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CHIRAL $D = 10$, $N = 2$ SUPERGRAVITY ON THE GROUP MANIFOLD:

I. - FREE DIFFERENTIAL ALGEBRA AND SOLUTION OF BIANCHI IDENTITIES

L. Castellani^{*)}

CERN - Geneva

A B S T R A C T

Chiral $D = 10$, $N = 2$ supergravity is derived by gauging a particular free differential algebra. This provides the necessary framework to construct its group manifold action. The situation parallels that of the $D = 6$ supergravity found by D'Auria, Fré and Regge, for which an action exists only on the group manifold. A short review on free differential algebras and their gauging is included for self-consistency.

^{*)} On leave of absence from the Istituto Nazionale di Fisica Nucleare, Sezione di Torino, Italy.

1. Introduction

The existence of supergravity (SG) in ten dimensions was first inferred in ref. 1) , as the zero-slope limit of the supersymmetric string. The lagrangian for $D = 10$, $N = 1$ SG , the field theory limit of type I superstrings ²⁾ , was constructed in refs. 3) . $D = 10$, $N = 2$ nonchiral SG , the field theory limit of type IIA superstrings, can be obtained ⁴⁾ by $11 \rightarrow 10$ dimensional reduction from $D = 11$ supergravity ⁵⁾ .

Here we prepare the ground for constructing the lagrangian of $D = 10$, $N = 2$ chiral SG , the zero slope limit of type IIB superstrings. Although the covariant field equations are known ⁶⁾ , a Lorentz invariant action in $D = 10$ spacetime does not exist, the reason being that one of the field equations is a selfduality condition $F_{a_1 \dots a_5} = \frac{1}{5!} \epsilon_{a_1 \dots a_5 b_1 \dots b_5} F_{b_1 \dots b_5}$. This condition cannot be obtained from a manifestly covariant action principle ⁷⁾ .

However, the lagrangian does exist ⁸⁾ on the group manifold, an extension of $D = 10$ spacetime where the fields of theory are interpreted as a complete cotangent basis ⁹⁾ . The variational equations, restricted to spacetime, correctly reproduce all of the field equations derived in ref. 6) , including the selfduality condition $F = \tilde{F}$. This is very similar to what happens in the $D = 6$ SG of ref. 10) : the selfduality of F_{abc} , necessary to match bosonic and fermionic degrees of freedom, can be obtained by varying the group manifold lagrangian $\mathcal{L}$, but not its spacetime restriction. In other words, the spacetime restriction of $\mathcal{L}$ is not supersymmetric ¹⁰⁾ .

The geometrical construction of lagrangian theories by the so-called group-manifold method ⁹⁾ can be summarized as follows. The starting point is the choice of a Lie algebra $\mathcal{G}$. The vielbein 1-form μ^A (A runs on the adjoint of $\mathcal{G}$) on the group manifold $G = \exp \mathcal{G}$ is taken to be the fundamental field of the theory. For example if $\mathcal{G}$ = Poincaré Lie algebra, μ^A is identified with the ordinary vielbein for A corresponding to the translation generators, and with the spin connection for A corresponding to the rotation generators.

In general supergravity theories may contain scalars and/or antisymmetric tensors, whose interpretation as vielbeins of an appropriate $G = \exp \mathcal{G}$ seems difficult. However, in the framework of free differential algebras ¹¹⁾⁻¹³⁾ (FDA's) , the dual formulation of Lie algebras is generalized to include p -forms ($p > 1$) , besides the 1-form μ^A . This elegant generalization is

just what is needed to describe the algebraic structure of any supergravity theory¹²⁾. A short review on FDA's is given in Section 2.

A dynamical theory of the generalized "potentials" μ^A requires the notion of "soft" group manifold, i.e. a deformation of $G = \exp \mathfrak{G}$ whose potentials μ^A do not need to satisfy the Cartan-Maurer equations of G . The extent of deformation is measured by the curvature $R^A = d\mu^A + \frac{1}{2} C^A_{BC} \mu^B \wedge \mu^C$, with $C^A_{BC} = \mathfrak{G}$ -structure constants. A lagrangian D-form can now be constructed in terms of R^A and μ^A following some general building principles⁹⁾, D being the number of spacetime dimensions. The field equations for μ^A are obtained by considering the variations $\delta \mathcal{L}_D / \delta \mu^A = 0$ on the whole group manifold. The selfduality conditions of the $D = 6$ supergravity of ref. 10) and of the chiral $D = 10$ supergravity discussed in the present paper are indeed obtained by projecting $\delta \mathcal{L} / \delta \mu^A = 0$ on "outside" directions (as opposed to "inside" or spacetime directions).

The theory described by $\mathcal{L}$ is automatically invariant under diffeomorphisms on the soft group manifold, since all operations used in $\mathcal{L}$, i.e. wedge products and exterior derivatives, are diffeomorphic invariant. These diffeomorphisms encompass the whole invariance structure of the theory, and are interpreted in spacetime as general coordinate, supersymmetry, and gauge transformations.

