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COHERENCE EFFECTS IN INITIAL JETS AT SMALL Q^2/s

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A B S T R A C T

The QCD evolution of initial jets (corresponding to space-like partons) is analyzed for small Bjorken x . Coherent effects are described by a gluon emission current which also embodies internal line insertions to double log accuracy. It is found that gluons are emitted off the space-like leg in the order of their angles with respect to the jet axis. The modified branching vertex probability is given and differences with previous prescriptions are pointed out. The relation to transverse momentum ordering in structure functions is also discussed.

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1. INTRODUCTION

Future hadron colliders will be characterized by the fact that active partons mostly interact with a small fraction x of the incoming hadron momenta. In this semihard regime, hadronic physics involves various scales (energy $\sqrt{s}$, hard scattering scale $Q = \langle x \rangle \sqrt{s}$, parton transverse momentum $\langle k_{\perp} \rangle$, hadronization scale Q_0) which may be of different order. This opens up a few theoretical questions in QCD which are not yet settled.

Firstly, the cross-section for Drell-Yan processes or jet production will enter the semihard "Regge" region where both Q and $\sqrt{s}/Q$ are large. The occurrence of sizeable cross-sections of "minijet" type makes this region quite interesting.

Secondly, the bremsstrahlung of active partons degrading their momentum will be strong and will affect distributions of both electroweak bosons and hadronic jets.

Finally, the coherent effects in jets arising from active partons scattering will be modified by non-vanishing impact parameter effects and multi-parton interactions.

On the theoretical side, considerable work on total cross-sections and structure functions has been performed¹⁾⁻⁵⁾, following earlier work of Lipatov and collaborators¹⁾ on Regge behaviour. Although the gluon distribution in hadrons is not known, its Q^2 -evolution at small x is calculable in the perturbative regime $\alpha_s(k_{\perp}) \log 1/x \lesssim 1$, and has also a chance of being isolated⁵⁾. Generalizations of the Altarelli-Parisi (A-P) equations in this region have been given^{2),3)} and unitarization corrections for large gluon density have been described by Gribov, Levin and Ryskin²⁾. It turns out^{3),4)} that in the perturbative regime the gluon structure function is dominated at small x by ladder type diagrams with transverse momentum ordering⁴⁾. As a consequence, the anomalous dimension shows, for each power of α_s , only one inverse power $(N-1)^{-1}$ of the moment index (conjugated to $\log 1/x$).

On the other hand, properties of the associated parton production have been mostly studied for time-like evolution^{3),6)-9)}, i.e., for partons which are produced with a large squared mass and hadronize in cascade (as for e_+e_- annihilation). The active parton bremsstrahlung is of different type, since the incom-

ing partons degrade their momentum by building up a negative squared mass, up to $(-Q^2)$ (space-like evolution).

The main result⁶⁾⁻⁸⁾ in the time-like case is that the jet evolution, at double log level and including interference terms, is given by a classical branching process with three-parton vertices, A-P probability distribution, and ordering in decay angles (Fig. 1a). This procedure includes a considerable amount of single logs^{6),10)} and is suitable for Monte Carlo simulations¹¹⁾, with a good description of e_+e_- data.

The space-like case is not expected to be different for $x = O(1)$, except for the fact that parton momenta and squared masses $|k^2|$ evolve in opposite directions (Fig. 1b). From various approaches^{12,13)} it is found that in both cases decay angles θ increase towards the hard vertex. It is then convenient to let the jet evolve in the direction of decreasing masses down from $|Q^2|$, i.e., of increasing momenta in the space-like case ["backward" evolution scheme¹⁴⁾]. An angular ordering of this type has been proposed by Marchesini and Webber¹²⁾ in a recent Monte Carlo simulation.

When x becomes small, the space-like jet evolution is expected to show substantial differences. We have mentioned the fact that, for $x \rightarrow 0$, the structure functions show $k_{\perp}$ -ordering which may be much different from angular ordering, due to the occurrence of small momentum fractions [z or $(1-z)$ at the vertex in Fig. 1b]. Ordering in masses (i.e., of $k_{\perp}$ for $z \rightarrow 0$) has also been taken as a possible prescription in Monte Carlo simulations¹⁴⁾.

The purpose of the present paper is to study the space-like jet evolution for small x , by including some loop contributions in QCD where leading powers of $\log Q^2/k^2$, $\log 1/(1-z)$ and $\log 1/z$ are kept. It has already been pointed out by the author¹⁵⁾ that coherence effects at small x single out the angles ϕ of the emitted gluons with respect to the jet axis, rather than the decay angles θ along the momentum conservation chain (Fig. 1b).

Here this proposal is substantiated by a two-loop calculation, which is generalized to higher orders through the introduction of a suitable soft gluon insertion current. The latter embodies internal line insertions, when needed, also if the exchanged gluons are softest ($z \ll 1$). Gluon emission becomes in fact coherent at large ϕ -angles (i.e., can be computed by summing amplitudes in phase), due to dominance of a single polarization in the two limiting cases ($z \rightarrow 1$ and $z \rightarrow 0$).

Our results are the following:

(a) The classical branching picture holds, provided ordering in ϕ -angles (with respect to the jet axis) is imposed: e.g., $\phi > \phi' > \theta''$ in Fig. 1b. This prescription reduces to the ordering in decay angles θ for $x \rightarrow 0(1)$, but differs from $k_{\perp}$ -ordering for $x \rightarrow 0$ in the region R: $\phi' < \phi < \phi'/z'$.

(b) Virtual corrections contain vertex type terms (in a physical gauge) of order $\alpha_s \log^2(1/z)$. They cancel in the total cross-section (or structure function) with gluon emission in the region R, thus recovering in this case the absence of double log $1/x$ terms and $k_{\perp}$ -ordering.

(c) The branching probability contains an additional form factor of new type, related to (b), which provides higher order damping in the region R, and is here calculated to lowest order in α_s . It is conceivable that the latter is important only for angular distributions and not for integrated y -distributions. Outside this region ϕ -ordering is equivalent¹⁵⁾ to the simpler requirement $x\theta > x'\theta' > \theta''$, which is now being used¹²⁾.

Following (a) - (c), the single parton inclusive distribution in, say, DIS at small x , should be given by a "jet-calculus" diagram¹⁶⁾ with ϕ -ordering at the vertex and $k_{\perp}$ -ordering in the propagators. This picture seems consistent with results of different authors, briefly anticipated in a recent work¹⁷⁾.

The paper is organized as follows. In Section 2 we give an account of coherent gluon emission to order α_s^2 in the example of DIS, and we derive the emission currents which include internal line insertions for $z \rightarrow 1, 0$. In Section 3 we generalize our results to higher orders, by including one-loop virtual corrections. In Section 4 we give the modified branching vertex probability, and we discuss some remaining problems. The structure function equation with ϕ -ordering and a few calculational details are given in the Appendices.

2. COHERENT FORM OF TWO-GLUON EMISSION

We want to discuss here how the coherence of two-gluon emission off the space-like jet arises, even in the small x limit. Although we discuss first the lowest order diagrams, our argument is meant to derive an emission current which is applicable to virtual corrections and multiple emission as well.

Let us then consider the process

$$\gamma(\omega) + q(p) \rightarrow q(\tilde{p}) + g(q') + g(q), \quad (2.1)$$

where we shall identify q_μ as the softest gluon. As is well known³⁾, in the limit $q_\mu \rightarrow 0$, the amplitude for Eq. (2.1) can be constructed by the gluon insertion formula on external legs

$$M_a^\mu(p, q', q) = J_a^\mu(q) M(p, q'), \quad (a)$$

$$J_a^\mu(q) = g \left(\frac{\tilde{p}^\mu}{\tilde{p} \cdot q} \tilde{t}_a - \frac{p^\mu}{p \cdot q} t_a + \frac{q'^\mu}{q' \cdot q} T'_a \right), \quad (b) \quad (2.2)$$

where the notation is as in Fig. 2 and extra indices and matrix product in colour space are understood.

Note that internal line insertions [not included in (2.2)] are not negligible if some pair of the external momenta ($p, \tilde{p}, q'$) becomes massless. Since the modifications to (2.2) in this collinear limit will be discussed later, let us first recall the main features of the simple formula (2.2).

