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A B S T R A C T

We investigate the evolution of higher dimensional, homogeneous, anisotropic cosmologies. The natural extension to higher dimensions of the 3-dimensional mixmaster cosmologies are analysed, and it is found -- contrary to the 3-dimensional case -- that the higher dimensional cosmologies exhibit no chaotic behaviour.

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1. INTRODUCTION

In the past few years higher dimensional cosmologies have received a great deal of attention [1]. This is due to the theoretical possibility of a unification of the fundamental forces in a higher dimensional space-time. In these theories the extra dimensions are usually assumed to compactify at some energy scale of the order of the Planck mass, while the remaining dimensions expand into our observable universe. In the type of models commonly considered, the extra dimensions are curled up in a maximally symmetric n -sphere. This rather arbitrary assumption is usually made because of the simple geometry of maximally symmetric spaces. It is well known from 3+1 dimensions that even if one considers a less symmetric geometry, there are interesting models, such as the mixmaster cosmologies [2], which can evolve into our observable universe but which, close to the initial singularity, enter an infinite succession of Kasner epochs during which the three scale factors oscillate with arbitrarily large amplitude and frequency. It is also well known that this type of chaotic behaviour is exhibited in the most general (inhomogeneous) cosmologies close to the initial singularity [3]. Recently it has been shown that the same result is true for spatial dimensions between four and nine, and that the chaotic behaviour disappears for dimension ten and above [4]. For $n+1$ dimensional cosmology with $n \geq 10$, the general approach to the singularity is the generalized Kasner solution. In this case the universe enters a finite succession of Kasner epochs, of which the last one is stable and determines the approach to the singularity. For homogeneous cosmologies the problem of the approach to the singularity is still unsolved for $4 \leq n \leq 9$, where n is the spatial dimension. Owing to the constraints imposed by the structure constants it is expected that even for $4 \leq n \leq 9$ the general approach to the singularity is the Kasner solution, i.e. a finite succession of Kasner epochs. In this paper, we shall show that this is indeed the case for a large class of homogeneous cosmologies. All the examples of models previously examined in the literature [5] fall within this class, and our result -- obtained in a simpler way -- agrees in each case with those obtained in the literature.

2. HOMOGENEOUS COSMOLOGY

The extension to $n+1$ dimensions of the usual 3+1 dimensional Bianchi models consists of the cosmologies with a one-parameter family of n -dimensional spatial hypersurfaces with a Lie Group acting simply transitively, i.e. the group motion may carry any point into any other point. The Lie Group G has n generators X_a , $a = 1, \dots, n$ and corresponding invariant forms $e^a = e^a_\mu dx^\mu$, $\mu = 1, \dots, n$, which satisfy $de^a = -1/2 C^a_{bc} e^b \wedge e^c$, where C^a_{bc} are the structure constants of G . We shall assume that the metric is diagonal in the e^a -basis:

$$ds^2 = dt^2 - R_a^2 (e^a)^2 .$$

Then we find the following 0-0 and a-a Einstein equations in vacuum:

$$\sum_a \left(\frac{\dot{R}_a}{R_a} \right) + \left(\frac{\dot{R}_a}{R_a} \right)^2 = 0 \quad (1a)$$

$$\frac{1}{2\sqrt{g}} \left(\sqrt{g} \frac{\dot{R}_a}{R_a} \right) = -P^a_a \quad (\text{no sum}) , \quad (1b)$$

where $g = R_a^2 \dots R_n^2$ and P_{ab} is the n -dimensional Ricci tensor. We find that

$$P^i_i = \frac{1}{2} \left(\frac{R_c^2}{R_b^2 R_d^2} (C^c_{di}) - \frac{1}{2} \frac{R_i^2}{R_c^2 R_d^2} (C^i_{cd})^2 + R_i^{-2} C^c_{di} C^d_{ci} + 2R_d^{-2} C^c_{cd} C^i_{id} \right) \quad (\text{no sum over } i) . \quad (2)$$

Now, if the right-hand side of Eq. (1b) can be ignored, then we recover the famous Kasner solution: taking the trace of Eq. (1b) we find $g \sim t^2$. Using this again in Eq. (1b) we find that $R_a \sim t^{p_a}$ with $\sum p_a = 1$. From Eq. (1a) it then follows that $\sum p_a^2 = 1$; this is the Kasner solution.