In Section 3 we present the free differential algebra of $D = 10$, $N = 2$ chiral supergravity. Section 4 is devoted to the FDA Bianchi identities, whose information content is twofold:

- i) they completely determine the curvatures R^A , and hence the transformation laws of all the fields,
- ii) they imply the spacetime field equations of all the fields.

These transformation laws and field equations coincide of course with those found at every order in the fermi fields in ref^{6b)}. Our vantage point consists in having reformulated $D = 10$, $N = 2$ chiral supergravity as a gauging of the FDA discussed in Section 3. This allows to construct an action on the corresponding group manifold⁸⁾.

2. Free differential algebras

Given a group element of G , $g(y) = \exp y^A T_A$ (T_A = generators of G), the left-invariant 1-form

$$\sigma(y) = g^{-1}(y) dg(y) \quad (2.1)$$

belongs to the Lie algebra of G , i.e.

$$\sigma(y) = \sigma^A(y) T_A \quad (2.2)$$

$\sigma^A(y)$ defines a cotangent basis on G . The Cartan-Maurer equations

$$d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0 \quad (2.3)$$

follow immediately from the definition (2.1), and can be considered as the dual formulation of the Lie algebra

$$[T_A, T_B] = C^C_{AB} T_C \quad (2.4)$$

Jacobi identities on C^A_{BC} translate into $d^2 = 0$ in the dual formulation. Eq. (2.3) deals with forms, and is therefore better suited to the purpose of constructing integrands (lagrangians). Also, it naturally extends to p -forms ($p \geq 1$):

$$d\vartheta^i_{(p)} + \sum \frac{1}{m} C^i_{i_1 \dots i_m} \vartheta^{i_1}_{(p_1)} \wedge \dots \wedge \vartheta^{i_m}_{(p_m)} = 0 \quad (2.5)$$

($p+1 = p_1 + \dots + p_m$)

$p, p_1, \dots, p_m$ are respectively the degrees of the forms $\vartheta^i, \vartheta^{i_1}, \dots, \vartheta^{i_m}$, $i, i_1, \dots, i_m$ are indices in irreps of G , and $C^i_{i_1 i_2 \dots i_m}$ are generalized structure constants satisfying generalized Jacobi identities (from $d^2 = 0$). When $p = p_1 = p_2 = 1$ and i, i_1, i_2 belong to the adjoint representation, eqs. (2.5) reduce to the ordinary Cartan-Maurer equations. The (anti)symmetry properties of the indices $i_1, \dots, i_m$ depend on the bosonic or fermionic character of the forms $\vartheta^{i_1}, \dots, \vartheta^{i_m}$.

In general eqs. (2.5) define a free differential algebra¹¹⁾ (FDA). The possible FDA extensions G' of a Lie algebra G have been studied in refs. 11)-13), and rely on the existence of Chevalley cohomology classes in G ¹⁴⁾.

Suppose that, given an ordinary Lie algebra $\mathcal{G}$, there exists a p-form

$$\Omega_{(p)}^i(\sigma) = \Omega_{A_1 \dots A_p}^i \sigma^{A_1} \wedge \dots \wedge \sigma^{A_p} \quad \begin{array}{l} i: \text{runs on a } \mathcal{G}\text{-irrep} \\ \Omega_{A_1 \dots A_p}^i = \text{constants} \end{array} \quad (2.6)$$

which is covariantly closed but not covariantly exact, i.e.

$$\nabla \Omega_{(p)}^i \equiv d\Omega_{(p)}^i + \sigma^A \wedge \mathcal{D}(T_A)^i_j \Omega_{(p)}^j = 0 \quad (2.7a)$$

$$\Omega_{(p)}^i \neq \nabla \Phi_{(p-1)}^i \quad (2.7b)$$

Then $\Omega_{(p)}^i$ is said to be a representative of a Chevalley cohomology class in the D_j^i irrep of $\mathcal{G}$. ∇ is the boundary operator satisfying $\nabla^2 = 0$ (use eqs. (2.6) and (2.3)). $\Omega_{(p)}^i(\sigma)$ allows the extension of the Lie algebra (2.3) to the FDA $\mathcal{G}'$:

$$d\sigma^A + \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0 \quad (2.8a)$$

$$\nabla \Sigma_{(p-1)}^i + \Omega_{(p)}^i(\sigma) = 0 \quad (2.8b)$$

where $\Sigma_{(p-1)}^i$ is a new (p-1)-form, not contained in $\mathcal{G}$. Closure of (2.8) is ensured because $\nabla \Omega_{(p)}^i(\sigma) = 0$. Note that (2.8a) can be cast in a form analogous to (2.8b):

$$\nabla \sigma^A - \frac{1}{2} C^A_{BC} \sigma^B \wedge \sigma^C = 0 \quad (2.9)$$

(recall that $(T_A)^B_C = C_{AC}^B$ in the adjoint representation). Closure of (2.9) again implies Jacobi identities on the structure constants.