Due to charge conservation ($\underline{t} = \underline{\tilde{t}} + \underline{T}'$) in the three-parton amplitude M , the colour and polarization sums in (2.2) lead to a factorized insertion formula

$$d\sigma(p, q', q) = d\sigma(p, q') \frac{\alpha_s}{2\pi^2} \frac{d^3q}{\omega^3} \frac{1}{2} \langle JJ \rangle, \quad (a)$$

(2.3)

$$\frac{1}{2} \langle JJ \rangle = \frac{N_c}{2} (\langle 01 \rangle + \langle 21 \rangle) + \left(C_F - \frac{N_c}{2} \right) \langle 02 \rangle, \quad (b)$$

where

$$\langle ij \rangle = \omega^2 \frac{q_i \cdot q_j}{q_i \cdot q \cdot q_j \cdot q} = \frac{\mathcal{Z}_{ij}}{\mathcal{Z}_i \mathcal{Z}_j},$$

(2.4)

$$\mathcal{Z}_{ij} = 1 - \cos \theta_{ij}, \quad \mathcal{Z}_i = 1 - \cos \theta_{iq}, \quad (i = 0, 1, 2)$$

denotes the two-body bremsstrahlung function and we have introduced the indices 2 (0) for the initial (final) quark respectively, a notation which is convenient in the many-gluon case (Sect. 3).

We shall distinguish in Eq. (2.3) two phase-space regions:

(1) Incoherent emission region. This is defined by q_μ being close in angle to either $\tilde{p}$, p or q' , i.e., $\theta_0 (\theta_2) \ll \theta_{10}$ or $\theta_1 \ll \theta_{12}$. It is easy to check that in each case the angular insertion factor (2.3b) reduces to the one expected from the corresponding colour charge:

$$\frac{1}{2} \langle JJ \rangle \simeq \frac{C_i}{3_i}, \quad (C_0 = C_2 = C_F = \frac{4}{3}, \quad C_1 = N_c) . \quad (2.5)$$

(2) Coherent emission region. This is the region where q is emitted at large angle, i.e., $\theta_2 \approx \theta_1 \gg \theta_{12}$. Then the p - and q' -currents in (2.2) become degenerate, so that, by charge conservation

$$J_a^r(q) = g \tilde{T}_a \left(\frac{\tilde{p}_r}{\tilde{p}q} - \frac{p_r}{pq} \right), \quad (2.6)$$

$$\frac{1}{2} \langle JJ \rangle \simeq C_F \langle 02 \rangle .$$

Therefore, one sees the quark charge, instead of the incoherent sum $(C_F + N_c)$.

In order to describe the p -jet region, it is convenient to set p_μ along the z -axis and to introduce the Sudakov parametrization

$$q_\mu = y p_\mu + \bar{y} \bar{p}_\mu + q_{\perp\mu}, \quad (2.7)$$

$$q_\perp^2 = q^2 = y \bar{y} q^2, \quad y \bar{y} = q^2/s,$$

where $\bar{p}_\mu$ is massless and along the negative z axis (with $2p\bar{p} = s = 1$), and similarly for q'_μ . We shall also define the momentum transfers

$$k'_\mu = p_\mu - q'_\mu = x' p_\mu - \bar{x}' \bar{p}_\mu + k'_{1\mu} ,$$

$$(x' = 1-y' , \bar{x}' = -q'^2/y' , k'_1 = -q') , \quad (2.8)$$

and $k = p - q - q'$. The regions (1,2) are then defined by

$$(1) \quad \varphi \ll \varphi' , \text{ or } |\varphi - \varphi'| \ll \varphi' , \quad (2.9)$$

$$(2) \quad 1 \gg \varphi \gg \varphi' ,$$

and the coherent region result, by (2.3) and (2.6), reads

$$d\sigma(p, q', q) = d\sigma(p, q) \frac{\alpha_s C_F}{\pi} \frac{dy}{y} \frac{d^2 q}{\pi q^2} . \quad (2.10)$$

However, as already remarked, the preceding treatment of soft emission is incomplete, precisely in the collinear region $q' \sim p$ where (2.10) is important, because $k' = p - q'$ becomes massless. In such a case we must also consider the insertion of q on the k' -leg (Fig. 3). We shall show that this can be done by a proper modification of the (bremsstrahlung) current (2.2).

In computing the q -insertion on k' , we shall distinguish two cases, according to whether (a) $y \ll x$, where $x = x' - y = 1 - y' - y$, or (b) $y \gg x$. Case (b) is unimportant for the process (2.1) at order α_s^2 , because the k -leg must be a quark (Fig. 3c) and the Altarelli-Parisi densities do not have soft fermion singularities. However, it becomes important for gluon exchange at higher orders and even dominant in the $x = x_B \rightarrow 0$ limit. Therefore, we shall also consider a fictitious process

$$s(q) + g(p) \rightarrow g(\tilde{p}) + g(q') + g(q) , \quad (2.11)$$

in which gluons are directly coupled to an external current s , and compute the q -insertion of Fig. 2d as well.

(a) $y \ll x$. This means that $1-z = y/x' \rightarrow 0$, and the q -insertion can be evaluated with the eikonal vertex. Since $k'^2 = -2pq'$ may be small in the collinear limit, we must distinguish it from

$$(k'-q)^2 = k^2 = -x \left(\frac{q^2}{y} + \frac{q'^2}{y'} \right) - (q+q')^2$$

even for small y . Then the total q -emission current from diagrams (a-c) in Fig. 3 becomes, in the coherent region

$$J_{\mu a}^{\text{tot}}(q) = g \tilde{T}_a \left(\frac{\bar{p}_\mu}{\bar{p} \cdot q} - \frac{p_\mu}{p \cdot q} \frac{k'^2}{k^2} - \frac{k'_\mu}{k' \cdot q - k'^2/2} \right), \quad (y \ll x). \quad (2.12)$$

Note that this current is conserved, and reduces to (2.6) for $k^2 \approx k'^2$, i.e., $2k'q \ll |k'^2|$, or

$$\frac{q^2}{1-z} \ll q'^2, \quad \left(\frac{\varphi'}{x\sqrt{1-z}} \gg \varphi \gg \varphi' \right). \quad (2.13)$$

In the remaining angular region ($1 \gg \phi \gg \phi'/x\sqrt{1-z}$) diagram (c) is dominant, i.e.,

$$J_{\mu a}^{\text{tot}} \approx g \tilde{T}_a \left(\frac{\bar{p}_\mu}{\bar{p} \cdot q} - \frac{k'_\mu}{k' \cdot q} \right), \quad \left(\frac{q^2}{1-z} \gg q'^2 \right), \quad (2.14)$$

but yields the same insertion formula (2.10), evaluated for $q^2 \approx (1-z) |k^2|$.

In other words, the coherent action of p and q' and k' -emission exchange their roles, with the same final result, Eq. (2.10).

(b) $y \gg x$. This region corresponds to $z = x/x' \rightarrow 0$ and is important (i.e., contributes leading logs) only in the gluon exchange case of Fig. 3d.

One would think at first sight that in this case the q_μ -insertion is not factorizable because q is not soft with respect to k' and is not emitted via the eikonal vertex. However, since both z and x' are small, helicity is conserved along the (q, k') and (q', p) directions (Fig. 3d), and q is coupled to a single k' polarization, the one in the (p, q') plane^{7),18)}.

