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STUDIES OF THE SPECTRAL DIMENSION FOR BRANCHED KOCH CURVES *

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ABSTRACT

The scaling behaviour of Koch curves with rather general branched structure is studied by means of the exact renormalization method and the general formula of the spectral dimension is obtained. The Einstein relation between the random walk dimension, fractal dimension and scaling exponent of dc conductivity is also verified.

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1. Introduction

Recently, there has been increasing interest in the system with a dilatation symmetry or fractal (Mandelbrot, 1977, 1982). In general, there are two kinds of fractals which can be distinguished: deterministic fractals (self-similar) such as Koch curve (Gefen et al., 1983), Sierpinski gasket (Gefen et al., 1981, 1984a, Rammal et al., 1982, Hilfer et al., 1984) and Sierpinski carpets (Gefen et al., 1984b), and random fractals (statistically self-similar) such as a random walk in free space and percolation cluster at threshold (Gefen et al., 1981, Mandelbrot, 1982). There are also many disordered systems with dilatation invariants such as linear or branched polymers (Havlin et al., 1982), diffusion limited aggregates (Witten et al., 1981, 1983), porous materials (Evan et al., 1984) and others.

The deterministic fractals have become an important and attractive area of study due to the great current theoretical interest in the properties of a system with self-similar structure and because they constitute simple models of realistic physical systems with statistical self-similar structure. For example, the non-branched and branched Koch curves can be related, as a model, to the linear or branched polymers.

Stinchcombe (1985) studied the diffusion in a branched Koch curve with the structure as shown in figure 1(a). Using the renormalization method, Stinchcombe obtained the analytic formula of the spectral dimension and found that the following Einstein relation is valid for such a kind of Koch curve:

$$\tilde{d}_w = \tilde{\tau} + d_f \quad (1)$$

where d_w, d_f and $\tilde{\tau}$ are the random walk dimension, fractal dimension and scaling exponent of dc conductivity. Marian and Stella (1986) gave the studies of the spectral dimension

of non-branched Koch curve with long-range interactions as shown in figure 1(b) and found that the Einstein relation (1) also is satisfied in such a case.

In this paper, we present further studies of the spectral dimension for more general branched Koch curve with the structure shown in figure (2). In our case, there are p branches at each branched point and the number p can be any integer ($p \geq 3$). Using the exact renormalization method, we obtain a general analytic formula of the scaling exponent and the spectral dimension. The calculations show that the Einstein relation is also valid for our branched Koch curves. The analytic compact formula of the spectral dimension of branched Koch curve has been presented. We show that the spectral dimensions are exactly equal for the fractals with same topological properties even though the fractal dimensions are very different.

2. Formulation

The branched Koch curve shown in figure 2 is studied. The fractal model is constructed recursively as the limit of a hierarchy of a system, each obtained from the previous one by replacing in it each bond by a cluster with many bonds as shown in figure (2). It is easy to find the fractal dimension

$$d_f = \ln[2(n+1) + \sum_{i=1}^{p-1} (m_i+1)] / \ln[(1+1)+2(n+1)] \quad (2)$$

where the self-similar scaling unit b is $[2(n+1)+1+1]$. We assume that there are interactions only between the points linked with a bond. We will study the vibrating model in which the displacements of the points are vertical to the bonds. The spectral dimension d is defined as follows:

$$\rho(\omega) \sim \omega^{d-1} \quad (3)$$

where the $\rho(\omega)$ is the density of the vibrating states. If the frequency ω has the following scaling behaviour:

$$\omega(bL) = b^{-d} \omega(L) \quad (4)$$

where the L is the size of the system. The spectral dimension d_s can be calculated as the following relation (Rammal, et al. 1983).

