Equivalence of Dirac Quantization and
Schwinger's Action Principle Quantization

- 6 APR. 1987

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Abstract

We show that the method of Dirac quantization is equivalent to Schwinger's action principle quantization. The relation between the Lagrange undetermined multipliers in Schwinger's method and Dirac's constraint bracket matrix is established and it is explicitly shown that the two methods yield identical (anti)commutators. This is demonstrated in the non-trivial example of supersymmetric quantum mechanics in superspace.

I. Introduction

There are various ways of obtaining canonical (anti)commutation relations in a quantum theory involving constraints. One such method is due to Dirac[1], who developed a systematic treatment of constrained Hamiltonian dynamics, which provides consistent (anti)commutation relations for the quantum theory corresponding to the constrained classical theory. This construction of the (anti)commutation relations is known as Dirac quantization.

Almost simultaneously an alternative method for constructing quantum brackets via a quantum action principle was developed by Schwinger[2]. Rather than with a classical phase space this method works with quantum states and operators from the beginning and the (anti)commutation relations for the theory are derived within this context. This is called Schwinger's action principle quantization and can be applied to constrained systems as well. This later method is not quite a canonical one although its constructions resemble the classical picture very much.

The two methods are seemingly different, yet when applied to any physical system they should yield the same (anti)commutation relations. However, a general proof for this equivalence has been lacking so far. In this paper we present such a proof in which we establish the relation between the Lagrange undetermined multipliers in Schwinger's method and the constraint bracket matrix needed to define the Dirac brackets. We also show explicitly that the two methods lead to identical (anti) commutators. The proof is given for first order systems, but the result applies in general since all quantum systems can be described in terms of a first order Lagrangian[2].

In Sec. II we briefly review what exactly is meant by Dirac quantization. In Sec. III we discuss the Schwinger action principle approach and present the above mentioned proof. For reasons of clarity we show the equivalence of the two methods for systems with a finite number of degrees of freedom, but the extension to field theories is straightforward[10].

Recently it was shown[5] that the method of Dirac quantization can be extended to superspace in a non-trivial manner and for that reason we also prove the above stated equivalence for supersymmetric quantum mechanics formulated in superspace. This is done in Sec. IV.

II. Dirac Quantization

As pointed out in Sec. I it is sufficiently general to consider only Lagrangians of the form

$$L = q^i A_{ij} \dot{q}^j - H(q) \quad (2.1)$$

where the $q^i (i=1, \dots, N)$ are the N coordinates of the configuration space Q ,

$\dot{q}^i = \frac{dq^i}{dt}$ their respective velocities, A is a constant anti-Hermitian

matrix and H is a function of the q 's only[2]. In (2.1) and in what follows the summation convention is employed.

We can describe the system equivalently by the Lagrangian

$$L = \frac{1}{2} \left(q^i A_{ij} \dot{q}^j - \dot{q}^i A_{ij} q^j \right) - H(q) \quad (2.2)$$

Lagrangians as in (2.2) give rise to N primary Hamiltonian constraints

$$\Gamma_\alpha(q, p) := p_\alpha - \frac{1}{2} q^\beta A_{\beta\alpha} + \frac{1}{2} A_{\alpha\beta} q^\beta = 0 \quad (\alpha, \beta=1, \dots, N) \quad (2.3)$$

(where the momentum p_i is defined as usual: $p_i := \frac{\partial L}{\partial \dot{q}^i}$) and some T ℓ -ary

($\ell \geq 2$) Hamiltonian constraints[6]

$$\Gamma_\alpha(q) = 0 \quad (\alpha = N+1, \dots, N+T) \quad (2.4)$$

For the canonical transition to the quantum theory we wish to remove all arbitrariness in the time evolution of the system. This may require the imposition of additional constraints[3,4]

$$\Gamma_\alpha(q, p) = 0 \quad (\alpha = N+T+1, \dots, M) \quad (2.5)$$

such that all constraints are satisfied at all times and that $\det C_{\alpha\beta} \neq 0$ where

$$C_{\alpha\beta} \approx \{\Gamma_\alpha, \Gamma_\beta\} \quad (\alpha, \beta = 1, \dots, M) \quad (2.6)$$

and $\approx$ means weakly equal. Here $\{A, B\}$ are the usual Poisson brackets.