Therefore, one can still write an effective q-emission current, e.g., in the $\bar{p}$ -gauge (see Appendix A)

$$M_a^\mu(p, q', q) = g \tilde{T}_a j^\mu(k', q) M(p-q, q'), \quad (a) \quad (2.15)$$

$$j^\mu(k', q) = \frac{\bar{p}^\mu}{\bar{p}q} - \frac{p^\mu}{pq} - \frac{2}{|q'|} \left(\varepsilon^\mu(k') - \varepsilon(k') \cdot q \frac{\bar{p}^\mu}{\bar{p}q} \right), \quad (b)$$

where

$$\begin{aligned} \varepsilon_\mu(k') &= \frac{1}{|q'|} \left(-q'_{\perp\mu} + \bar{p}_\mu 2q'^2/x'v \right) \\ &= |q'|^{-1} (k'_\mu - x' p_\mu + \bar{p}_\mu \text{-terms}) \end{aligned} \quad (2.16)$$

is the physical k' -polarization in the (p, q') plane. By replacing (2.16) into (2.15), we get

$$j^\mu(k', q) = \frac{1}{k'^2} \left[2(k'^\mu - y p^\mu + \bar{y} \bar{p}^\mu) + k^2 \frac{\bar{p}^\mu}{\bar{p}q} - k'^2 \frac{p^\mu}{pq} \right]. \quad (2.17)$$

The current (2.17) generalizes in a sense those in (2.6) and (2.12), and reduces to the one of Lipatov¹⁾ in the Regge region (Appendix A). However, one should be careful in distinguishing [2.15a)] from [2.2a)]: the momentum change $p \rightarrow p-q$ is non-trivial because in the case of (b) both (i) $k'^2 \neq k^2$ and especially (ii) $x \ll x'$. Therefore, if we want to completely factorize the q -dependence we must use in the double log region for the gluon case (2.11), the rescaling

$$d\sigma(p-q, q') = \frac{q'^2}{(q+q')^2} \frac{x'}{x} d\sigma(p, q') \quad (2.18)$$

as well as the polarization sum

$$-j_\mu j^\mu = \frac{(q+q')^2}{q^2 q'^2}, \quad (2.19)$$

where we have used, in the case of (b), the approximation $k^2 \approx -(q+q')^2$.

By replacing (2.18) and (2.19) in (2.15), we finally obtain

$$\begin{aligned} d\sigma(p, q', q) &= \frac{\alpha_s}{2\pi^2} \frac{d^3q}{\omega} \frac{1}{2} (-j_\mu j^\mu) \frac{q'^2}{(q+q')^2} \frac{1}{z} d\sigma(p, q') \quad (a) \\ &= \bar{\alpha}_s \frac{d^2q}{\pi q^2} \frac{dz}{z(1-z)} d\sigma(p, q'), \quad (\bar{\alpha}_s = \frac{N_c \alpha_s}{\pi}). \quad (b) \end{aligned} \quad (2.20)$$

Let us note that Eq. (2.20) follows not only by the soft emission formula (2.15), but also by the collinear singularity relation (2.18) and is therefore valid in the double log region. It exhibits the interesting feature that the collinear singularity at $q+q' = 0$ in (2.18) has disappeared because of the coherent emission factor (2.19) and is replaced by the one at $q = 0$. The longitudinal momentum rescaling has instead provided, with respect to (2.10), an additional z^{-1} factor which is needed to describe the $x = 0$ singularity.

The interpretation of Eq. (2.20) can then be given in terms of the probability branching diagram of Fig. 4c. It corresponds to (q, y) -emission, followed by (q', y') emission, convoluted with proper Altarelli-Parisi (A-P) densities in the double log region, and with transverse momenta always defined with respect to the fastest parton z -axis.

We can therefore summarize our results by the probability branching diagrams of Fig. 4:

- (1) In the regions $\phi \ll \phi'$, $|\phi - \phi'| \ll \phi'$ we have, starting from the hard vertex, q' -emission followed by incoherent q -emission off (a) the q' -leg and (b) the p -leg, described by Eqs. (2.3a) and (2.5).

(2) In the coherent emission region $1 \gg \phi \gg \phi'$ we have q-emission followed by q'-emission off the k'-leg (Fig. 4c), as described by Eqs. (2.10) and (2.20b) for $y \ll x$ and $x \ll y$ respectively.

(3) The singular angular distributions are given in terms of decay angles in the case of Fig. 4a (time-like branching) and of ϕ -angles with respect to the jet axis in the case of Figs. 4b and 4c (space-like branching).

Notice that the space-like branching follows the rule that ϕ -angles decrease with decreasing off-shellness. Care should be taken in distinguishing them from Θ -angles, defined as decay angles in the momentum conservation chain (Fig. 4c): in the part R of the coherent region ($\phi'/z' > \phi > \phi'$) Θ -angles give a different picture of collinear singularities, as remarked after Eq. (2.20).

For instance, in the double branching diagram of Figs. 4c and 1b, one has¹⁵⁾

$$z\Theta = |\varphi + z'^{-1}\varphi'| = |\varphi + \Theta'|, \quad (2.21)$$

and the collinear singularities are in the variables ϕ and ϕ' . Therefore, only for $\phi > \Theta' \approx \phi'/z'$ is ϕ -ordering equivalent to the prescription $z\Theta > \Theta'$, or else to the one $x\Theta > x'\Theta' > \Theta''$, which is being adopted¹²⁾. The region R ($\phi \ll \Theta'$) corresponds to the boundary $z\Theta \approx \Theta'$, which in the Θ -angle approach is assumed to be not singular and is therefore neglected.

3. VIRTUAL CONTRIBUTIONS AND HIGHER ORDERS

Here we complete the analysis of the $O(\alpha_s^2)$ contributions to DIS by discussing virtual corrections, we give a partial generalization to higher orders, and we compare with known^{2),3)} results for the structure functions.

We shall first consider (Fig. 5) virtual corrections to the processes

$$\gamma(\omega) + q(b) \rightarrow q(\tilde{p}) + q(q') \quad , \quad (a) \quad (3.1)$$

$$s(\omega) + q(b) \rightarrow q(\tilde{p}) + q(q') \quad (b)$$

arising from a soft gluon q_μ , parametrized as in (2.7). By always assuming $y \ll y' = 1-x$, we shall again distinguish the cases (a) $y \ll x$ and (b) $y \gg x$. (Notice here the definition $1-y' = x$, to be eventually identified with $x_B = |Q^2|/s$.)

A simple analysis of the $\bar{y}$ -integration in the Feynman integrals (Appendix B) shows that one can go on-shell at the pole $q^2 = 0$. In case (a) q -emission is given by the current (2.12) and the first-order virtual corrections factorize in the form^{*)}

$$\begin{aligned} d\sigma_a^{(1)}(p, q') &= d\sigma(p, q') \bar{\alpha}_s \int \frac{d^3 q}{4\pi\omega} (j^{\text{tot}})^2 \\ &= -d\sigma(p, q') \bar{\alpha}_s \int \frac{dz}{1-z} \frac{d^2 \underline{q}}{\pi q^2}, \end{aligned} \quad (3.2)$$

where $\bar{\alpha}_s = \alpha_s C_i / \pi$ for $i = q, g$. This contribution is the one needed to regularize the $z \rightarrow 1$ singularity of A-P densities¹⁸⁾.

In case (b), on the other hand, the leading diagrams in the $\bar{p}$ -gauge are of vertex type (Fig. 5b), [of rescattering type in the Feynman gauge (Fig. 5c)], yielding the result (Appendix B)

$$\begin{aligned} d\sigma_b^{(1)}(p, q') &= d\sigma(p, q') \left(-\bar{\alpha}_s \int \frac{dy}{y} \int \frac{d^2 \underline{q}}{\pi q^2} \Theta(|\underline{q}| - y |\underline{q}'|) \right) \\ &= -\bar{\alpha}_s \left(\log \frac{1}{z} \right)^2 d\sigma(p, q'). \end{aligned} \quad (3.3)$$

This additional term is needed to regularize the $q^2 = 0$ singularity in A-P-type equations^{2), 3)} (see below).

The generalization of the preceding results to higher orders is not obvious: we shall only discuss the insertion of the softest gluon of the p -jet (either real or virtual) in the collinear singular region, by exploiting the emission currents found in Sect. 2.

*) The kinematical boundary in (3.2) involves the momentum fraction $y/x = 1-z > \varepsilon \simeq Q_0/Q$, where Q_0 is a cut-off mass. Therefore the x -dependence can be rescaled away at fixed $Q = x \sqrt{s}/2$.

A - Real Emission. Since we are mostly interested in the $x \rightarrow 0$ region, let us consider the purely gluonic case, i.e., the process

$$s(q) + g(p) \rightarrow g(\tilde{p}) + g(q) + g(q_1) + \dots + g(q_n), \quad (3.4)$$

which generalizes (2.11). (The addition of a $q\bar{q}$ pair which couples to weak currents is straightforward.)

