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A NEW INTERPRETATION OF THE πN σ -TERM

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A B S T R A C T

An explicit relation between the σ -term, the S-wave πN scattering length $a^{(+)}$ and the range parameter $b^{(+)}$ is established. The range term is shown to be the dominant feature. The important P-wave background terms in the extrapolation of the physical amplitude to the Cheng-Dashen point are shown to be very small if a path $v^2 + 2m_N v_B \approx 0$, i.e., with $\cos\theta \approx 0$, is chosen. This avoids the compensating cancellations between large numbers typical of other procedures. Correction terms are also shown to be small using physical arguments. The present relation is in a convenient form for phenomenological applications.

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Chiral symmetry breaking in low energy πN scattering is phenomenologically described in terms of the parameter $\sigma_{NN} \approx 60 \text{ MeV}$ ¹⁾⁻³⁾. This quantity is important not only for the understanding of the πN amplitude itself. It also enters in a considerable number of applications such as the NN interaction, the nuclear three-body force, the conditions for nuclear π -condensation, etc. On the quark level its value is crucial to the recent discussions of the strange quark content of the nucleon⁴⁾. The empirical determination of the σ_{NN} -term is achieved using dispersion relations by an analytical extrapolation of the information from physical πN scattering to the unphysical soft point or, more frequently, to the Cheng-Dashen point, which has an amplitude closely related to the soft amplitude. A classical example is the Altarelli-Cabibbo-Maiani (ACM) procedure⁵⁾ using forward dispersion relations in which the Δ -dominated total cross-sections enter as important ingredients. The final result appears as a sum of several terms with large mutual cancellations: only the high accuracy of πN data makes a satisfactory final result possible and no explanation for the cancellations is given. This situation is typical also of other extrapolations even when these are made along other paths in the analytic (ν, t) plane^{1)-3), 6)}: individual terms are (4-5) times the deduced value for the σ -term. The final result for σ_{NN} , on the other hand, is nowadays stable with $\sigma_{NN} \approx 60 \text{ MeV}$. This value causes theoretical problems^{3), 7)}, since it is nearly twice the value $\sigma_{NN} \approx (35 \pm 5) \text{ MeV}$, which follows from perturbative QCD with chiral symmetry breaking due to current quark mass terms

$$\sigma_{NN} = \frac{1}{2} (m_u + m_d) \langle \vec{p} | \bar{u}u + \bar{d}d | \vec{p} \rangle. \quad (1)$$

One possible explanation for this discrepancy is that the proton has a 21% probability of containing strange $\bar{s}s$ quark pairs, as compared to all $\bar{q}q$ pairs⁴⁾.

Consider now elastic πN scattering with the variables defined in Fig. 1. The isoscalar spin averaged invariant amplitude $F^{(+)}(\nu, t)$ [frequently denoted by $C^{(+)}(\nu, t)$ in the literature] has the usual partial wave expansion to leading order

$$\begin{aligned} \text{Re } F^{(+)}(\nu, t) &\approx 4\pi \frac{\sqrt{s}}{M} \sum_{\ell=0}^{\infty} [(\ell+1) \text{Re } f_{\ell+}^{(+)}(\nu) + \ell \text{Re } f_{\ell-}^{(+)}(\nu)] P_{\ell}(\cos \theta) \\ &\equiv 4\pi \frac{\sqrt{s}}{M} [a^{(+)} + b^{(+)} \vec{q}^2 + \dots + c^{(+)} \vec{q} \cdot \vec{q}' + \dots]. \end{aligned} \quad (2)$$

Here $f_1 = e^{i\delta_1} \sin \delta_1 / q$ has the expansion $(\text{tg } \delta_{l=0}/q)^{(+)} = a^{(+)} + b^{(+)} q^2 + \dots$.