We can then define the Dirac brackets[2,3,4]

$$\{A, B\}_D := \{A, B\} - \{A, \Gamma_\alpha\} C_{\alpha\beta}^{-1} \{\Gamma_\beta, B\} \quad (2.7)$$

for any phase space functions A and B .

In the quantum theory corresponding to this dynamical system one then represents the phase space functions $A(q, p)$ by some hermitian operators $\hat{A}$ acting on vectors in a Hilbert space H . The (anti)commutation relations of these operators are prescribed by the Dirac brackets[1]:

$$[\hat{A}, \hat{B}] = i\hbar \{A, B\}_D \quad (2.8)$$

The procedure for systems with infinitely many degrees of freedom is similar and the extension to systems with anticommuting coordinates does

not pose any difficulty either[3,5].

It should be noted that this procedure does not specify the Hilbert space H of the states of the system, neither the mapping $A \rightarrow \hat{A}$ of classical observables to quantum operators. Neither Dirac's method nor Schwinger's method provides this information[1] and one might argue that the term quantization is not fully justified. For the purpose of this publication we may assume the mapping $A \rightarrow \hat{A}$ and H to be known. The program of geometric quantization[7] provides these specifications (of H and the mapping $\hat{\cdot}$). Its starting point is a symplectic manifold and it can be shown that the Dirac bracket in (2.7) is essentially the symplectic form on the final constraint submanifold[8] here defined by $\Gamma_a = 0 (a=1, \dots, M)$.

Eqns. (2.7) and (2.8) therefore give the crucial elements for a quantization and the evaluation of the Dirac brackets together with the prescription given in (2.8) is what we call Dirac quantization. Finally we remark that for theories in superspace the matrix $C_{a\beta}$ is generally not invertible. However, with suitable modifications Dirac quantization can be extended to theories in superspace [5].

III. Schwinger's Action Principle Quantization

In this section we will show that the (anti)commutation relations derived from the action principle quantization as proposed by Schwinger[2] are identical with the ones given by (2.8).

From the operator principle of stationary action[2] for a quantum system with N degrees of freedom one infers the operator equations of motion

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right)^{\wedge} = \left(\frac{\partial L}{\partial q^i} \right)^{\wedge} \quad (i=1, \dots, N) \quad (3.1)$$

and the generators of the variations

$$\begin{aligned} \hat{G}_{\delta q} &= \hat{p}_i \delta q^i \\ G_{\delta p} &= \delta p_i \hat{q}^i \\ \hat{G}_{\delta t} &= \hat{H} \delta t \end{aligned} \quad (3.2)$$

Here the $\hat{q}^i$ and $\hat{p}_i := \left(\frac{\partial L}{\partial \dot{q}^i} \right)^{\wedge}$ are considered to be hermitian operators acting

on the quantum mechanical state space, i.e. the Hilbert space H . All functions of $\hat{q}$ and $\hat{p}$ are operators as well (the Lagrange operator

$L(\hat{q}, \dot{\hat{q}})$ has to be hermitian[2]) and for any such operator $\hat{A}(\hat{q}, \hat{p})$ one has the (anti)commutation relation

$$[\hat{A}, \hat{G}_{\delta}] = i\hbar \delta_G \hat{A} \quad (3.3)$$

Let us first examine (3.3) in the absence of any constraints. In this case the right hand side of (3.3) is an unconstrained variation and considering only variations proportional to the identity operator one finds using (3.2):

$$\begin{aligned} [\hat{A}, \hat{p}_i] &= i\hbar \left(\frac{\partial A}{\partial q^i} \right)^{\wedge} \\ [\hat{A}, \hat{q}^i] &= -i\hbar \left(\frac{\partial A}{\partial p_i} \right)^{\wedge} \end{aligned} \quad (3.4)$$

$$[\hat{A}, \hat{H}] = i\hbar \frac{d\hat{A}}{dt}$$

where we have excluded operators $\hat{A}$ which depend explicitly on time. Also the non-trivial question of ordering on the right hand side shall not concern us here.