The insertion of the softest gluon q_μ in (3.6) is done by following the same steps as in Sect. 2. In the incoherent emission regions (1) of Sect. 2 the eikonal current

$$\begin{aligned} J_\mu^{eik} &= \sum_{i=1}^{n+1} t_i \frac{q_i r}{q_i q} + \tilde{T} \frac{\bar{p} r}{\bar{p} q} \\ &= \sum_{i=1}^{n+1} t_i \left(\frac{q_i r}{q_i q} - \frac{\bar{p} r}{\bar{p} q} \right), \quad (q_{n+1} = p, t_{n+1} = -t), \end{aligned} \quad (3.5)$$

applies, and q -emission factorizes as in (2.3a) and (2.5), by exhibiting a collinear singularity whenever q is close in angle to each of the q_i .

Coherent emission is instead obtained when several q_i 's become degenerate in a given direction, thus forming a jet. If this sub-jet is time-like, i.e., only emitted gluons are concerned, the situation is described by a branching process with ordering in decay angles^{7),3)}.

If instead several emitted gluons (or sub-jets) q_i become degenerate with the incoming one, the various coherent regions C_j are defined in terms of ϕ -angles with the z -axis, i.e., referring to Fig. 6,

$$C_j : \varphi_1, \dots, \varphi_{j-1} > \varphi \gg \varphi_j, \dots, \varphi_n, \quad (3.6)$$

where the ϕ_i 's are in decreasing order: $1 \gg \phi_1 > \phi_2 \dots > \phi_n$.

If then some $k_i = (p - q_i - \dots - q_n)$ becomes nearly massless, in the sense that $|k_i^2| \ll 2 k_i q$, q -insertions on the k_i legs are not negligible. In the coherent region C_j we need to consider the chain of collinear singularities in the momentum transfers $k_i (i > j)$ for some q_i ordering (Fig. 6). Let us show that

the sum of all insertions on the sub-jet $(q_j, \dots, q_n, p)$, including internal lines, is given by the current $-(\underline{t} - \sum_1^j \underline{t}_i) p_\mu / pq$, i.e., by the total colour charge $\underline{T}_j = \underline{t} - \sum_1^j \underline{t}_i$ of the k_j -leg ($j > 1$).

In fact, we can do the q -insertions on the chain of Fig. 6 recursively, starting from (p, q_n, k_n) . Since $y \ll y_i$ ($i = 1, \dots, n$) by assumption, we also have $y \ll x_i$, except possibly for $i = 1$. From the current (2.12) we then get the recursion formula

$$\begin{aligned} M_\mu(p; q_i, q) = & -g \frac{p_\mu}{pq} (\underline{t} - \underline{t}_n) \frac{(k_n - q)^2}{k_n^2} M(p - q; q_i) \\ & + (2n-1) \text{ insertions} \end{aligned} \quad (3.7)$$

The first term in (3.7) replaces the denominator $(k_n - q)^2$ by k_n^2 and builds up the charge $(\underline{t} - \underline{t}_n)$. We then repeat the operation on (k_n, q_{n-1}, k_{n-1}) , and so on. Eventually, we obtain for $q \in C_j$,

$$M_\mu(p; q_i, q) = \underline{T}_\mu^{(j)}(q) M(p; q_i), \quad (a)$$

$$\underline{T}_\mu^{(j)} = g \sum_1^{j-1} \left(\frac{q_i \cdot r}{q \cdot q} - \frac{\bar{p} \cdot r}{\bar{p} \cdot q} \right) \underline{t}_i + g \underline{T}_j \left(\frac{\bar{p} \cdot r}{\bar{p} \cdot q} - \frac{p \cdot r}{p \cdot q} \right), \quad (b)$$

i.e., an insertion formula on external legs only. It is known³⁾ that the first term in (3.8b) gives a negligible contribution to the polarization sum for $\phi \ll \phi_i$ ($i < j-1$). Therefore, we obtain the factorized result

$$\begin{aligned} d\sigma(p; q_i, q) = & \bar{\alpha}_s \frac{dy}{y} \frac{d^2 q}{\pi q^2} d\sigma(p; q_i), \\ & (y \ll x_j, \quad q \in C_j, \quad j > 1), \end{aligned} \quad (3.9)$$

where we have used $\underline{T}_j^2 = N_c$ in the purely gluonic case of Eq. (3.4).

If instead $q \in C_1$ (i.e., all q_i 's are degenerate), case (b) $y \gg x = x_1 - y = (1 - \sum_1^n y_i - y)$ can occur, which is described again by the diagrams in Fig. 3, with $k_2 = k_1 + q_1$ replacing p . By the same token we obtain

$$d\sigma(p; q_i, q) = \bar{\alpha}_s \frac{dz}{z(1-z)} \frac{d^2q}{\pi q^2} d\sigma(p; q_i), \quad (3.10)$$

$$(q \in C_1, \quad 1-z = y/x, \quad x_1 = 1 - \sum_1^n y_i).$$

This is the "first-gluon" factorization analogous to Eq. (2.20).

Equation (3.10) is consistent with the branching picture of Sect. 2 and generalizes Eq. (3.9) to the case $z \ll 1$. In particular, in the strong-ordering region $\phi \gg \phi_1 \dots \gg \phi_n$ it gives rise to ladder diagrams with ϕ -ordering.

The factorization pattern found above can be checked, for the case $y \ll x$, on some exact results for tree amplitudes^{19),20)}. Firstly, it is consistent with a recent analysis¹⁹⁾ of $2 \rightarrow 3$ amplitudes in the strict $q_\mu \rightarrow 0$ limit. Secondly, it also arises in the n -gluon amplitudes first found in Ref. (3) in the soft limit ($y, y_i \ll x$). In fact, in this case

$$d\sigma^{(n)} = \prod_{i=1}^n \left(\bar{\alpha}_s \frac{d^3q_i}{2\pi \omega_i^2} \right) \frac{1}{2} \zeta_{0,n+1}^2 \sum'_{\text{perm}} \left(\zeta_{01} \zeta_{12} \dots \zeta_{n+1,0} \right)^{-1}, \quad (3.11)$$

where $q_0 = \tilde{p}$, $q_{n+1} = p$, $\zeta_{ij} = 1 - \cos\theta_{ij}$, and Σ' runs over permutations of $(0, \dots, n+1)$ yielding inequivalent denominators.

In Eq. (3.11), q occurs in the angular factors in the form $(\zeta_{kq} \zeta_{qi})^{-1}$, where - in order to maximize the number of small factors - i belongs to C_j and k to the remaining set. Factorization comes from being $\zeta_{kq} \zeta_{qi} \approx \zeta_{ki} \zeta_{qp}$ in the coherent region C_j . On the other hand, the factorization argument given before includes internal line insertions, thus giving a partial explanation of why (3.11) coincides with the exact result²⁰⁾ in some particular cases.

B - One-gluon virtual corrections. If the softest gluon is virtual, one can classify its contributions in the incoherent (coherent) regions as in Eqs. (2.9), (3.6) by going to the mass-shell $q^2 = 0$ and taking positive and negative energy contributions, when needed (Appendix B).

The incoherent region contributions regularize the emission of time-like sub-jets, and give rise to the corresponding Sudakov form factors, as is known³⁾.

In the coherent regions C_j ($j > 1$) the virtual gluon must satisfy condition (a) $y \ll x_j$. Since the counting of the mass-shell cut is the one of a squared vertex, the double q -insertion is factorized by the squared current (3.8), thus yielding the result

$$d\sigma_{\alpha j}^{(1)}(p; q_i) = -\bar{\alpha}_s \int_0^{1-\varepsilon_j} \frac{dz}{1-z} \int_{C_j} \frac{d^2 q}{\pi q^2} d\sigma(p, q_i), \quad (3.12)$$

$$(q \in C_j, \varepsilon_j = Q_0/E_j, E_j = x_j \sqrt{s}/2),$$

which generalizes Eq. (3.2).