Numerically $a^{(+)} = -0.010(3)m_\pi^{-1}$; $b^{(+)} = -0.044(7)m_\pi^{-3}$; $c^{(+)} = 0.21m_\pi^{-3}$, etc.⁸⁾.

$$\begin{aligned} v &= \frac{PQ}{M} \\ P_\mu &= \frac{(p+p')_\mu}{2} \\ Q_\mu &= \frac{(q+q')_\mu}{2} \\ t &= (q-q')^2 \\ v_B &= \frac{t-q^2-q'^2}{4M} = -\frac{qq'}{2M} \end{aligned}$$

It is convenient to remove the nucleon Born contributions in pseudovector form from the low-energy expansion. These are in the threshold region

$$a_{N,PV}^{(+)} \simeq -\left(1 + \frac{m_\pi}{M}\right)^{-1} \frac{g^2}{4\pi} \frac{m_\pi^2}{4M^3} \simeq -0.010m_\pi^{-1}; \quad b_{N,PV}^{(+)} \simeq \frac{g^2}{4\pi} \frac{1}{4M^3} \simeq 0.012m_\pi^{-3}, \quad (3)$$

where $g^2/4\pi \simeq 14.26$ is the πNN coupling constant. We denote the quantities with the pole term subtracted with a bar ($\bar{F}$, $\bar{a}$, etc.), so that

$$Re \bar{F}^{(+)} = 4\pi \frac{\sqrt{3}}{M} [\bar{a}^{(+)} + \bar{b}^{(+)} \vec{q}^2 + \dots + \bar{c}^{(+)} \vec{q} \cdot \vec{q}' + \dots] \quad (4)$$

with

$$\bar{a}^{(+)} = (0 \pm 0.003)m_\pi^{-1}; \quad \bar{b}^{(+)} = (-0.056 \pm 0.007)m_\pi^{-3}; \quad \bar{c}^{(+)} = 0.21 m_\pi^{-3}. \quad (5)$$

We have omitted small terms generated by unitarity for the time being. As will become obvious shortly the exact numerical value of the P-wave coefficient $\bar{c}^{(+)}$ is physically irrelevant in the extrapolation although its value is very large in the low-energy expansion. It is usual in the literature to determine the σ -term from the value of the amplitude at the Cheng-Dashen point characterized by the kinematics $v = v_B = 0$ and $q^2 = q'^2 = m_\pi^2$.

$$\bar{F}^{(+)}(v = v_B = 0; q^2 = q'^2 = m_\pi^2) = \frac{1}{f_\pi^2} \sigma_{NN} + \mathcal{O}(m_\pi^4). \quad (6)$$

Here the pion decay constant is $f_\pi \simeq 93.2$ MeV. The explicit kinematics of the Cheng-Dashen point is in the limit $E = E' = M$ for the nucleon energy:

$$q_0 = q'_0 = \frac{m_\pi^2}{2M}; \quad \vec{q}^2 = \vec{q}'^2 = -m_\pi^2; \quad \cos\theta = \hat{q} \cdot \hat{q}' = -\frac{m_\pi^2}{4M^2} \approx -\frac{1}{200}. \quad (7)$$

(The Breit system with $\vec{p} + \vec{p}' = 0$ nearly coincides with the centre-of-mass system in the present case. At the Cheng-Dashen point the variables are in this system

$$q_0^B = q_0'^B = 0; \quad \vec{q}_B^2 = \vec{q}_B'^2 = -m_\pi^2; \quad \cos\theta_B = 0.)$$

The very small value for $\cos\theta$, which vanishes in the limit $m_\pi = 0$ means that P-wave contributions are heavily suppressed at the Cheng-Dashen point (and so are contributions from any odd partial wave.)

This observation suggests that there is a great advantage to insist on $|\cos\theta| \approx 0$ in the extrapolation of the physical amplitude. This condition is not imposed on extrapolations in the literature. Indeed, the classical ACM extrapolation even maximizes the importance of the P-wave term by starting from a forward dispersion relation, i.e., $\cos\theta = 1$. This fact is the origin of the compensating cancellations between large terms which are implicit or explicit in current extrapolation procedures^{1)-3), 6)}.