The obvious extension of (3.3) to arbitrary generators $\hat{B}(\hat{q}, \hat{p})$ is

$$\begin{aligned} [\hat{A}, \hat{B}] &= i\hbar \delta_{\hat{B}}^{\hat{A}} \\ &= i\hbar \left(\left(\frac{\partial \hat{A}}{\partial \hat{q}_1} \right)^{\wedge} \delta_{\hat{B}}^{\hat{q}_1} + \left(\frac{\partial \hat{A}}{\partial \hat{p}_1} \right)^{\wedge} \delta_{\hat{B}}^{\hat{p}_1} \right) \end{aligned} \quad (3.5)$$

Here we use that, by definition (3.3)

$$i\hbar \delta_{\hat{B}}^{\hat{q}_1} = [\hat{q}_1^{\wedge}, \hat{B}] = -[\hat{B}, \hat{q}_1^{\wedge}] = i\hbar \left(\frac{\partial \hat{B}}{\partial \hat{p}_1} \right)^{\wedge}$$

where we have used (3.4) in the last step. Similarly one finds

$$i\hbar \delta_{\hat{B}}^{\hat{p}_1} = -i\hbar \left(\frac{\partial \hat{B}}{\partial \hat{q}_1} \right)^{\wedge}$$

such that (3.5) now becomes:

$$\begin{aligned} [\hat{A}, \hat{B}] &= i\hbar \left(\left(\frac{\partial \hat{A}}{\partial \hat{q}_1} \right)^{\wedge} \left(\frac{\partial \hat{B}}{\partial \hat{p}_1} \right)^{\wedge} - \left(\frac{\partial \hat{A}}{\partial \hat{p}_1} \right)^{\wedge} \left(\frac{\partial \hat{B}}{\partial \hat{q}_1} \right)^{\wedge} \right) \\ &= i\hbar \{A, B\}^{\wedge} \end{aligned} \quad (3.6)$$

Eq. (3.6) is just the canonical quantization rule for systems with no constraints. In other words Schwinger's action principle quantization is equivalent to the canonical quantization for systems with no constraints.

Let us now consider the case of a constrained system with a Lagrangian as given in (2.2) where $\hat{H}$ has to be hermitian to assure the hermiticity of $\hat{L}$. We wish to retain the $\hat{p}_i$ as canonical quantum variables, however, there are the same N constraints as the primary Hamiltonian ones in (2.3):

$$\hat{\Gamma}_a(\hat{q}, \hat{p}) = \hat{0} \quad (a=1, \dots, N) \quad (3.7)$$

where $\hat{0}$ is the null operator. Moreover, it has been shown that from the requirement of the solubility of the Euler-Lagrange equations (3.1) alone the same T constraints

$$\hat{\Gamma}_a(\hat{q}) = \hat{0} \quad (a=N+1, \dots, N+T) \quad (3.8)$$

as the ℓ -ary ($\ell \geq 2$) Hamiltonian constraints of (2.4) follow[6].