If $q \in C_1$ (i.e., $\phi \gg \phi_1 > \dots > \phi_n$) we must also consider case (b), $x \ll y \ll x_1$, where x (x_1) are the momentum fractions of $k = p - \sum_1^n q_i$ ($k_1 = k + q_1$). Here, the current (2.15) is the relevant one; however, q cannot be emitted and reabsorbed by the k leg, due to cancellation of the self-energy contributions (Appendix B).

Therefore, the q -insertion factor becomes

$$\left(\frac{\bar{p}_r}{\bar{p}q} - \frac{p_r}{pq} \right) j^r(k+q, q) = -4 \frac{k(k+q)}{q^2(k+q)^2}, \quad (3.13)$$

and, by azimuthal average [cf. Eq. (3.3)], we obtain

$$d\sigma_b^{(1)}(p; q_i) = -\bar{\alpha}_s \int_x^{x_1} \frac{dy}{y} \int_{\tilde{q}}^{|k|} \frac{d^2 q}{\pi q^2} \otimes (|q| - \frac{y}{x_1} |q_1|) \cdot dx_1 \frac{d\sigma}{dx_1}(p; q_i), \quad (k = p - \sum_1^n q_i, q \in C_1), \quad (3.14)$$

which generalizes Eq. (3.3).

Notice that the dependence of (3.14) on x/x_1 implies a rescaling of the structure function, when integrating over emitted momenta. These are therefore

the virtual corrections which regularize the $(q^2)^{-1}$ singularity in A-P-type equations for $x \rightarrow 0^{2),3)}$ and are related to gluon "Reggeization" in the region $s \gg k^{21)}$.

In fact, as discussed in Refs. 1)-3), the unintegrated gluon structure function satisfies the small x equation

$$\mathcal{F}(k^2, x) = \delta(k^2) \delta(x-1) + \bar{\alpha}_s \int_x^1 dz P(z) \int_0^{k^2/z} \frac{dk'^2}{(k^2 - k'^2)_R} \mathcal{F}(k'^2, \frac{x}{z}), \quad (3.15)$$

which takes into account all powers of $\bar{\alpha}_s/(N-1)$ in the anomalous dimension. The complete structure function is given by $F = x^{-1} \int \mathcal{F} dk^2$, and we have introduced the distributions

$$P(z) = \frac{1}{z} \left(\frac{1}{1-z} \right)_+, \quad (a)$$

$$\frac{1}{(k^2 - k'^2)_R} = \frac{1}{|k^2 - k'^2|} - \delta(k^2 - k'^2) \lim_{\lambda^2 \rightarrow 0} \int_0^{k^2} \frac{dq^2}{q^2 + \lambda^2} \quad (b) \quad (3.16)$$

It turns out that (3.16b), which follows from (3.12) and (3.14), regularizes the singularity at $k^2 = k'^2$ in (3.15), by leading to dominance of the region $k \gg k'$ in the perturbative regime $\bar{\alpha}_s \log 1/x \ll 1^{3)}$. Notice that the space-like anomalous dimension does not show singularities higher than $[\bar{\alpha}_s/(N-1)]^k$, and has the expansion

$$\gamma_N = \frac{\bar{\alpha}_s}{N-1} + \left(c_1 \frac{\bar{\alpha}_s^2}{N-1} + \dots \right) + \frac{\bar{\alpha}_s^4}{(N-1)^4} + \dots, \quad (3.17)$$

where the first two terms follow from explicit two-loop calculations²¹⁾ and the third from Eq. (3.15).

The relatively smooth behaviour (3.17) is related to transverse momentum ordering in the space-like case ($|k| > |k'| \dots$). On the other hand, unintegrated angular distributions show $\bar{\alpha}_s (\log 1/x)^2$ contributions in both real [Eq. (3.10)] and virtual terms [Eq. (3.14)]. The regularization (3.16b) is the expression of a mutual cancellation⁴⁾ in the region R ($\phi' < \phi < x'\phi'/y$) which is ordered in angles, but not in transverse momenta.

4. MODIFIED BRANCHING PROBABILITY AND DISCUSSION

We are now in a position to specify how virtual corrections affect the branching probability at the vertex

$$\begin{aligned}
 (E, k) &\rightarrow \left(\frac{E}{z}, k' \right) + \left(\frac{E}{z} (1-z), q \right), \\
 q &= k' - k, \quad k = E X, \quad q = E \frac{1-z}{z} \varphi,
 \end{aligned}
 \tag{4.1}$$

in the backward evolution scheme (Fig. 7).

We need to combine the no-emission probabilities in the incoherent and coherent regions which, to lowest order in α_s , yield

$$\Delta_v(E, X, \varphi, z) = 1 - \bar{\alpha}_s (\ell' + \ell'' + \ell_a + \ell_b),
 \tag{4.2}$$

where the logarithmic factors

$$\ell'' = \left(\log \frac{E \varphi}{Q_0} \frac{1-z}{z} \right)^2, \quad \ell' = \left(\log \frac{E \varphi}{Q_0 z} \right)^2,
 \tag{4.3}$$

come from the incoherent regions around the q (time-like) and k' (space-like) legs respectively, and

$$\begin{aligned}
 \ell_a &= \int \frac{dz'}{1-z'} \int \frac{d^2 \varphi'}{\pi \varphi'^2} \Theta \left(\varphi' - \frac{Q_0}{(1-z')E} \right) = \left(\log \frac{E}{Q_0} \right)^2 - \left(\log \frac{E \varphi}{Q_0} \right)^2, \quad (a) \\
 \ell_b &= \int \frac{dz'}{z'} \int \frac{d^2 \varphi'}{\pi \varphi'^2} \Theta (z' X - \varphi') \\
 &= \left(\log \frac{X}{\varphi} \right)^2 - \left(\log \frac{z X}{\varphi} \right)^2 \Theta (z X - \varphi), \quad (b)
 \end{aligned}
 \tag{4.4}$$

come from the coherent regions (a) $y \ll z$ and (b) $y \gg z$ respectively. Due to (4.3) and (4.4), Eq. (4.2) takes the form

$$\Delta_V = \frac{\Delta(E, Q_0)}{\Delta(E\varphi, Q_0)} \Delta_1\left(\frac{k}{\tilde{q}}, z\right) \cdot \Delta\left(E\varphi \frac{1-z}{z}, Q_0\right) \Delta\left(\frac{E\varphi}{z}, Q_0\right), \quad (4.5)$$

where we have introduced the notation $\tilde{q} = |q|(1-z) = E\phi/z$, and the factor

$$\Delta_1\left(\frac{k}{\tilde{q}}, z\right) = \begin{cases} 1 - \bar{\alpha}_s \log \frac{1}{z} \log \frac{k^2}{z \tilde{q}^2} & , \quad (\tilde{q} \leq k) & (a) \\ 1 - \bar{\alpha}_s \left(\log \frac{k}{z \tilde{q}}\right)^2 & , \quad (k < \tilde{q} < k/\sqrt{z}) & (b) \end{cases} \quad (4.6)$$

arises from the vertex-type corrections (3.13) and has no analogue in the time-like case.

The effect of Δ_1 is to provide extra damping, depending on the ratio $|k|/\tilde{q} \approx |k|/|k-k'|$, which becomes $\approx (1 - \bar{\alpha}_s \log^2 1/z)$ for $|k'| \ll |k|$, as found in Eq. (3.3). If instead $|q| = |k-k'| \ll |k|$, this damping becomes stronger, showing that the collinear singularity at $q = k-k' = 0$ is somewhat depressed at higher orders. The remaining region $|k| < |k'| < |k|/\sqrt{z}$, close to the kinematical boundary, corresponds to $\phi \approx -\phi'/z$ (Fig. 7) and has no phase space, because there is no $|k| = 0$ singularity.

The problem now arises of how to combine several vertices of type (4.5) in a branching scheme in order to describe multiple emission. The simplest way is to introduce Δ_1 as a multiplicative factor in the probability distribution for the branching process (4.1), i.e.,

$$dw^{(1)} = \bar{\alpha}_s dz P(z) \Delta_1\left(\frac{k}{\tilde{q}}, z\right) \frac{d^2\varphi}{\pi \varphi^2} \frac{\Delta(E\varphi_0, Q_0)}{\Delta(E\varphi, Q_0)} \bigcirc_{\varphi} \bigcirc_{\left[\frac{E\varphi}{z}(1-z) - Q_0\right]}, \quad (4.7)$$

$$\bigcirc_{\varphi} = \bigcirc_{(\varphi_0 - \varphi)} \bigcirc_{(\varphi - \varphi')} \bigcirc_{(\varphi - \theta'')},$$

where the superscript (1) recalls that (4.7) is obtained from the lowest order form of Δ_1 given before.