It is clear that $\Omega_{(p)}^i(\sigma)$ differing by exact pieces $\nabla \Phi_{(p-1)}^i$ lead to equivalent FDA's, via the redefinition $\Sigma_{(p-1)}^i \rightarrow \Sigma_{(p-1)}^i + \Phi_{(p-1)}^i$.

The whole game can be repeated on the free differential algebra $\mathcal{G}'$ which contains now $\sigma^A, \Sigma_{(p-1)}^i$. One looks for the existence of polynomials in $\sigma^A, \Sigma_{(p-1)}^i$:

$$\Omega_{(q)}^i(\sigma, \Sigma) = \Omega_{A_1 \dots A_k i_1 \dots i_s}^i \sigma^{A_1} \wedge \dots \wedge \sigma^{A_k} \Sigma_{i_1}^{i_1} \wedge \dots \wedge \Sigma_{i_s}^{i_s} \quad (2.10)$$

satisfying the cohomology conditions (2.7). If such a polynomial exists, the FDA $\mathbb{G}'$ of eqs. (2.8) can be further extended to $\mathbb{G}''$, and so on.

In constructing D-dimensional supergravity theories we can choose as starting point the super-Poincaré Lie algebra, whose dual formulation is:

$$d\omega^{ab} - \omega^{ac} \wedge \omega^{cb} = 0 \quad (2.11a)$$

$$D V^a - \frac{i}{2} \bar{\psi} \wedge \Gamma^a \psi \equiv dV^a - \omega^{ab} \wedge V^b - \frac{i}{2} \bar{\psi} \wedge \Gamma^a \psi = 0 \quad (2.11b)$$

$$D\psi \equiv d\psi - \frac{1}{4} \omega^{ab} \wedge \Gamma_{ab} \psi = 0 \quad (2.11c)$$

ω^{ab} , V^a and ψ are the 1-forms associated to Lorentz rotations, translations and supersymmetry charge(s) respectively. The possible $\mathbb{G}'$ extensions to FDA's depend on D. For example in D = 11 there is a cohomology class on $\mathbb{G} : \{V^a, \omega^{ab}, \psi\}$ in the identity representation¹²⁾:

$$\Omega(V, \omega, \psi) = \frac{1}{2} \bar{\psi} \wedge \Gamma^{ab} \psi \wedge V^a \wedge V^b \quad (2.12)$$

$d\Omega = 0$ holds because of the D = 11 Fierz identity

$$\bar{\psi} \wedge \Gamma^{ab} \psi \wedge \bar{\psi} \wedge \Gamma^a \psi \wedge V^b = 0 \quad (2.13)$$

This allows the extension of (2.11) by means of a 3-form A :

$$dA - \Omega(V, \omega, \psi) = 0 \quad (2.14)$$

We now turn to consider the gauging of FDA's. Defining as curvatures R^A , $R_{(p+1)}^i$ the left-hand sides of eqs. (2.3) and (2.5) respectively, we can say that the FDA p-forms have vanishing curvature. In the case of ordinary Lie algebras, this corresponds to a "rigid" group manifold $\exp \mathbb{G}$

A dynamical theory of the FDA p-forms can be constructed by relaxing the conditions $R^A = 0$, $R_{(p+1)}^i = 0$, i.e. by considering FDA "potentials" μ^A , $\mu_{(p)}^i$ whose curvature is nonvanishing. The μ^A potentials are then a cotangent basis in a "soft" group manifold, a deformation of the

original $G = \exp G'$.

The Jacobi identities for the structure constants, combined with the definitions

$$R^A = d\mu^A + \frac{1}{2} C^A_{BC} \mu^B \wedge \mu^C \quad (2.15a)$$

$$R^i_{(p+1)} = d\mu^i_{(p)} + \sum \frac{1}{m} C^i_{i_1 \dots i_m} \mu^{i_1}_{(p_1)} \wedge \dots \wedge \mu^{i_m}_{(p_m)} \quad (2.15b)$$

imply the Bianchi identities:

$$dR^A + C^A_{BC} \mu^B \wedge R^C \equiv \nabla R^A = 0 \quad (2.16a)$$

$$dR^i_{(p+1)} + \sum \frac{1}{m} C^i_{i_1 \dots i_m} R^{i_1}_{(p_1)} \wedge \mu^{i_2}_{(p_2)} \wedge \dots \wedge \mu^{i_m}_{(p_m)} \quad (2.16b)$$

$$(-)^{p_1} \sum \frac{1}{m} C^i_{i_1 \dots i_m} \mu^{i_1}_{(p_1)} \wedge R^{i_2}_{(p_2+1)} \wedge \dots \wedge \mu^{i_m}_{(p_m)}$$

$$(-)^{p_1+p_2+\dots+p_{m-1}} \sum \frac{1}{m} C^i_{i_1 \dots i_m} \mu^{i_1}_{(p_1)} \wedge \dots \wedge \mu^{i_{m-1}}_{(p_{m-1})} \wedge R^{i_m}_{(p_m+1)} = 0$$