The explicit linear extrapolation of the invariant amplitude to the Cheng-Dashen point gives approximately

$$\sigma_{NN}(t=2m_\pi^2) \approx 4\pi f_\pi^2 [\bar{a}^{(+)} - m_\pi^2 \bar{b}^{(+)} + \dots] \quad (8)$$

The empirical values for $\bar{a}^+$ and $\bar{b}^+$ give in this approximation^{*)}

*) A recent measurement at the π^-p atomic level shift allows one to deduce⁹⁾ $a_{\pi^-p} = 0.059(6)m_\pi^{-1}$ at variance with the value $a_{\pi^-p} = 0.081(3)m_\pi^{-1}$ quoted in the data tables⁸⁾. This would lead to $a^{(+)} = \pi^-p = -0.021(3)$ and therefore to a value of σ_{NN} about 10 MeV lower than normally deduced (both in our case here as well as in any other analysis). This new value must, however, be taken with great caution for the moment, since both π^-d and in particular $\pi^-^3\text{He}$ atomic level shifts are in better agreement and consistent with the normally quoted values for $a^{(+)}$ and a_{π^-p} . This value is also contradicted by a very recent determination from π^+p and π^-p Coulomb interference, which implies $a^{(+)} = (-0.0097 \pm 0.0017)m_\pi^{-1}$ ¹⁰⁾.

$$\sigma_{NN}(t=2m_\pi^2) \approx (0 \pm 3 + 44 \pm 5) \text{ MeV} \approx (44 \pm 6) \text{ MeV.} \quad (9)$$

In comparison to the characteristic values $\sigma_{NN} \approx (58 \text{ to } 64) \text{ MeV}$ obtained from elaborate extrapolations¹⁾⁻³⁾, it is clear that our simple procedure already gives the essence of the result. The σ -term is heavily dominated by the S-wave range term $\bar{b}^{(+)}$. Further, it explicitly vanishes in the chiral limit $m_\pi \rightarrow 0$, since it is proportional to m_π^2 . It is also noteworthy that $\bar{b}^{(+)}$ is a quantity which is well defined even in the chiral limit $m_\pi = 0$ and has a non-vanishing value there. The extrapolation to this limit should be smooth. This means that the physical connection between σ_{NN} and $\bar{b}^{(+)}$ can be established directly in any model of πN scattering which respects chiral symmetry in this limit without explicit reference to u and d quarks.

In the discussion up to this point we have been somewhat cavalier about smaller corrections which will now be discussed. The principal corrections to the result (8) are due to kinematics and to unitarity. We note first that the Born contribution from the pion pole term is regular at the Cheng-Dashen point: the extrapolation of $\bar{F}^{(+)}$ and $F^{(+)}$ to this point gives rather similar results. As a consequence it is not much of a limitation to use physical amplitudes directly in the discussion of corrections. For instance, if the physical condition $\cos\theta \approx 0$ is respected along the path $v^2 + 2Mv_B = 0$ [i.e., $\cos\theta_B = 0$] the pole term is to a good approximation $1/4 (g^2/4\pi)(2m_\pi^2 - v^2) M^{-3}$.

Unitarity corrections. The S-wave partial wave amplitudes

$$f_0 \equiv \frac{e^{i\delta_0} \sin \delta_0}{q} \equiv \frac{tg \delta_0}{q(1 - i tg \delta_0)} \quad (10)$$

have a cusp at threshold. As a consequence there is an additional contribution $\Delta F^{(+)}$ to $\text{Re} F^{(+)}$ below threshold, where q becomes imaginary with $q \equiv i\kappa$:

$$\Delta F^{(+)} \simeq -4\pi \kappa \left[\frac{t_0 \delta_0 / q}{1 + \kappa (t_0 \delta_0 / q)} \right]^{(+)} \quad (11)$$

This expression simplifies greatly at the Cheng-Dashen point since the isovector amplitude vanishes there and the $I = 1/2$ and $3/2$ S-wave amplitudes become equal. This is a consequence of crossing on the one hand and of the vanishing of the Born terms on the other. Thus

$$\Delta F^{(+)}(\nu = \nu_B = 0) \simeq -4\pi m_\pi (a^{(+)} - b^{(+)} m_\pi^2)^2 \simeq -0.015 m_\pi^{-1}. \quad (12)$$

This -2% term is implicitly included in all good extrapolation procedures.