A closer look at eqs. (1) shows that the Kasner solution is a good approximation as long as $t^2 p_a^2$ remains small for small times t . We notice that this is always

the case for the two last terms in $t^2 p_a^a$ since these have the form $t^{2(1-p_c)}$. The first two terms, however, have the form $t^{2\alpha_{abc}}$ with $\alpha_{abc} = 1 + p_a - p_b - p_c$ and could possibly be negative. It is therefore interesting to investigate which constraints one must impose to demand that the α_{abc} be positive. If all the α_{abc} remain positive for arbitrary small times, then the approach to the singularity is that of the Kasner solution, while if negative α_{abc} exist in $t^2 p_i^i$ the corresponding term grows indefinitely for small t and the universe will 'bounce' and enter a new Kasner epoch described by new p_a . Now, notice from eq. (2) that each term containing $t^{2\alpha_{abc}}$ is proportional to C_{bc}^a , i.e. only terms with $C_{bc}^a \neq 0$ appear in P_d^d . We may then ask which groups have structure constants such that for a suitable choice of the p_a all the α_{abc} corresponding to non-vanishing C_{bc}^a are positive. We shall show that if one among the generators X_a of G can find a closed subset of generators -- a subalgebra -- of dimension d with $2 \leq d \leq n - 1$, then it is possible to find a region of finite measure in p -space such that all the α_{abc} in $t^2 p_i^i$ are positive for p_a in this region. This shows that the approach to the initial singularity in this case cannot be chaotic -- after going through a (perhaps large) number of Kasner epochs the universe will eventually bounce into the region with all the α_{abc} positive. Once described by a set of p_a from this region the universe evolves for arbitrary small times according to the Kasner solution corresponding to this particular set of p_a .

3. THE APPROACH TO THE INITIAL SINGULARITY

Let us assume it is possible to divide the generators X_a into a set $\{X_A\}$, $A = 1, \dots, d$ and $\{X_\sigma\}$, $\sigma = 1, \dots, D$, where $D = n - d$ and where the X_A span a d -dimensional subalgebra of G , i.e. $C_{AB}^\sigma = 0$ for all A, B, σ . Now, we shall provide a set of p_a such that all α_{abc} corresponding to non-vanishing C_{bc}^a are positive: let p_a be the Kasner index corresponding to X_a , and define $p_A = p$, $A = 1, \dots, d$ and $p_\sigma = q$, $\sigma = 1, \dots, D$, where

$$p = \frac{\frac{1}{dD} + \frac{1}{d} \sqrt{\frac{1}{d} + \frac{1}{D} - \frac{1}{dD}}}{\frac{1}{d} + \frac{1}{D}}, \quad q = \frac{\frac{1}{dD} - \frac{1}{D} \sqrt{\frac{1}{d} + \frac{1}{D} - \frac{1}{dD}}}{\frac{1}{d} + \frac{1}{D}}.$$

Here p and q are defined as the solution to the equations $dp + Dq = dp^2 + Dq^2 = 1$ (i.e. $\Sigma p_a = \Sigma p_a^2 = 1$). Let us now check the possible values of α_{abc} :

$$\alpha_{ABC} = 1 - p, \quad \alpha_{AB\sigma} = \alpha_{A\sigma B} = 1 - q, \quad \alpha_{A\sigma\sigma} = 1 + p - 2q,$$

$$\alpha_{\sigma A\sigma} = \alpha_{\sigma\sigma A} = 1 - p, \quad \alpha_{\sigma\sigma\tau} = 1 - q.$$

These are all positive since $q < p < 1$; $\alpha_{\sigma AB}$ does not appear in $t^2 P_1^i$ since $C_{AB}^\sigma = 0$. Now, since all the α_{abc} are positive for this particular choice of p_i they are also positive for a whole surrounding region I of finite measure in p -space. This shows that the region corresponding to all the α_{abc} being positive has finite measure. Therefore, we can conclude that models which allow for diagonal solutions, and for which the generators can be split as devised above, are non-chaotic and the general approach to the initial singularity is the Kasner solution. All the models already analysed allow such a split of the generators [5], so it follows from our simple argument that these are non-chaotic, in agreement with the earlier analysis.

Any Lie algebra L can be split into two parts $L = Q + M$, where M and Q are subalgebras with Q semisimple and M solvable with $[M, Q] \subset M$ [6]. If neither Q nor M are trivial, then the generators of L may be chosen according to the split into Q and M , and it follows from our result, that any existing diagonal solution close to the singularity is the (non-chaotic) Kasner solution. If Q is trivial then $L = M$ is solvable, so $L \neq [L, L]$, i.e. L may be split into $L' = [L, L]$ plus remaining generators, and our result shows also in this case that the general solution is non-chaotic. Finally if $L = Q$, then $L = L_1 \oplus \dots \oplus L_k$ with L_i being simple. Obviously if $k \geq 2$ we may split the generators and draw the same conclusion. If $k = 1$, then L is simple; but the only simple, real algebras without subalgebras are $SO(3)$ and $SO(2,1)$, i.e. the Bianchi models IX and VIII. These are therefore the only candidates for chaotic behaviour (it is well known that these models exhibit the chaotic mixmaster behaviour close to the singularity). The natural candidates as higher dimensional mixmaster cosmologies, namely the higher dimensional simple Lie groups, all contain subalgebras, i.e. it is possible to split the generators as divided

above. Therefore even for these groups the chaotic behaviour -- which is characteristic for the 3-dimensional simple groups -- is absent.

In conclusion, we have shown that no obvious generalization to higher dimensions of the usual 3-dimensional diagonal mixmaster cosmologies exist. We have shown that whenever diagonal solutions exist they will eventually evolve into the Kasner solution corresponding to a set of p_a with all the α_{abc} positive.

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