As we will explicitly see in Section 4, the curvatures R^A , $R^i_{(p+1)}$ are determined as solutions of the Bianchi identities. Knowing R^A , $R^i_{(p+1)}$, we have complete information on the invariance structure of the theory, even without knowing the explicit form of the lagrangian. Indeed it is sufficient to use the fact that the lagrangian is diffeomorphic invariant on the group manifold. The transformation of μ^A under diffeomorphisms is given by:

$$\begin{aligned} \delta \mu^A(\vec{y}) &= \mu^A(\vec{y} + \delta \vec{y}) - \mu^A(\vec{y}) = \\ &= \mathcal{L}_{\delta \vec{y}} \mu^A(y) = \left(\underline{\delta \vec{y}} \rfloor d + d \underline{\delta \vec{y}} \rfloor \right) \mu^A(y) = \\ &= d \delta y^A + \underline{\delta \vec{y}} \rfloor R^A - C^A_{BC} \delta y^B \mu^C = (\nabla \delta y)^A + \underline{\delta \vec{y}} \rfloor R^A \end{aligned} \quad (2.17)$$

where $\mathcal{L}_{\delta \vec{y}}$ is the Lie derivative along the infinitesimal vector $\delta \vec{y}$, and the contraction $\underline{\delta \vec{y}} \rfloor$ is defined on p-forms $\omega_{(p)} = \omega_{B_1 \dots B_p} \mu^{B_1} \wedge \dots \wedge \mu^{B_p}$ as:

$$\vec{t} \int \omega_{(p)} = p \, t^A \omega_{A B_1 \dots B_p} \mu^{B_1} \wedge \dots \wedge \mu^{B_p} \quad (2.18)$$

We see from (2.17) that the knowledge of R^A is sufficient to determine the transformation rules $\delta \mu^A$ leaving the theory invariant.

3. The free differential algebra of $D = 10$, $N = 2$ chiral supergravity

Chiral $D = 10$, $N = 2$ supergravity⁶⁾ (or 2b supergravity for short) contains a complex Weyl gravitino ψ_μ and a complex Weyl spinor λ of opposite handedness. The bosonic fields are: a graviton $g_{\mu\nu}$, a complex antisymmetric tensor $A_{\mu\nu}$, a real antisymmetric tensor $B_{\mu\nu\rho\lambda}$ (restricted by a selfduality condition) and a complex scalar ϕ . The theory has a global $U(1)$ symmetry that rotates the two supersymmetry charges into each other. The $U(1)$ charges of the fields are $U = \frac{1}{2}$ for the gravitino and $U = \frac{3}{2}$ for the spinor; $U = 0$ for $g_{\mu\nu}$, $U = 1$ for $A_{\mu\nu}$, $U = 0$ for $B_{\mu\nu\rho\lambda}$ and $\frac{1}{2}U = 2$ for ϕ .

Scalar fields in supergravity theories are interpreted as coordinates of noncompact coset spaces. Here the complex scalar ϕ can be seen as parametrizing the coset $SU(1,1)/U(1)$ ⁶⁾. In the formulation of refs. 6) that we adopt here, a real auxiliary field a is added to the complex scalar ϕ , and both are combined in a $SU(1,1)$ group matrix:

$$\begin{pmatrix} V_-^1 & V_+^1 \\ V_-^2 & V_+^2 \end{pmatrix} = \exp \begin{pmatrix} ia & \phi \\ \phi^* & -ia \end{pmatrix}, \quad \epsilon_{\alpha\beta} V_-^\alpha V_+^\beta = 1 \quad (3.1)$$

The $U(1)$ becomes then a local symmetry, necessary to compensate for the extra field a .

We give in Table 1 a list of the fields with their $SU(1,1)$ and $U(1)$ transformation properties. Indices α, β refer to $SU(1,1)$, subscripts $\pm$ to $U(1)$ charges $U = \pm 1$. In this $SU(1,1)$ basis we have:

$$(A^1)^* = A^2 \quad (3.2a)$$

$$(V_+^1)^* = V_-^2 \quad (3.2b)$$

It is moreover convenient to define the $SU(1,1)$ invariant quantities

$$\phi_\mu = -i \epsilon_{\alpha\beta} V_-^\alpha \partial_\mu V_+^\beta \quad (3.3a)$$

$$P_\mu = -i \epsilon_{\alpha\beta} V_+^\alpha \partial_\mu V_+^\beta \quad (3.3b)$$

Q_μ transforms as a $U(1)$ gauge field under $U(1)$, and will be used as a $U(1)$ -connection in covariant derivatives.