P-wave contributions. The extrapolation of $c^{(+)}_{\vec{q} \cdot \vec{q}'}$ to the Cheng-Dashen point gives a small but non-vanishing contribution since $\cos\theta \neq 0$. The corresponding term $\delta F^{(+)}$ is

$$\delta F^{(+)}(\nu = \nu_B = 0) \simeq 4\pi c^{(+)} \frac{m_\pi^4}{4M^2} \simeq 0.015 m_\pi^{-1}. \quad (13)$$

This +2% contribution to the σ -term determined at the Cheng-Dashen point is presumably included in all the usual extrapolations. In view of the physical origin I strongly believe this term should be eliminated in the further extrapolation of $\sigma_{NN}(2m_\pi^2)$ to the true value $\sigma_{NN}(0)$ at the soft point $\nu = \nu_B = q^2 = q'^2 = 0$. This means that the true value of σ_{NN} is about 1 MeV smaller than the Cheng-Dashen value due to this effect.

Higher partial waves. These contributions are easily obtained using the same methods. Odd waves give no contributions since $\cos\theta = 0$. The even waves can be extrapolated "as if" there were no pion Born terms. One can easily verify that the most dangerous term, the D-wave contribution proportional to $(3a_2^{(+)} + 2a_2^{(-)})$, represents less than 1% in the present procedure.

The addition of the unitarity correction and the P-wave term gives finally the value for the σ -term at the Cheng-Dashen point

$$\sigma_{NN}(t=2m_\pi^2) \approx (44 \pm 6) \text{ MeV.}$$

I insist that this result is not intended to rival the elaborate extrapolation procedures which are used but to clarify the relation of the σ -term to physical πN scattering. Our result now raises another question. We have made the extrapolation using the physical pion mass. Let us now assume this quantity to be variable and also that the property $\bar{a}/\bar{b}m_\pi^2 \ll 1$ is retained in the limit $m_\pi \rightarrow 0$. If this is the case and in the limit $m_\pi \rightarrow 0$ the σ -term is explicitly proportional to m_π^2 and is exactly equivalent to

$$\sigma_{NN}(t=0) = -4\pi f_\pi^2 m_\pi^2 \bar{b}(m_\pi=0). \quad (14)$$

In this case an extrapolation to the Cheng-Dashen point and to the soft point $v = v_B = q^2 = q'^2 = 0$ give identical results. There are no unitarity corrections nor P-wave corrections in this limit.

The present result explains the origin of the compensation between large terms in the standard procedures based on variations of the ACM approach⁴⁾. The final result should be insensitive to contributions from πN P-waves and the Δ -isobar. In the physical forward scattering amplitude the P-wave amplitude is typically four times larger than the amplitude related to σ_{NN} , and these huge "background" terms must be eliminated in the final answer. This is implicitly achieved by a dispersion integral over total cross-sections, which largely gives the same P-wave amplitude from other physical data, but with opposite sign.

Here we have found a value for $\sigma_{NN} \approx 44 \text{ MeV}$ which is lower than the values quoted in the literature. Both the S-wave scattering length and range parameter have been taken from the same analysis of the data by the Karlsruhe group, which gives a value for $\sigma_{NN} \approx 64 \text{ MeV}$. Consistency therefore suggests that nearly the same value should result if the analysis is perfect.

I have only a partial explanation for the difference. In the expansion in $\vec{q}^2$ higher order terms were neglected in the S-wave expansion and the extrapolation to the Cheng-Dashen point is linear. In particular, terms of order $q_0/M \vec{q}^2$ have not been included in the discussion.