The right hand side of (3.3) now has to be treated as a constrained variation and for this the method of Lagrange undetermined multipliers $\hat{\Lambda}_a$ is applied[2]*:

$$\begin{aligned} \frac{1}{i\hbar} [\hat{A}, \hat{B}]_S &= (c) \delta_B^{\hat{A}} \\ &= \delta_B^{\hat{A}} + \hat{\Lambda}_a(\hat{A}) \delta_B^{\hat{\Gamma}_a} \quad (a=1, \dots, N+T) \end{aligned}$$

Here the subscript (c) stands for constrained variations on the right hand side and the subscript $[\cdot]_S$ denotes the modified (anti)commutators in the presence of constraints. By introducing the Lagrange undetermined multipliers we have written the constrained variation $(c) \delta_B^{\hat{A}}$ in terms of the unconstrained ones

$$\delta_B^{\hat{A}} = \frac{1}{i\hbar} [\hat{A}, \hat{B}]$$

and

$$\delta_B^{\hat{\Gamma}_a} = \frac{1}{i\hbar} [\hat{\Gamma}_a, \hat{B}] \quad ,$$

such that we can write the consistent (anti)commutator as

* We wish to thank Prof. C. R. Hagen for clarification of this point.

$$[\hat{A}, \hat{B}]_S = [\hat{A}, \hat{B}] + \hat{\Lambda}_\alpha(\hat{A}) [\hat{\Gamma}_\alpha, \hat{B}] \quad (3.9)$$

Here we may use (3.6):

$$[\hat{A}, \hat{B}]_S = i\hbar(\{\hat{A}, \hat{B}\} + \hat{\Lambda}_\alpha(\hat{A}) \{\hat{\Gamma}_\alpha, \hat{B}\}) \quad (3.10)$$

The consistent (anti)commutator must satisfy

$$[\hat{A}, \hat{\Gamma}_\alpha]_S = [\hat{A}, \hat{0}]_S = \hat{0}$$

for all $\alpha=1, \dots, N+T$. Using (3.10) this leads to

$$\hat{\Lambda}_\beta(\hat{A}) \{\hat{\Gamma}_\beta, \hat{\Gamma}_\alpha\} = -\{\hat{A}, \hat{\Gamma}_\alpha\} \quad (\alpha, \beta=1, \dots, N+T) \quad (3.11)$$

From (3.11) we see that the matrix $\{\hat{\Gamma}_\alpha, \hat{\Gamma}_\beta\}$ has to be invertible for all multipliers to be determined. If the matrix $\{\hat{\Gamma}_\alpha, \hat{\Gamma}_\beta\}$ does not have maximal rank some of the multipliers $\hat{\Lambda}_\beta(\hat{H})$ remain arbitrary, which leads to arbitrariness in the time evolution of the operators $\hat{q}^1$ and $\hat{p}_1$ since, e.g.

$$i\hbar \frac{d\hat{q}^1}{dt} = [\hat{q}^1, \hat{H}]_S = [\hat{q}^1, \hat{H}] - \hat{\Lambda}_\beta(\hat{H}) [\hat{\Gamma}_\beta, \hat{q}^1] \quad .$$

We remove this unphysical arbitrariness as in Sec. II by the introduction of additional constraints similar to the ones in (2.4)

$$\hat{\Gamma}_\alpha(\hat{q}, \hat{p}) = \hat{0} \quad (\alpha=N+T+1, \dots, M)$$

such that $C_{\alpha\beta} = \{\hat{\Gamma}_\alpha, \hat{\Gamma}_\beta\}$ ($\alpha, \beta=1, \dots, M$) is invertible. Accordingly we will have more Lagrange multipliers, but they will all become determined now.

The consistent (anti)commutator is now given by

$$[\hat{A}, \hat{B}]_S = [\hat{A}, \hat{B}] + \hat{\Lambda}_\alpha(\hat{A}) [\hat{\Gamma}_\alpha, \hat{B}] \quad (\alpha=1, \dots, M) \quad (3.12)$$

As before in (3.11) the requirement of consistency leads to

$$\hat{A}_\beta(\hat{A}) \hat{C}_{\beta\alpha} = -\{A, \Gamma_\alpha\}^\wedge \quad (\alpha, \beta=1, \dots, M) \quad (3.13)$$

where now $C_{\beta\alpha}$ is invertible and thus

$$\hat{A}_\alpha(\hat{A}) = -(\{A, \Gamma_\beta\} \bar{C}_{\beta\alpha}^1)^\wedge \quad (3.14)$$