The FDA extension of the chiral $N = 2$ super-Poincaré Lie algebra in $D = 10$ is given by the following generalized Cartan-Maurer equations:

$$R^a = D V^a - i \bar{\psi} \Gamma^a \psi = 0 \quad (3.4a)$$

$$R^{ab} = d\omega^{ab} - \omega^{ac} \omega^{cb} = 0 \quad (3.4b)$$

$$\rho = D\psi = d\psi - \frac{1}{4} \omega^{ab} \Gamma_{ab} \psi - \frac{i}{2} \phi \psi = 0 \quad (3.4c)$$

$$R(A)^\alpha = dA^\alpha + 2i V_+^\alpha \bar{\psi} \Gamma^a \psi^* V^a + 2i V_-^\alpha \bar{\psi}^* \Gamma^a \psi V^a = 0 \quad (3.4d)$$

$$R(B) = dB + \bar{\psi} \Gamma^{a_1 \dots a_3} \psi V^{a_1 \dots a_3} + \frac{3}{4} \epsilon_{\alpha\beta} A^\alpha (V_+^\beta \bar{\psi} \Gamma^a \psi^* + V_-^\beta \bar{\psi}^* \Gamma^a \psi) V^a = 0 \quad (3.4e)$$

$$D\lambda = d\lambda - \frac{1}{4} \omega^{ab} \Gamma_{ab} \lambda - \frac{3}{2} i \phi \lambda = 0 \quad (3.4f)$$

$$D V_\pm^\alpha = dV_\pm^\alpha \mp i \phi V_\pm^\alpha = 0 \quad (3.4g)$$

The fields are all written as forms and we have omitted the wedge symbols.

The closure of the FDA in (3.4) is self-evident using $\tilde{D}^2 = 0$ and $\tilde{D}\psi = 0$, except for (3.4d) and (3.4e). The Cartan-Maurer equation for A^α is closed because of the $D = 10$ Fierz identities for complex Weyl gravitini :

$$\bar{\psi} \Gamma^a \psi^* \bar{\psi} \Gamma^a = 0 \quad (3.5a)$$

$$\Gamma^a \psi \bar{\psi}^* \Gamma^a \psi = 0 \quad (3.5b)$$

The closure of dB relies on eqs. (3.5) and on the identity

$$\bar{\psi} \Gamma^{abc} \psi \bar{\psi} \Gamma^c \psi = 2 \bar{\psi} \Gamma^a \psi^* \bar{\psi}^* \Gamma^b \psi + 2 \bar{\psi}^* \Gamma^a \psi \bar{\psi} \Gamma^b \psi^* \quad (3.6)$$

All the Fierz identities for complex Weyl spinors ψ can be derived straightforwardly by decomposing ψ as

$$\psi = \psi^1 + i \psi^2 \quad (3.7)$$

with ψ^1, ψ^2 Majorana-Weyl of same chirality, and by applying the known Fierz decompositions for $D = 10$ Majorana-Weyl spinors (see for ex. refs. 15) and 16)).

To summarize: the possibility of extending the $N = 2$ super-Poincaré Lie algebra $\mathcal{G}$ with "central charge" Q of (3.4a,b,c,g) is due to the existence of the cohomology classes in the $(V, \omega, \psi, V_1^\alpha)$ system:

$$V_+^\alpha \bar{\psi} \Gamma^a \psi^* V^a \quad (3.5a)$$

$$V_-^\alpha \bar{\psi}^* \Gamma^a \psi V^a \quad (3.5b)$$

and to the existence of the cohomology class in the $(V, \omega, \psi, V_1^\alpha, A^\alpha)$ system:

$$\bar{\psi} \Gamma^{a_1 \dots a_3} \psi V^{a_1 \dots a_3} + \frac{3}{4} \epsilon_{\alpha\beta} A^\alpha (V_+^\beta \bar{\psi} \Gamma^a \psi^* + V_-^\beta \bar{\psi}^* \Gamma^a \psi) V^a \quad (3.6)$$

4. Bianchi identities and determination of the curvatures.

The Bianchi identities for the FDA in (3.4) read:

$$\mathcal{D}R^a + R^{ab}V^b - i\bar{\psi}\Gamma^a\rho + i\bar{\rho}\Gamma^a\psi = 0 \quad (4.1a)$$

$$\mathcal{D}R^{ab} = 0 \quad (4.1b)$$

$$\mathcal{D}\rho + \frac{1}{4}R^{ab}\Gamma_{ab}\psi = 0 \quad (4.1c)$$

$$\begin{aligned} dR(A)^\alpha - 2i\mathcal{D}V_+^\alpha\bar{\psi}\Gamma^a\psi^*V^a - 2i\mathcal{D}V_-^\alpha\bar{\psi}^*\Gamma^a\psi V^a \\ - 2iV_+^\alpha\bar{\rho}\Gamma^a\psi^*V^a + 2iV_+^\alpha\bar{\psi}\Gamma^a\rho^*V^a \\ - 2iV_-^\alpha\bar{\rho}^*\Gamma^a\psi V^a + 2iV_-^\alpha\bar{\psi}^*\Gamma^a\rho V^a = 0 \end{aligned} \quad (4.1d)$$