$$d_s = d_f / a \quad (5)$$

The exponent a can be exactly calculated by a decimation procedure. In our case, the Koch curves only have two kinds of points, p -branching points and non-branching (2-branching) points so that two renormalization equations can be obtained. After long algebra, the renormalization equations of the frequency can be written as follows:

$$\begin{cases} \tilde{\alpha}_p' = \frac{\alpha_p' - p + pW}{-U} + p \\ \alpha_p' = \frac{\alpha_p' - p + pC_n(\alpha_s)}{B_n(\alpha_s)} + p \end{cases} \quad (6)$$

$$\begin{cases} \tilde{\alpha}_s' = \frac{\alpha_s' - 2 + 2W}{-U} + 2 \\ \alpha_s' = \frac{\alpha_s' - 2 + 2C_n(\alpha_s)}{B_n(\alpha_s)} + 2 \end{cases} \quad (7)$$

where $\alpha_i' = \omega^2 / \omega_i^2$, $i=p, s$. The $\omega_i = (K_i / M_i)^{1/2}$ and the K_i and M_i is the effective renormalized elastic constant and mass of the i -branching point at n th stage. In equation (3), the $\tilde{\alpha}$ is the renormalized frequency at $(n-1)$ th stage and the $W(\alpha_s, \alpha_p)$, $U(\alpha_s, \alpha_p)$, $C_n(\alpha_s)$ and $B_n(\alpha_s)$ are defined as

$$\begin{cases} C_n(\alpha_s) = \frac{-1}{\alpha_s - 2 + C_{n-1}(\alpha_s)} \\ B_n(\alpha_s) = \prod_{i=1}^n C_i(\alpha_s) \end{cases}, \quad C'_n(\alpha_s) = \frac{-1}{\alpha_s - 2} \quad (8)$$

$$\begin{cases} W = W(\alpha_s, \alpha_p) = \frac{-B_n(\alpha_s)}{\Delta(\alpha_s, \alpha_p)} (\alpha_p - p + C_n(\alpha_s) + \sum_{j=1}^{p-1} C_{nj}(\alpha_s)) \\ U = U(\alpha_s, \alpha_p) = \frac{-B_n(\alpha_s)}{\Delta(\alpha_s, \alpha_p)} (\sum_{j=1}^{p-1} B_{nj}(\alpha_s)) \\ \Delta = \Delta(\alpha_s, \alpha_p) = (\alpha_p - p + C_n(\alpha_s) + \sum_{j=1}^{p-1} C_{nj}(\alpha_s) - \sum_{j=1}^{p-1} B_{nj}(\alpha_s)) \\ \cdot (\alpha_p - p + C_n(\alpha_s) + \sum_{j=1}^{p-1} C_{nj}(\alpha_s) - \sum_{j=1}^{p-1} B_{nj}(\alpha_s)) \end{cases} \quad (9)$$

Because we are interested in the region of frequency near zero, we can expand the function $\alpha'_2, \alpha'_p, W, U, C$ and B near the zero ω and obtain