For simple examples this has been verified in ref. [9]. Reinserting this in (3.12) and making use of (3.6) gives

$$\begin{aligned} [\hat{A}, \hat{B}]_S &= [\hat{A}, \hat{B}] - (\{A, \Gamma_\alpha\} \bar{C}_{\alpha\beta}^1)^\wedge [\hat{\Gamma}_\beta, \hat{B}] \\ &= i\hbar(\{A, B\}^\wedge - (\{A, \Gamma_\alpha\} \bar{C}_{\alpha\beta}^1)^\wedge \{\hat{\Gamma}_\beta, B\}^\wedge) \\ &= i\hbar(\{A, B\} - \{A, \Gamma_\alpha\} \bar{C}_{\alpha\beta}^1 \{\hat{\Gamma}_\beta, B\})^\wedge \\ &= i\hbar\{A, B\}_D^\wedge \end{aligned} \quad (3.15)$$

where we have used the definition of the Dirac bracket (2.7) in the last line. Eq. (3.15) states that the (anti)commutator consistent with the constraints obtained by Schwinger's action principle is identical to the (anti)commutator obtained by Dirac quantization (2.8). This shows that the two methods of quantization are in fact equivalent.

As can be seen in (3.14), the inversion of the matrix $C_{\alpha\beta}$ amounts to finding the Lagrange "undetermined" multipliers that satisfy the consistency requirement.

Finally it is clear that an extension of this proof to systems with an infinite number of degrees of freedom follows the same steps as outlined above.

IV. The Equivalence in Superspace

Recently it has been shown that the method of Dirac quantization can be extended to theories in superspace[5]. Schwinger's action principle

quantization can be applied to such theories as well and here too both methods lead to the same (anti)commutation relations.

We will demonstrate this in the case of supersymmetric quantum mechanics whose Dirac quantization has been done in ref. [5] (whose notation we also employ here).

The action for supersymmetric quantum mechanics in superspace is

$$S = \int dt d\theta d\bar{\theta} \left(-\frac{1}{2} \bar{D}\Phi^1 D\Phi^1 - V(\Phi) \right)$$

and in order to write it as a first order system we introduce N auxilliary fields[6] ϕ^1 :

$$\begin{aligned} L(\Phi^1, \phi^1; \dot{\Phi}^1) = & \bar{\theta}\theta\dot{\Phi}^1\phi^1 - \frac{1}{2}\bar{\theta}\theta\dot{\phi}^1\phi^1 + \frac{1}{2}(\bar{\theta}\dot{\Phi}^1\bar{\theta}\phi^1 - \theta\dot{\phi}^1\theta\phi^1) \\ & + \frac{1}{2}\bar{\theta}\dot{\phi}^1\theta\phi^1 - V(\phi^1) \end{aligned} \quad (4.1)$$

where we introduced the notation $\partial := \frac{\partial}{\partial\theta}$ and $\bar{\partial} := \frac{\partial}{\partial\bar{\theta}}$.

Besides the supersymmetries

$$\delta\Phi^1 = \xi(\partial + i\theta\partial_t)\Phi^1, \quad \delta\phi^1 = \xi\bar{\partial}\Phi^1$$

$$\bar{\delta}\Phi^1 = \bar{\xi}(\bar{\partial} - i\bar{\theta}\partial_t)\Phi^1, \quad \bar{\delta}\phi^1 = \bar{\xi}\partial\phi^1$$

the Lagrangian in (4.1) also possesses the following invariance:

$$\delta\Phi^1 = 0, \quad \delta\phi^1 = \alpha(t)\theta + \beta(t)\bar{\theta} + \gamma(t)\bar{\theta}\theta \quad (4.2)$$