$$\begin{aligned} dR(B) - \bar{\rho}\Gamma^{a_1-a_3}\psi V^{a_1-a_3} + \bar{\psi}\Gamma^{a_1-a_3}\rho V^{a_1-a_3} \\ - \frac{3}{4}\epsilon_{\alpha\beta}R(A)^\alpha(V_+^\beta\bar{\psi}\Gamma^a\psi^* + V_-^\beta\bar{\psi}^*\Gamma^a\psi)V^a \\ - \frac{3}{4}\epsilon_{\alpha\beta}A^\alpha(\mathcal{D}V_+^\beta\bar{\psi}\Gamma^a\psi^* + \mathcal{D}V_-^\beta\bar{\psi}^*\Gamma^a\psi)V^a \\ - \frac{3}{4}\epsilon_{\alpha\beta}A^\alpha(V_+^\beta\bar{\rho}\Gamma^a\psi^* + V_-^\beta\bar{\rho}^*\Gamma^a\psi)V^a \\ + \frac{3}{4}\epsilon_{\alpha\beta}A^\alpha(V_+^\beta\bar{\psi}\Gamma^a\rho^* + V_-^\beta\bar{\psi}^*\Gamma^a\rho)V^a = 0 \end{aligned} \quad (4.1e)$$

$$\mathcal{D}(\mathcal{D}\lambda) + \frac{1}{4}R^{ab}\Gamma_{ab}\lambda = 0 \quad (4.1f)$$

$$\mathcal{D}(\mathcal{D}V_\pm^\alpha) = 0 \quad (4.1g)$$

The curvatures $R^a, R^{ab}, \rho, R(A)^\alpha, R(B), \mathcal{D}\lambda$ and $\mathcal{D}V_\pm^\alpha$ satisfying eqs. (4.1) are found as follows. First we write their most general parametrization on the V, ψ directions, compatible with:

- i) Lorentz covariance
- ii) $SU(1,1)_{\text{global}} \times U(1)_{\text{local}}$ covariance
- iii) invariance under the scaling symmetry of the Bianchi identities (4.1):

$$\begin{aligned}
 V^a &\rightarrow e V^a & V_{\pm}^{\kappa} &\rightarrow V_{\pm}^{\kappa} \\
 \psi &\rightarrow e^{1/2} \psi & A &\rightarrow e^{\epsilon} A \\
 \omega &\rightarrow \omega & B &\rightarrow e^4 B \\
 \lambda &\rightarrow e^{-1/2} \lambda
 \end{aligned}
 \tag{4.2}$$

Note that the group manifold directions are really spanned by the whole cotangent basis V, ψ, ω . However requiring Lorentz covariance restricts the curvatures to have only $V-\psi$ nonvanishing components, and all the fields to depend only on superspace coordinates x, Θ 9).

We also assume vanishing torsion, i.e. $R^a = 0$. This ansatz is indeed compatible with the Bianchi identities, since they do admit a solution with $R^a = 0$ (see below). $R^a \neq 0$ would simply result in a redefinition of the spin connection. To make contact with refs. 6), $R^a = 0$ is necessary, since there the spin connection is the torsionless one.

The most general parametrization of the curvatures satisfying the conditions i), ii), iii) is:

$$R^{ab} = R^{ab}_{cd} V^{cd} + \bar{\psi} \Theta^{ab}_c V^c + \bar{\psi} W^{ab} \psi + \bar{\psi} W^{ab'} \psi^* + \text{herm. conj.}
 \tag{4.3a}$$

$$\begin{aligned}
 \rho = & \rho_{ab} V^{ab} + \kappa_0 \Gamma^{a_1-a_4} \psi V^{a_5} F_{a_1-a_5} + \kappa_1 \Gamma^{a_1-a_6} \psi V^{a_6} F_{a_1-a_5} \\
 & + \kappa_2 \Gamma^{a_1-a_4} \psi^* V^{a_5} V^{\kappa}_{+} F^{\beta}_{a_2-a_4} \in \kappa\beta \\
 & + \kappa_3 \Gamma^{a_2 a_3} \psi^* V^{a_1} V^{\kappa}_{+} F^{\beta}_{a_1-a_3} \in \kappa\beta \\
 & + \kappa_4 \Gamma^{a_1} \lambda \bar{\psi} \Gamma^{a_2} \psi^* + \kappa_5 \Gamma^{b_1-b_5} \lambda \bar{\psi} \Gamma^{b_1-b_5} \psi^* \\
 & + \kappa_6 \Gamma^{a_1 a_2} \psi \bar{\lambda} \Gamma^{a_2} \lambda V^{a_1} + \kappa_7 \psi \bar{\lambda} \Gamma^{a_2} \lambda V^{a_1} \\
 & + \kappa_8 \Gamma^{a_1-a_4} \psi \bar{\lambda} \Gamma^{a_2-a_4} \lambda V^{a_1} + \kappa_9 \Gamma^{a_1 a_2} \psi \bar{\lambda} \Gamma^{a_1-a_3} \lambda V^{a_3} \\
 & + \kappa_{10} \Gamma^{a_1-a_6} \psi \bar{\lambda} \Gamma^{a_1-a_5} \lambda V^{a_6} + \kappa_{11} \Gamma^{a_1-a_4} \psi \bar{\lambda} \Gamma^{a_1-a_5} \lambda V^{a_5}
 \end{aligned}
 \tag{4.3b}$$