However, the Karlsruhe group has also made an expansion of the invariant amplitude in a power series of v^2 and t . The analysis is very closely related to the one in Ref. 2 and the coefficients are explicitly tabulated in Ref. 8, Table 2.4.7.1. The expansion uses the constraints of scattering data in a wide range of energies as well as dispersion relations. It should be a good guide for understanding the difference in the extrapolation. To this end I have identically rearranged the (v^2, t) expansion in terms of the more "natural" Breit frame quantities $\vec{q}_B \cdot \vec{q}'_B \equiv [(t - q^2 - q'^2)/2 + v^2]$ and $\vec{q}_B^2 \equiv (v^2 - m_\pi^2)$ on the mass shell. The linear terms with a small constant term describe the amplitude quite well. Since the Breit frame nearly coincides with the CM frame, the coefficient of the term $\vec{q}_B \cdot \vec{q}'_B$ coincides closely with the P-wave scattering volume $\bar{c}^{(+)}$, which is not surprising. In the reduction to the CM amplitude this term with a large, accurately known coefficient then produces an S-wave contribution $q_0/(M+q_0)\bar{c}^{(+)}\vec{q}^2$ with a contribution $m_\pi/(M+m_\pi)\bar{c}^{(+)} \approx 0.027 \text{ m}_\pi^{-3}$ to $\bar{b}^{(+)}$. This S-wave term gives a negligible contribution when extrapolated to the Cheng-Dashen point, since it is proportional to q_0 . When included in $\bar{b}^{(+)}$ its sign and magnitude accounts for the difference between the σ -term in the two approaches: it restores the value of σ_{NN} from 44 MeV to ≈ 65 MeV in agreement with the analysis of Koch²⁾ with $\sigma_{NN} \approx 64$ MeV.

While this explains the mathematics of the difference, it does not necessarily mean that (v^2, t) power series expansion gives the correct extrapolation of the range term $\bar{b}^{(+)}$. It is not obvious that the Δ -isobar dominated term should have the behaviour $\vec{q}_B \cdot \vec{q}'_B$ at low energy generating the above S-wave contribution. Indeed, there are ambiguities in the isobar terms as discussed, e.g., in Ref. 3), Appendix 8.1.2 and Ref. 11).

Caution should therefore be exerted with coefficients from the (v^2, t) expansion. The dominant features determining amplitudes are of P-wave origin, and no special precautions have been made to insist on particularly good consistency with S-wave data. This can have the effect that minor systematic contributions can be important. For instance, in the ACM procedure small inconsistencies between total cross-sections and measured amplitudes of only (3 to 4)% give differences which are about 15% in the σ -term. A detailed experimental study of the energy dependence of the S-wave amplitude would thus be most welcome. It is important for the understanding of the extrapolation procedures.

In conclusion: I have given here a physical, very direct argument that the low-energy πN scattering amplitude should be extrapolated so as to deemphasize the Δ -dominated P-wave term. It is then apparent that the σ -term corresponds mostly to the S-wave range term $\bar{b}^{(+)}$. The experiments determining this quantity are thus the essential ones for constraining the σ -term. The extrapolation from the physical region to the Cheng-Dashen point is very stable using the present procedure contrary to many methods in the literature.

It may be argued that the range term $\bar{b}^{(+)}$ is not directly measured but is also implicitly determined using analytical extrapolation. I disagree with this position for two reasons. First, the effect of $b^{(+)}$ is clearly observed in the behaviour of the phase shifts δ_1 and δ_3 with energy, and it is thus at least partially directly observable. In addition, the observation that the S-wave channel is the crucial one, means that one must pay particular attention to consistency with experiments in this channel in determining optimal analytical fits; such a special emphasis is not introduced at present to my knowledge. Furthermore, the very fact that the specific $b^{(+)}$ -term in the πN interaction is so closely related to the σ -term implies that it is conceivable to use, for example, low-energy πd scattering as an additional source of information. Experiments can now be designed to isolate this particular feature. The present relation also suggests simple schemes to incorporate the essence of the σ -term in nuclear problems involving pions.

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