One finds with $\Pi^1 := \frac{\partial L}{\partial\dot{\Phi}^1}$ and $\pi^1 := \frac{\partial L}{\partial\dot{\phi}^1}$ the following constraints:

$$\Gamma_1^1 = \delta^2(\theta) \Pi^1$$

$$\Gamma_2^1 = \delta^2(\theta) \left(\partial \bar{\partial} \phi^1 - \frac{\partial v}{\partial \phi^1} \right)$$

$$\Gamma_3^1 = \delta^2(\theta) \partial (\Pi^1 + \frac{1}{2} \phi^1)$$

$$\Gamma_4^1 = \delta^2(\theta) \bar{\partial} (-\Pi^1 + \frac{1}{2} \phi^1)$$

$$\Gamma_5^1 = \delta^2(\theta) \pi^1 \quad (4.3)$$

$$\Gamma_6^1 = \delta^2(\theta) (\partial \bar{\partial} \Pi^1 - \phi^1)$$

$$\Gamma_7^1 = \delta^2(\theta) \partial \pi^1, \quad \Gamma_8^1 = \delta^2(\theta) \partial \phi^1$$

$$\Gamma_9^1 = \delta^2(\theta) \bar{\partial} \pi^1, \quad \Gamma_{10}^1 = \delta^2(\theta) \bar{\partial} \phi^1$$

$$\Gamma_{11}^1 = \delta^2(\theta) \partial \bar{\partial} \pi^1, \quad \Gamma_{12}^1 = \delta^2(\theta) \partial \bar{\partial} \phi^1$$

Here Γ_8^1 , Γ_{10}^1 and Γ_{12}^1 are "gauge fixing" constraints which are imposed to eliminate the arbitrariness coming from the three first class primary constraints Γ_7^1 , Γ_9^1 and Γ_{11}^1 , which in turn stem from the invariance given in (4.2).

Applying the method described in ref. [5] one obtains the following commutation relations from Dirac quantization:

$$[\hat{\phi}^1(\tau, \theta, \bar{\theta}), \hat{\phi}^j(\tau, \theta', \bar{\theta}')] = \hbar \delta^{1j} (\bar{\theta} \theta' - \bar{\theta}' \theta) \hat{1}$$

$$[\hat{\Pi}^1(\tau, \theta, \bar{\theta}), \hat{\Pi}^j(\tau, \theta', \bar{\theta}')] = \frac{\hbar}{4} \delta^{1j} (\bar{\theta} \theta' - \bar{\theta}' \theta) \hat{1} \quad (4.4)$$

$$\begin{aligned}
[\hat{\phi}^i(t, \theta, \bar{\theta}), \hat{\Pi}^j(t, \theta', \bar{\theta}')] &= -\frac{i\hbar}{2} \delta^{ij} (\bar{\theta}\theta' + \bar{\theta}'\theta - 2\bar{\theta}'\theta') \hat{I} + \\
&\quad + i\hbar \delta^2(\theta) \delta^2(\theta') \left(\frac{\partial^2 v}{\partial \phi^i \partial \phi^j} \right)^\wedge \\
[\hat{\phi}^i(t, \theta, \bar{\theta}), \hat{\phi}^j(t, \theta', \bar{\theta}')] &= i\hbar \delta^{ij} \delta^2(\theta') \left(\frac{\partial^2 v}{\partial \phi^i \partial \phi^j} \right)^\wedge
\end{aligned}$$

Here $\hat{I}$ is the identity operator on H . We will now illustrate how the same commutators arise from action principle quantization.

The generators for the different variations are

$$\hat{G}_{\delta\phi} = \int d^2\theta' \hat{\Pi}^j \delta\phi^j$$

$$\hat{G}_{\delta\phi} = \int d^2\theta' \hat{\pi}^j \delta\phi^j$$

$$\hat{G}_{\delta\Pi} = - \int d^2\theta' \hat{\phi}^j \delta\Pi^j$$

$$\hat{G}_{\delta\pi} = - \int d^2\theta' \hat{\phi}^i \delta\pi^i$$

where $d^2\theta' = d\theta' d\bar{\theta}'$ and we consider only variations proportional to the unit operator. The consistent commutator $[\hat{\phi}^i, \hat{\Pi}^j]_S$ is then derived from the action principle by

$$\frac{1}{i\hbar} [\hat{\phi}^j(t, \theta, \bar{\theta}), \hat{G}_{\delta\phi}]_S = \delta\phi^j(t, \theta, \bar{\theta}) + \int d^2\theta' \hat{\Lambda}_{\alpha, \phi^j}(\theta, \bar{\theta}; \theta', \bar{\theta}') \delta_{\phi} \hat{\Gamma}_{\alpha} \quad (4.5)$$