$$\begin{cases} \alpha'_2 \approx A_2(n) \alpha_2 \\ \alpha'_p \approx A_p(n) \alpha_2 + D_p(n) \alpha_p \\ A_2(n) = (n+1)^2 \left[\frac{(1+2C'_n(\omega))}{n+1} + \frac{2}{n+1} B'_n(\omega) \right] \\ A_p(n) = \frac{p}{B_n(\omega)} [C'_n(\omega) + (-1 - C'_n(\omega)) \frac{B'_n(\omega)}{B_n(\omega)}] \\ D_p(n) = n+1 \\ W \approx W(\omega) + W'_2(\omega) \alpha_2 + W'_p(\omega) \alpha_p \\ U \approx U(\omega) + U'_2(\omega) \alpha_2 + U'_p(\omega) \alpha_p \\ \Delta \approx \Delta(\omega) + \Delta'_2(\omega) \alpha_2 + \Delta'_p(\omega) \alpha_p \\ W(\omega) = \frac{-B_n(\omega)}{\Delta(\omega)} (-\phi + C_n(\omega) + \sum_{j=1}^{p-1} C_{nj}(\omega)) \\ W'_2(\omega) \approx -\frac{1}{\Delta(\omega)^2} [B'_n(\omega) \Delta(\omega) - B_n(\omega) \Delta'(\omega)] [-\phi + C_n(\omega) + \sum_{j=1}^{p-1} C_{nj}(\omega)] \\ + \frac{B_n(\omega)}{\Delta(\omega)} [C'_n(\omega) + \sum_{j=1}^{p-1} C'_{nj}(\omega)] \\ W'_p(\omega) \approx \frac{B_n(\omega)}{\Delta(\omega)^2} (-\phi + C'_n(\omega) + \sum_{j=1}^{p-1} C'_{nj}(\omega)) \Delta'_p(\omega) - \frac{B_n(\omega)}{\Delta(\omega)} \\ U(\omega) = -\frac{B_n(\omega)}{\Delta(\omega)} \sum_{j=1}^{p-1} B_{nj}(\omega) \\ U'_2(\omega) = -\frac{1}{\Delta^2(\omega)} \left\{ [B'_n(\omega) \sum_{j=1}^{p-1} B_{nj}(\omega) + B_n(\omega) \sum_{j=1}^{p-1} B'_{nj}(\omega) \Delta(\omega)] \right. \\ \left. - B_n(\omega) \sum_{j=1}^{p-1} B_{nj}(\omega) \Delta'_2(\omega) \right\} \\ U'_p(\omega) = \frac{B_n(\omega)}{\Delta(\omega)^2} \left[\sum_{j=1}^{p-1} B_{nj}(\omega) \right] \Delta'_p(\omega) \\ \Delta(\omega) = B_n(\omega) [-\phi + \sum_{j=1}^{p-1} (C_{nj}(\omega) - B_{nj}(\omega)) + C_n(\omega)] \\ \Delta'_2(\omega) = -B_n(\omega) [C'_n(\omega) + \sum_{j=1}^{p-1} (C'_{nj}(\omega) - B'_{nj}(\omega))] + \\ [-\phi + C_n(\omega) + \sum_{j=1}^{p-1} C_{nj}(\omega) - B_{nj}(\omega)] [C'_n(\omega) + \sum_{j=1}^{p-1} (C'_{nj}(\omega) + B'_{nj}(\omega))] \\ \Delta'_p(\omega) = -B_n(\omega) + (-\phi + C_n(\omega) + \sum_{j=1}^{p-1} (C_{nj}(\omega) - B_{nj}(\omega))) \end{cases}$$

We have the renormalization equations near the zero frequency:

$$\begin{cases} \tilde{\alpha}_2 = E_2 \alpha_2 + \tilde{E}_2 \alpha_p \\ \tilde{\alpha}_p = E_p \alpha_2 + F_p \alpha_p \end{cases} \quad (10)$$

where

$$\begin{cases} E_2 = \frac{-1}{W(\omega)-1} \left[\frac{A_2(n)+2W'_2(\omega)}{2} - U'_2(\omega) \right] \\ \tilde{E}_2 = \frac{-2}{W(\omega)-1} [W'_p(\omega) - U'_p(\omega)] \\ E_p = \frac{-1}{W(\omega)-1} [A_p(n) + \phi (W'_2(\omega) - U'_p(\omega))] \\ F_p = \frac{-1}{W(\omega)-1} [D_p(n) + \phi (W'_p(\omega) - U'_p(\omega))] \end{cases} \quad (11)$$

For the fractal, it has self-similar structure and the stages will be extended to infinite. That means the decimation procedure will be infinite from the infinite stage (the fixed point) to finite n th- or $(n-1)$ th stage. We have to solve the eigen-equation of M and find two eigenvalues of the matrix M . The largest one corresponds to the fixed point which we just need. In that case, we have

$$M \begin{pmatrix} \alpha_2^* \\ \alpha_p^* \end{pmatrix} = \begin{pmatrix} E_2 & \tilde{E}_2 \\ E_p & F_p \end{pmatrix} \begin{pmatrix} \alpha_2^* \\ \alpha_p^* \end{pmatrix} = \lambda_{max} \begin{pmatrix} \alpha_2^* \\ \alpha_p^* \end{pmatrix} \quad (12)$$