$$R(A)^{\kappa} = F^{\kappa}_{abc} V^{abc} + a_1 \left[V^{\kappa}_{+} \bar{\psi}^* \Gamma^{ab} \lambda^* V^{ab} + V^{\kappa}_{-} \bar{\psi} \Gamma^{ab} \lambda V^{ab} \right]
 \tag{4.3c}$$

$$\begin{aligned}
 R(B) &= F_{a_1-a_5} V^{a_1-a_5} + b_1 \bar{\lambda} \Gamma^{a_1-a_5} \lambda V^{a_1-a_5} + \\
 &+ b_2 \epsilon_{\alpha\beta} A^\alpha V^\beta \bar{\lambda} \Gamma^{a_1 a_2} \psi V^{a_1 a_2} + b_2^* \epsilon_{\alpha\beta} A^\alpha V^\beta \bar{\psi} \Gamma^{a_1 a_2} \lambda V^{a_1 a_2} \\
 \mathcal{D}\lambda &= l_0 i P_a \Gamma^a \psi^* + l_1 \Gamma^{a_1-a_3} \psi \epsilon_{\alpha\beta} V^\alpha F^\beta_{a_1-a_3} \quad (4.3d) \\
 \mathcal{D}V_+^\alpha &= \mu_0 V_-^\alpha \bar{\psi}^* \lambda + \mu_1 V_-^\alpha P_a V^a \quad (4.3e) \\
 \mathcal{D}V_-^\alpha &= \mu_0^* V_+^\alpha \bar{\psi} \lambda^* + \mu_1^* V_+^\alpha P_a^* V^a \quad (4.3f)
 \end{aligned}$$

Inserting (4.3) into the Bianchi identities (4.1) fixes the coefficients to be

$$\begin{aligned}
 \pi_0 &= \frac{5i}{96} & \pi_1 &= \frac{i}{96} & \pi_2 &= -\frac{1}{32} & \pi_3 &= \frac{9}{32} & \pi_4 &= \frac{7}{16} & \pi_5 &= -\frac{7}{16 \cdot 1680} \\
 \pi_6 &= -\frac{3i}{128} & \pi_7 &= \frac{2i}{128} & \pi_8 &= -\frac{1}{8} & \pi_9 &= \frac{5}{8} & \pi_{10} &= -\frac{1}{960} & \pi_{11} &= \frac{5}{960} \\
 a_1 &= 1 & b_1 &= \frac{1}{80} & b_2 &= \frac{3i}{8} \\
 l_0 &= 1 & l_1 &= -\frac{i}{8} & \mu_0 &= 1 & \mu_1 &= 1
 \end{aligned} \quad (4.4)$$

Moreover, from the $\mathcal{D}\rho$ and $\mathcal{D}\mathcal{D}\lambda$ Bianchi identities we find, after using $R^a = 0$ to retrieve second order formalism, the same propagation equations of refs. 6). The selfduality condition

$$F_{a_1-a_5} = -\frac{1}{5!} \epsilon_{a_1-a_5 b_1-b_5} F_{b_1-b_5} \quad (4.5)$$

arises in the $\bar{\psi} \Gamma^{a_1} \psi V^{a_2-a_5} F_{a_1-a_5}$ projection of the $dR(B)$ Bianchi identity.

Table 2 summarizes our results for the curvatures. The transformation rules of the fields under all the symmetries of the theory (general coordinate transformations in spacetime, supersymmetry, A and B gauge transformations) can be easily read off, using eq. (2.17).

5. Conclusions

We have found the free differential algebra underlying $D = 10$, $N = 2$ chiral supergravity. The whole information on the theory is contained in the FDA : indeed, by solving the corresponding Bianchi identities in Section 4, we obtain the transformation rules and the propagation equations of all the fields. This is due to the known fact that the supersymmetry algebra closes only on shell⁶⁾.

The next step, the actual construction of the group manifold lagrangian, requires only the use of the standard techniques described in refs. 9). This will be discussed in a forthcoming paper⁸⁾.