Inserting the form of the generator given above and keeping in mind that all variations are independent we find from (4.5)

$$\begin{aligned}
\frac{1}{i\hbar} [\hat{\Phi}^j(t, \theta, \bar{\theta}), \hat{\Pi}^i(t, \theta', \bar{\theta}')]_S &= \delta^{ij} \delta^2(\theta - \theta') \hat{I} + \hat{\Lambda}_{\Gamma_2^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) \\
&- \hat{\Lambda}_{\Gamma_2^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) \left(\frac{\partial^2 v}{\partial \Phi^i \partial \Phi^j} \right) \delta^2(\theta') \\
&- \frac{1}{2} \hat{\Lambda}_{\Gamma_3^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) \bar{\theta}' \\
&+ \frac{1}{2} \hat{\Lambda}_{\Gamma_4^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) \theta'
\end{aligned} \tag{4.6}$$

Multiplying (4.6) by $\delta^2(\theta')$ we find

$$\delta^{ij} \hat{I} \delta^2(\theta) \delta^2(\theta') + \hat{\Lambda}_{\Gamma_2^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) \delta^2(\theta') =$$

$$\frac{1}{i\hbar} [\hat{\Phi}^j(t, \theta, \bar{\theta}), \hat{\Gamma}_1^i]_S = 0$$

and consequently

$$\hat{\Lambda}_{\Gamma_2^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) = -\delta^{ij} \hat{I} \delta^2(\theta) \tag{4.7}$$

In order to determine $\hat{\Lambda}_{\Gamma_3^i, \Phi^j}$ and $\hat{\Lambda}_{\Gamma_4^i, \Phi^j}$ we consider

$$\begin{aligned}
\frac{1}{i\hbar} [\hat{\Phi}^j(t, \theta, \bar{\theta}), \hat{G}_{\Pi}]_S &= \int d^2\theta' (\hat{\Lambda}_{\Gamma_1^i, \Phi^j}(\theta, \bar{\theta}; \theta' \bar{\theta}') \delta_{\Pi} \hat{\Gamma}_1^i \\
&+ \hat{\Lambda}_{\Gamma_3^i, \Phi^j}(\theta, \bar{\theta}; \theta', \bar{\theta}') \delta_{\Pi} \hat{\Gamma}_2^i + \hat{\Lambda}_{\Gamma_4^i, \Phi^j}(\theta, \bar{\theta}; \theta', \bar{\theta}') \delta_{\Pi} \hat{\Gamma}_3^i \\
&+ \hat{\Lambda}_{\Gamma_6^i, \Phi^j}(\theta, \bar{\theta}; \theta', \bar{\theta}') \delta_{\Pi} \hat{\Gamma}_6^i)
\end{aligned}$$

from which we derive

$$\begin{aligned}
\frac{1}{i\hbar} [\hat{\Phi}^j(\tau, \theta, \bar{\theta}), \hat{\Phi}^i(\tau, \theta', \bar{\theta}')]_S &= -\hat{\Lambda}_{\Gamma_1^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) \delta^2(\theta') \\
&+ \hat{\Lambda}_{\Gamma_3^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) \bar{\theta}' + \hat{\Lambda}_{\Gamma_4^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) \theta' \\
&+ \hat{\Lambda}_{\Gamma_6^i, \Phi^j}(\theta, \bar{\theta}; 0, 0)
\end{aligned}$$