where the eigenvalue λ_{max} is

$$\lambda_{max} = \frac{1}{2} [E_2 + F_p + \sqrt{(E_2 + F_p)^2 - 4(E_2 F_p - \tilde{E}_2 E_p)}] \quad (13)$$

and the ratio β is

$$\beta = \frac{\alpha_2^*}{\alpha_p^*} \quad (14)$$

The scaling exponent a can be obtained as

$$a = \ln(\lambda_{max}) / 2 \ln(b) \quad (15)$$

and the spectral dimension d_S is d_f/a . From the equation (2), we have

$$d_S = 2 \ln [2(n+1) + \sum_{k=1}^{p-1} (m_k+1)] / \ln(\lambda_{max})$$

If we change the self-similar scaling unit b and fix the n, p and all m , we can obtain many different fractals with the same topology. But the spectral dimensionality does not change because the equation which determines the λ_{max} is not changed. Figure 3 gives the calculating results of the d_S , d_f and a related to the branching number. In our calculations, we fix the n to be 2, m_i to be 4 and $m_{i+1} = m_i + 2$, and p has been changed from 3 to 200. The results show that all dimensions increase for increasing p . We also did the calculation for the cases of fixed $p = 15$, all m equal among them n variable from 1 to 10, with m from 1 to 200. When p and n are fixed and m is changed, the spectral dimensions and the fractal dimensions also change. In figure 4, we give a part of the calculations which have been plotted in the curves of $d_S \sim d_f$. We can find that the spectral dimensions have the maximum value at some value of the m . In any case, we have calculated the ratio and found that it exactly equals $2/p$.

The Einstein relation (1) can be rewritten as

$$2a - d_f = \tilde{\tau}$$

Due to that the dw equals $2d_f/d_S (=2a)$. The scaling exponent $\tilde{\tau}$ of the dc conductivity can be calculated easily as follows:

$$\tilde{\tau} = \ln [2(n+1) + (\frac{1}{\sum_{j=1}^{p-1} \frac{1}{m_j}})] / \ln(b)$$

In our calculations, we have calculated the $2a - d_f$ and $\tilde{\tau}$, and the results have shown that they are exactly equal in any case.

That means the Einstein relation is also valid for such rather general branched Koch curves. The results show that

the scaling exponent a and the spectral dimension of the branched Koch curve can be calculated by the following equations:

$$a = (\tilde{\tau} + d_f) / 2$$

$$d_S = 2d_f / (\tilde{\tau} + d_f)$$

where

$$d_f = \ln [2(n+1) + \sum_{k=1}^{p-1} (m_k+1)] / \ln(b)$$

$$\tilde{\tau} = \ln [2(n+1) + (\frac{1}{\sum_{k=1}^{p-1} \frac{1}{m_k}})] / \ln(b)$$

The b is the self-similar scaling unit. In figure 2, $b = 1 + 1/2(n+1)$. From equations (19) and (20), we can find that the fractal d_f and spectral dimension d_S can become equal and one only in the limit case of n . In that case the exponent $\tilde{\tau}$ also approaches to one. When the $p \rightarrow \infty$ or $m_i \rightarrow \infty$, the $d_f \rightarrow \infty$ and the $d_S \rightarrow 2$. We can find that the maximum value of the spectral dimension is 2 for such a kind of Koch curve.

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Figure 1a A branched Koch curve at first stage.

Figure 1b A non-branched Koch curve with long-range interaction at first stage.

Figure 2 A branched Koch curve with multi-branches at first stage.

Figure 3 The curves of the fractal dimension d_f , spectral dimension d_s and the scaling exponent α related to the branching number p .

Figure 4 The curve of $d_s \sim d_f$ at fixed the number $n, p=15$, $l=2$ and changing the m_c from 3 to 200.

Fig. 1

Fig. 2

Fig. 3

Fig. 4