However, due to the mixed $|k|$ -, $\tilde{q}$ - and z -dependence of Δ_1 , it is not clear which form it will take at higher orders. In particular, the Ward identity method used in the form factor^{22), 23)} and Regge²⁴⁾ regions seems inapplicable to

the jet region that we consider here. (This region differs from Regge's because angles may be sizeable, but momentum transfers are not yet purely transverse.)

If anyway Eq. (3.15) is assumed for the structure function, it suggests that the new form factor Δ_1 is not simply multiplicative at higher orders. In fact, if we isolate in (3.15) the virtual terms of Eq. 3.16b), we obtain a Green's function $\delta(k, x/x')$ in x space satisfying the equation

$$\delta(k, x) + \bar{\alpha}_s \int_x^1 \frac{dz}{z} \log \frac{k^2}{\lambda^2} \delta(k, \frac{x}{z}) = \delta(x-1) \quad (4.8)$$

and given by the formula

$$\delta(k, z) = \delta(z-1) - \bar{\alpha}_s \log \frac{k^2}{\lambda^2} \exp \left[-\bar{\alpha}_s \log \frac{k^2}{\lambda^2} \log \frac{1}{z} \right]. \quad (4.9)$$

The relation of δ to Δ_1 should be a convolution with the A-P density, i.e.,

$$\int_z^1 \frac{dz'}{z'} \delta(z') = \Delta_1 \left(\frac{k}{\tilde{q}}, z \right). \quad (4.10)$$

It is clear that the expression (4.9) does not work since, by (4.10), it is inconsistent with the form (4.6) of Δ_1 : this one is $\tilde{q}$ -dependent, while (4.9) is not. The reason for this discrepancy is that ϕ -ordering is not built-in in Eq. (3.15): in fact, $k_\perp$ -ordering eventually emerges because of large cancellations between real and virtual terms.

However, the feature that δ is a kernel in x -space should be kept in our coherent emission approach, because of the form (3.14) of the virtual terms, involving a convolution in x -space.

By then introducing ϕ -ordering in the structure function on the basis of Eqs. (3.10) and (3.14), we find in Appendix C an evolution equation which is equivalent to (3.15) at double-log level, but has built-in ϕ -ordering. The new Green's function for the virtual terms is

$$\delta\left(\frac{k}{\tilde{q}}, z\right) = \delta(z-1) - 2\bar{\alpha}_s \log \frac{kz}{\tilde{q}} \otimes \left(k - \frac{\tilde{q}}{z}\right) \exp \left[-\bar{\alpha}_s \left(\log \frac{k}{\tilde{q}}\right)^2 + \bar{\alpha}_s \left(\log \frac{kz}{\tilde{q}}\right)^2 \right] =$$

$$= \frac{\partial}{\partial \log 1/z} \left[\Theta(1-z) \exp \left(-\bar{\alpha}_s \left(\log \frac{k}{\tilde{q}} \right)^2 + \bar{\alpha}_s \left(\log \frac{kz}{\tilde{q}} \right)^2 \Theta \left(k - \frac{\tilde{q}}{z} \right) \right) \right]. \quad (4.11)$$

This one is now consistent, by (4.10), with the lowest-order result (4.6) and is the form of δ that we propose.

The meaning of $\delta(k/\tilde{q}; z/z')$ is that of a probability density for having a rapidity gap ($x \rightarrow x' = x/z$) without additional gluon emission. This fact is connected with multiple gluon exchange in the $2 \rightarrow n$ subprocesses, and is typical of the small x region.

We are thus led to introduce a modified branching distribution for the process (4.1) of the form

$$dw = \bar{\alpha}_s dz \int_z^1 \frac{dz'}{z'} \delta \left(\frac{k}{\tilde{q}}, z' \right) \frac{1}{z'} P \left(\frac{z}{z'} \right) \Theta \left[\frac{E\varphi}{z} \left(1 - \frac{z}{z'} \right) - Q_0 \right] \\ \cdot \frac{d^2\varphi}{\pi \varphi^2} \frac{\Delta(E\varphi_0, Q_0)}{\Delta(E\varphi, Q_0)} \Theta \varphi, \quad (4.12)$$

whose first moment is normalized to $[(1 - \Delta(E\varphi_0, Q_0))]$, in view of the definition of δ and Δ . However, more work is needed in order to check the guess (4.11), especially for its $\tilde{q}$ -dependence.

Let us remark that the exact form of (4.12) is mostly important for the angular distribution of bremsstrahlung gluons at small $p_\perp$. For integrated distributions, on the other hand, the region R of disordered momentum transfers is expected to cancel with virtual corrections. Outside this region one can set $\delta = \delta(z'-1)$ while restricting the phase space by requiring $z\theta \approx \phi > \theta$, where θ labels the decay angles along the chain [Fig. 1b and Eq. (2.21)]. One thus obtains the restricted branching distribution¹²⁾

$$dw_R = \tilde{\alpha}_s dz P(z) \frac{\Delta(E\varphi_0, \varphi_0)}{\Delta(E\theta, \varphi_0)} \frac{d^2\theta}{\pi \tilde{\theta}^2} \mathcal{H}_\theta \mathcal{H}(E\theta(1-z) - \varphi_0),$$

$$\mathcal{H}_\theta = \mathcal{H}(\varphi_0 - \theta) \mathcal{H}(z\theta - \theta') \mathcal{H}(z\theta - \theta''), \quad (4.13)$$

which interpolates between angular ordering ($z \rightarrow 1$) and $k_\perp$ -ordering ($z \rightarrow 0$).

To sum up, we have found in this paper that coherent effects for angular distributions at small Bjorken x require the ordering of ϕ -angles with respect to the jet axis (rather than θ -angles or momentum transfers). The total cross-sections (and presumably y -distributions) are characterized by further cancellations with virtual corrections from which $k_\perp$ -ordering at small x emerges [Eqs. (3.9)-(3.14)].

Correspondingly, a new type of form factor, δ , appears in the branching process (4.12) and modifies the gluon A-P densities at small z in a q - and k -dependent way. We have computed δ to $O(\alpha_s)$, and guessed its form to higher orders as a kernel in x -space [Eqs. (4.8)-(4.11)]. For integrated distributions, the form factor δ is presumably compensated by integration over part of the angular phase space, and the simpler branching scheme (4.13) can be used.

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APPENDIX A: SOFT EMISSION CURRENT FOR SMALL x

Here we want to derive the form (2.15b) of the soft emission current in the case (b) $x \ll y$ discussed in the text. We have essentially to establish phase and normalization of diagram (d) with respect to (a) and (b) in Fig. 3.

By using helicity conservation in the (p, q') and (k', q) directions (Fig. 3d) and the polarization sum in the $\bar{p}$ -gauge

$$d^{\lambda\nu}(q) = - \left(g^{\lambda\mu} - \frac{q^\lambda \bar{p}^\mu}{\bar{p} \cdot q} \right) \left(g_\mu^\nu - \frac{\bar{p}_\mu q^\nu}{\bar{p} \cdot q} \right), \quad (A.1)$$

we obtain the contributions to j_μ

$$j_\mu^{(a,b)} = \frac{\bar{p}_\mu}{\bar{p} \cdot q} - \frac{p_\mu}{p \cdot q}, \quad (a)$$

(A.2)

$$j_\mu^{(d)} = - \left(\varepsilon_\mu(k') - \varepsilon(k') \cdot q \frac{\bar{p}_\mu}{\bar{p} \cdot q} \right) \cdot \frac{\varepsilon(k') \cdot (p+q')}{q'^2} \cdot x', \quad (b)$$

where the last x' factor is needed in order to factorize the amplitude $M(p-q, q')$ as in (2.15a) and rescales the k' -coupling with respect to the p -coupling.

