqrt{3}}{2}$ $m_3 = \frac{3+\sqrt{3}}{\sqrt{2}}, \quad m_6 = \sqrt{3}$	$1 \leftrightarrow 5$ $2 \leftrightarrow 4$	$q_1 = 2 - q_5 = \frac{4}{3}$ $q_2 = 3 - q_4 = \frac{5}{3}$ $q_3 = 2, \quad q_6 = 1$
E_7	$\pi/9$	(*)	None	
E_8	$\pi/15$	(*)	None	

The parameters χ and m_ℓ (mass spectrum) for the models associated to simply-laced Lie algebras. For D_4 , notice that $m_1 = m_\pm = \frac{1}{2}$ and then $\Delta = S_3$.

(*) These values can be extracted from Ref. 12).

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$$e^{\Theta \operatorname{sign}(a-b)} \operatorname{sh} \gamma = c_{a-b}$$

$$\operatorname{sh} \Theta = b$$

$$\operatorname{sh}(\Theta + \gamma) = a$$

Fig. 1