and consequently

$$\begin{aligned}
\frac{1}{2} \delta^2(\theta') (-)^{\hat{A}} \hat{\Lambda}_{\Gamma_4^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) &= \frac{1}{i\hbar} [\hat{\Phi}^j(\tau, \theta, \bar{\theta}), \frac{1}{2} \delta^2(\theta') \partial' \hat{\Phi}^i(\tau, \theta', \bar{\theta}')]_S \\
&= \frac{1}{i\hbar} [\hat{\Phi}^j(\tau, \theta, \bar{\theta}), \hat{\Gamma}_3^i]_S - \frac{1}{i\hbar} [\hat{\Phi}^j(\tau, \theta, \bar{\theta}), \delta^2(\theta') \partial' \hat{\Pi}^i(\tau, \theta', \bar{\theta}')]_S \\
&= -\frac{1}{i\hbar} \delta^2(\theta') \partial' [\hat{\Phi}^j(\tau, \theta, \bar{\theta}), \hat{\Pi}^i(\tau, \theta', \bar{\theta}')]_S \\
&= -i \delta^{ij} \delta^2(\theta') \bar{\theta} - \frac{1}{2} \delta^2(\theta') (-)^{\hat{A}} \hat{\Lambda}_{\Gamma_4^i, \Phi^j}(\theta, \bar{\theta}; 0, 0)
\end{aligned} \tag{4.8}$$

where we used the notation

$$(-)^{\hat{A}} = \begin{cases} \hat{A} & \text{if } \hat{A} \text{ is bosonic} \\ -\hat{A} & \text{if } \hat{A} \text{ is fermionic} \end{cases}$$

and (4.6) was used to arrive at the last line. From this we find

$$i \hat{\Lambda}_{\Gamma_4^i, \Phi^j}(\theta, \bar{\theta}; 0, 0) = \delta^{ij} \hat{\Gamma} \bar{\theta} \tag{4.9}$$

Actually $\hat{\Lambda}_{\Gamma_4^i, \Phi^j}$ is not uniquely determined by (4.8) (terms containing

$\theta, \bar{\theta}, \bar{\theta}\theta$ might be added), but the non-unique parts in $\hat{\Lambda}_{\Gamma_4^i, \Phi^j}$ are irrelevant

for the brackets since the multipliers always appear multiplied by $\delta^2(\theta)$.

This is similar to what is encountered in the extension of Dirac quantization to superspace[5].

Similarly employing the consistency conditions one finds

$$i\hat{A}_{\Gamma_3^1, \Phi^j}(\theta, \bar{\theta}; 0, 0) = \delta^{ij} \hat{I}\theta \quad (4.10)$$

Inserting the solutions for the multipliers in (4.6) we find:

$$\begin{aligned} \frac{1}{i\hbar} [\hat{\Phi}^j(t, \theta, \bar{\theta}), \hat{\Pi}^i(t, \theta', \bar{\theta}')]_S = & -\frac{1}{2} (\bar{\theta}\theta' + \bar{\theta}'\theta - 2\bar{\theta}'\theta') \delta^{ij} \hat{I} \\ & + \delta^2(\theta) \delta^2(\theta') \left(\frac{\partial^2 v}{\partial \Phi^i \partial \Phi^j} \right)^{\wedge} \end{aligned} \quad (4.11)$$

which is the same commutator as in (4.4). Similarly one reproduces the other commutators given in (4.4) via Schwinger's action principle quantization. This concludes our illustration of the equivalence of the two methods.

V. Conclusion

We have shown that Dirac quantization and Schwinger's action principle quantization are equivalent. This is proven by demonstrating that the two methods give identical quantum (anti)commutation relations. Moreover we have shown that this equivalence extends to theories in superspace as well.

Acknowledgements

This work is supported in part by the U.S. Department of Energy Contract No. DE-AC02-76ER13065. One of us (A.D.) is supported by an Outstanding Junior Investigator Award.

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