From the explicit expression of the physical polarization $\varepsilon_\mu(k')$ in the (p, q') plane

$$\begin{aligned} \varepsilon_\mu(k') &= |q'|^{-1} (-q'_\mu + \bar{p}_\mu 2q'^2/x') \\ &= |q'|^{-1} (k'_\mu - x' p_\mu + \bar{p}\text{-terms}), \end{aligned} \quad \left(\varepsilon(k')^2 = -1 \right), \quad (A.3)$$

we get $\varepsilon \cdot (p+q') = 2|q'|/x'$ and Eq. (2.15b) follows. By then noting that the $\bar{p}_\mu$ -terms in (A.3) do not contribute to (A.2b), we obtain the alternative expression

$$j_\mu = \frac{1}{k'^2} \left[-k'^2 \frac{p_\mu}{p \cdot q} + k'^2 \frac{\bar{p}_\mu}{\bar{p} \cdot q} + 2 \left(k'_\mu - y p_\mu - \frac{\bar{p}_\mu}{\bar{p} \cdot q} (k' \cdot q - y p \cdot q) \right) \right], \quad (A.4)$$

from which the final form of j_μ in Eq. (2.17) follows:

$$j_\mu(k', q) = \frac{1}{k'^2} \left[2(k'_\mu - y p_\mu + \bar{y} \bar{p}_\mu) + k^2 \frac{\bar{p}_\mu}{\bar{p}q} - k'^2 \frac{p_\mu}{pq} \right]. \quad (A.5)$$

It is amusing to note that in the limit $x \ll x' \approx y$, $\bar{y} \approx \bar{x} \gg \bar{x}'$ ($\bar{p}q \approx -\bar{p}k$, $pq \approx pk'$) Eq. (A.5) reduces to the vertex function used by Lipatov¹⁾ in his treatment of Regge behaviour in gauge theories. However, we cannot use $\bar{x}$ -ordering here, because the k_i 's are not purely transverse. Furthermore, in the limit $y, \bar{y} \rightarrow 0$, we obtain that $k'^2 j_\mu / k^2$ reduces to the current J_μ^{tot} of Eq. (2.12), thus giving a good interpolation between these two limiting cases.

APPENDIX B: VIRTUAL TERMS FOR $y \gg x$

Here we shall exemplify the decomposition of Eqs. (3.2) and (3.3) for $d\sigma_{a,b}^{(1)}$ in the case of the 3g-diagram in the $\bar{p}$ -gauge (Fig. 5b).

In our notation the pole factors in the denominators are (1) q^2 , (2) $(p-q-q')^2$, (3) $(p-q)^2$, all linear in $\bar{y}$, with $i\varepsilon$ factors

$$(1) -i\varepsilon/y, (2) i\varepsilon/(x-y), (3) i\varepsilon/(1-y), \quad (B.1)$$

respectively. We have, therefore, $0 < y < 1$ and one can distinguish the two cases:

(A) $y < x$. One can close the $\bar{y}$ -contour on pole (1). The region $0 < y \ll x$ gives directly the eikonal contribution of q_μ while in the region $0 < x-y \ll x$, $\tilde{q} = p-q-q'$ becomes soft. In the latter region it is convenient to distort the contour over the poles (2) and (3): the (2)-contribution is eikonal, the (3)-contribution cancels with a similar one from case (B).

(B) $y > x$. Here one can close the $\bar{y}$ -contour on pole (3). The region $0 < 1-y \ll x$ gives directly the corresponding eikonal term and the one $0 < (y-x) \ll x$ cancels with (A).

The non-trivial regions are (B1) $x \ll 1-y \ll 1$ where (3) is eikonal and (B2) $x \ll y \ll 1$, where (1) and (2) have opposite p -component y and essentially transverse mass. The corresponding contributions in front of $d\sigma(p, q')$ are

$$(B1) \quad -\bar{\alpha}_s \int_x^1 \frac{dy}{y} \int \frac{d^2 q}{\pi q^2}, \quad (B.2)$$

$$(B2) \quad \bar{\alpha}_s \int_x^1 \frac{dy}{y} \int \frac{d^2 q}{\pi q^2} \frac{q \cdot (q + q')}{(q + q')^2}.$$

Let us then notice that self-energy diagrams contribute only to case (a), where they complete the coherent emission described by the current (2.12). By then summing the contributions (B.2) and by noting the azimuthal average

$$\left\langle \frac{\underline{q}'(\underline{q} + \underline{q}')}{(\underline{q} + \underline{q}')^2} \right\rangle_{\varphi_q} = \oplus (\underline{q}'^2 - \underline{q}^2) , \quad (\text{B.3})$$

we obtain Eq. (3.3) in the text.

APPENDIX C: STRUCTURE FUNCTION EQUATION WITH ϕ -ORDERING

Here we want to derive the analogue of Eq. (3.15) for the gluon structure function, by including the ϕ -ordering constraint according to Eqs. (3.10) and (3.14). We discuss in detail the small x region in which $P(z) \sim 1/z$.

Let us start by defining the unintegrated structure function $\mathcal{F}(\underline{k}, q, x)$, where we fix not only the momentum transfer $\underline{k} = -\sum_i \underline{q}_i$ and the momentum fraction $x = 1 - \sum_i y_i$, but also the upper momentum $q = E\phi$ (called $\tilde{q}$ in the text) which corresponds to the largest ϕ -angle.

It is also convenient to introduce the function $\dot{\mathcal{F}}$ with one unintegrated angle by the definition

$$\dot{\mathcal{F}}(\underline{k}, q, x) = \frac{1}{2} \frac{\partial}{\partial \log q} \mathcal{F}(\underline{k}, q, x) . \quad (C.1)$$

By then using the recurrence relations (3.10) and (3.14) and the ϕ -ordering condition $q > q'z'$, we obtain the analogue of the unintegrated A-P equation

$$\begin{aligned} \dot{\mathcal{F}}(\underline{k}, q, x) &= \frac{1}{2} \alpha_s \delta(q - Q_0) \delta^2(\underline{k}) \delta(x-1) - \bar{\alpha}_s \int \frac{dz}{z} 2 \log \frac{kz}{q} \Theta(k - \frac{q}{z}) \cdot \\ &\cdot \dot{\mathcal{F}}(\underline{k}, \frac{q}{z}, \frac{x}{z}) + \bar{\alpha}_s \int_x^1 \frac{dz}{z} \left\langle \mathcal{F}(\underline{k} + \frac{q}{z} \underline{n}, \frac{q}{z}, \frac{x}{z}) \Theta_R \right\rangle_{\underline{n}} , \end{aligned} \quad (C.2)$$

where we have introduced the phase space constraint $\Theta_R = \Theta[k^2/z - (\underline{k} + q\underline{n}/z)^2]$ and the azimuthal average $\langle \mathcal{F} \rangle_{\underline{n}}$ over the $\underline{n}$ direction.

In order to discuss Eq. (C.2), let us first write it in moment space, by collecting the virtual terms on the left-hand side:

$$\begin{aligned} \dot{\mathcal{F}}_N(\underline{k}, q) + \bar{\alpha}_s \int \frac{dz}{z} z^{N-1} 2 \log \frac{kz}{q} \Theta(kz - q) \dot{\mathcal{F}}_N(\underline{k}, \frac{q}{z}) = \\ = \delta^2(\underline{k}) \frac{1}{2} \alpha_s \delta(q - Q_0) + \bar{\alpha}_s \int \frac{dz}{z} z^{N-1} \left\langle \mathcal{F}_N(\underline{k} + \frac{q}{z} \underline{n}, \frac{q}{z}) \Theta_R \right\rangle_{\underline{n}} . \end{aligned} \quad (C.3)$$

Notice that for $k < q$ the virtual term vanishes ($q < zk$), and the real term too, by lack of phase space ($q \lesssim \sqrt{z} k$). Therefore, $\mathcal{F}_N(\underline{k}, q)$ is q-independent for

$q < k$: this will be useful in order to relate (C.3) to (3.15).

On the other hand, for $q > k$, $\dot{\mathcal{F}}_N$ is non-trivial. In order to recast (C.3) in the form of an evolution equation, we have to invert the z -integration kernel on the left-hand side, which is of Volterra type. The corresponding Green's function is obtained, by standard manipulations, in terms of the kernel δ of Eq. (4.11) in the text. Equation (C.3) then takes the form

$$\begin{aligned} \dot{\mathcal{F}}_N(k, q) &= \delta^2(k) \frac{1}{2} \omega_0 \delta(q - \omega_0) + \int_0^1 \frac{dz'}{z'} \delta\left(\frac{k}{q}, z'\right) \cdot \\ &\cdot \bar{\alpha}_s \int_0^{z'} \frac{d\bar{z}}{\bar{z}} \bar{z}^{n-1} \left\langle \mathcal{F}_N\left(\bar{k} + \frac{q}{\bar{z}} \bar{z}, \frac{q}{\bar{z}}\right) \oplus \left(\frac{z'}{\bar{z}} \bar{k}^2 - \left(\bar{k} + \frac{q}{\bar{z}} \bar{z}\right)^2\right) \right\rangle_{\bar{z}}. \end{aligned} \quad (C.4)$$

Note that the z' -dependence has almost disappeared in the last line by setting $\zeta = zz'$, due to the $d\zeta/\zeta$ measure. For a general A-P density one should replace it by $P(\zeta/z') d\zeta/z'$.

By integrating (C.4) over q , and reverting to x -space, we obtain the alternative equation

$$\begin{aligned} \mathcal{F}(k, q_0, x) &= \delta^2(k) \delta(x-1) + \int_0^{q_0} \frac{dq^2}{\pi q^2} \int_0^1 \frac{dz'}{z'} \delta\left(\frac{k}{q}, z'\right) \cdot \\ &\cdot \bar{\alpha}_s \int_0^{z'} \frac{d\bar{z}}{\bar{z}} P\left(\frac{\bar{z}}{z'}\right) \mathcal{F}\left(\bar{k} + \frac{1}{\bar{z}} q, \frac{q}{\bar{z}}, \frac{x}{\bar{z}}\right) \oplus_R. \end{aligned} \quad (C.5)$$

This form is the one that directly arises from the branching distribution (4.12) of the text, after regularization of the A-P density.

The form (C.5) is, however, not convenient for the actual solution of Eq. (C.3) because the kernel δ , analogous to a Sudakov factor, regularizes the $q = 0$ singularity only upon integration. We shall then resort to an explicitly subtracted form, which is obtained from (C.3) by a partial integration:

$$\frac{1}{2} \frac{\partial}{\partial \log q} \left[\mathcal{F}_N(k, q) + \bar{\alpha}_s \int_{q/k}^1 \frac{dz}{z} z^{n-1} \mathcal{F}_N\left(k, \frac{q}{z}\right) 2 \log \frac{kz}{q} \right] -$$

$$-\frac{1}{2}\alpha_0\delta(q-q_0)\delta^2(\underline{k}) = \bar{\alpha}_s \int \frac{dz}{z} z^{N-1} \left[\left\langle \mathcal{F}_N\left(\underline{k} + \frac{q}{z}\underline{n}, \frac{q}{z}\right) \Theta_R \right\rangle_n - \right. \\ \left. - \mathcal{F}_N\left(\underline{k}, \frac{q}{z}\right) \Theta\left(\underline{k} - \frac{q}{z}\right) \right]. \quad (C.6)$$

Note again that $\mathcal{F}_N$ takes the q -independent value $\mathcal{F}_N(\underline{k})$ for $k < q$. By q -integration of (C.6) we then obtain the equation for $\mathcal{F}_N(\underline{k})$:

$$\mathcal{F}_N(\underline{k}) = \delta^2(\underline{k}) + \frac{\bar{\alpha}_s}{N-1} \int \frac{d^2 q}{\pi q^2} \left[\mathcal{F}_N(\underline{k} + \underline{q}, q) \Theta_R - \mathcal{F}_N(\underline{k}, q) \Theta(\underline{k} - \underline{q}) \right], \quad (C.7)$$

which is very similar to (3.15), except for the q -dependence of the integrand on the right-hand side.

For $q \ll k$, we can exhibit the q -dependence of $\mathcal{F}_N$ at double-log level by evaluating the right-hand side of (C.6). By setting $q' = q/z$ and taking into account the phase space boundary, the latter takes the form

$$\frac{\bar{\alpha}_s}{2} \int_q^{k^2/q} \frac{d^2 q'}{\pi q'^2} \left(\frac{q}{q'}\right)^{N-1} \left[\mathcal{F}_N(\underline{k} + \underline{q}', q') - \mathcal{F}_N(\underline{k}, q') \Theta(\underline{k} - \underline{q}') \right], \quad (C.8)$$

which shows a subtraction similar to the right-hand side of (C.7).

Note now that the integration region $q \ll q' \ll k$ in (C.8) is suppressed by cancellation of real and virtual terms. One is then left with the regions $|\underline{k} + \underline{q}'| \ll q' \approx k$ and $q' \gg k$, which yield the contributions

$$\frac{\bar{\alpha}_s}{2} \left(\frac{q}{k}\right)^{N-1} \left[\frac{1}{k^2} \int_{\underline{k}}^{k^2} d^2 k' \mathcal{F}_N(k') + \int_k^\infty \frac{d^2 q'}{q'^2} \left(\frac{k}{q'}\right)^{N-1} \mathcal{F}_N(q') + \dots \right], \quad (C.9)$$

where use has been made of the q -independence of $\mathcal{F}_N(\underline{k}, q)$ for $k \lesssim q$. Since the q -dependence of (C.9) is $\sim q^{N-1}$, it is possible to invert explicitly (C.6) by the

δ -kernel, yielding the result

$$\mathcal{F}_N(\underline{k}, q) = \mathcal{F}_N(\underline{k}) \left[\Theta(q-k) + \Theta(k-q) \left(\frac{q}{k} \right)^{N-1} \exp\left(-\bar{\alpha}_s \left(\log \frac{k}{q}\right)^2\right) \right], \quad (C.10)$$

where, by (C.9),

$$\mathcal{F}_N(\underline{k}) = \delta^2(\underline{k}) + \frac{\bar{\alpha}_s}{N-1} \left(\frac{1}{k^2} \int^{k^2} dk'^2 \mathcal{F}_N(k'^2) + \dots \right), \quad (C.11)$$

and the dots stand for subleading terms coming from the regions $q' \gg k$ and $q' \ll k$ in (C.8).

Equation (C.11) is the small x form of the A-P equation. By the ansatz $\mathcal{F}_N \sim (k^2)^{\gamma-1}$ it yields the leading form $\gamma = \bar{\alpha}_s/(N-1)$ of the anomalous dimension. Therefore, Eqs. (C.7) and (3.15) are equivalent at double-log level.

On the other hand, the exact evaluation of the subleading terms in (C.8) or in (C.7) is a more delicate task which requires a full analysis of the q -dependence, beyond the double log result (C.10). Therefore it is not clear at present whether Eqs. (C.7) and (3.15) give the same expansion (3.17) of the anomalous dimension in powers of $\bar{\alpha}_s/(N-1)$, beyond the leading term.

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FIGURE CAPTIONS

- Fig. 1 (a) Time-like and (b) space-like branching schemes. In case (b) ϕ -angles with the jet axis and θ angles along the momentum-conserving chain are shown.
- Fig. 2 Two-gluon emission process in deep inelastic scattering (DIS).
- Fig. 3 Soft gluon insertion diagrams on (a)-(b) external lines and (c)-(d) internal ones in a physical gauge. Diagram (d) refers to a purely gluonic process.
- Fig. 4 Branching graphs for (a), (b) the incoherent regions and (c) the coherent one.
- Fig. 5 Virtual corrections to single gluon emission: (a) soft gluon insertion and (b) three-gluon vertex contribution in a physical gauge; (c) leading diagrams for $y \gg x$ in the Feynman gauge.
- Fig. 6 Chain of collinear singularities to be considered in the coherent region C_j .
- Fig. 7 The space-like branching vertex in energy and transverse momentum in the "backward" evolution scheme.

Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5

Fig. 6

Fig. 7