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APPENDIX - Conventions and gamma algebra in D = 10

For all index contractions we use the Minkowskian metric:

$$\eta_{ab} = (+ - - - - - - - - - -)$$

The notation $V^{abc\dots}$ is short for $V^a \wedge V^b \wedge V^c \wedge \dots$

Gamma algebra conventions:

$$\{\Gamma_a, \Gamma_b\} = 2\eta_{ab} \quad , \quad \Gamma_1^\dagger = \Gamma_1 \quad , \quad \Gamma_a^\dagger = -\Gamma_a \quad (a \neq 1) \quad , \quad \Gamma_1 \Gamma_a^\dagger \Gamma_1 = \Gamma_a$$

$$\Gamma_{11} = \Gamma_1 \dots \Gamma_{10} \quad , \quad \Gamma_{11}^\dagger = \Gamma_{11} \quad , \quad \Gamma_{11}^2 = 1$$

$$C^2 = -1 \quad , \quad C^* = C \quad , \quad C^T = -C$$

$$C \Gamma_a C^{-1} = -\Gamma_a^T \quad , \quad C \Gamma_{11} C^{-1} = -\Gamma_{11}^T$$

$$C \Gamma^{a_1 \dots a_m} C^{-1} = \begin{cases} + (\Gamma^{a_1 \dots a_m})^T & \text{if } m = 3, 4, 7, 8 \\ - (\Gamma^{a_1 \dots a_m})^T & \text{if } m = 1, 2, 5, 6, 9, 10 \end{cases}$$

Chirality of the spinor fields:

$$\Gamma_{11} \psi = -\psi \quad , \quad \Gamma_{11} \lambda = +\lambda$$

Reality properties of spinor currents:

$\bar{\psi} \Gamma^a \psi$	imaginary	$\bar{\lambda} \Gamma^a \lambda$	real
$\bar{\psi} \Gamma^{a_1 \dots a_3} \psi$	real	$\bar{\lambda} \Gamma^{a_1 \dots a_3} \lambda$	imaginary
$\bar{\psi} \Gamma^{a_1 \dots a_5} \psi$	imaginary	$\bar{\lambda} \Gamma^{a_1 \dots a_5} \lambda$	real

i.e. $(\bar{\psi} \Gamma^a \psi)^\dagger = -\bar{\psi} \Gamma^a \psi$ etc

Duality:

$$\epsilon_{a_1 \dots a_m b_1 \dots b_m} \Gamma_{b_1 \dots b_m} = (-1)^{\frac{m}{2}(m-3)} m! \Gamma_{a_1 \dots a_m} \Gamma_{11}$$

$$\epsilon_{12345678910} = +1$$

TABLE 1 - Fields of 2b supergravity in the SU(1,1)/U(1) formulation

field	SU(1,1) repr.	U(1) charge
V_μ^a	-	0
Ψ	-	$\frac{1}{2}$
$A_{\mu\nu}^\alpha$	doublet	-
$B_{\mu\nu\rho\lambda}$	-	-
λ	-	$\frac{3}{2}$
$V_\pm^\alpha$	doublet	± 1

TABLE 2 - Curvatures of 2b supergravity

$$R^a = 0$$

$$\begin{aligned} \rho = & \rho_{ab} V^{ab} + \frac{5i}{96} \Gamma^{a_1 a_4} \psi V^{a_5} F_{a_1 a_5} + \frac{i}{96} \Gamma^{a_1 a_6} \psi V^{a_6} F_{a_1 a_5} \\ & + \frac{1}{32} \left[-\Gamma^{a_1 a_4} \psi^* V^{a_5} + 9 \Gamma^{a_2 a_3} \psi^* V^{a_4} \right] V_+^\alpha F_{a_2 a_4}^\beta \epsilon_{\alpha\beta} \\ & + \frac{7}{16} \left[\Gamma^{a_1} \lambda \bar{\psi} \Gamma^{a_2} \psi^* - \frac{1}{1680} \Gamma^{b_1 b_5} \lambda \bar{\psi} \Gamma^{b_1 b_5} \psi^* \right] \\ & + \frac{i}{128} \left[-3 \Gamma^{a_1 a_2} \psi \bar{\lambda} \Gamma^{a_2} \lambda V^{a_1} + 2 \psi \bar{\lambda} \Gamma^{a_2} \lambda V^{a_1} \right] \\ & + \frac{1}{8} \left[-\Gamma^{a_1 a_4} \psi \bar{\lambda} \Gamma^{a_2 a_4} \lambda V^{a_1} + 5 \Gamma^{a_1 a_2} \psi \bar{\lambda} \Gamma^{a_1 a_3} \lambda V^{a_3} \right] \\ & + \frac{1}{960} \left[-\Gamma^{a_1 a_6} \psi \bar{\lambda} \Gamma^{a_1 a_5} \lambda V^{a_6} + 5 \Gamma^{a_1 a_4} \psi \bar{\lambda} \Gamma^{a_1 a_5} \lambda V^{a_5} \right] \end{aligned}$$

$$R(A)^\alpha = F_{abc}^\alpha V^{abc} + V_+^\alpha \bar{\psi}^* \Gamma^{ab} \lambda^* V^{ob} + V_-^\alpha \bar{\psi} \Gamma^{ab} \lambda V^{ab}$$

$$\begin{aligned} R(B) = & F_{a_1 a_5} V^{a_1 a_5} + \frac{1}{80} \bar{\lambda} \Gamma^{a_1 a_5} \lambda V^{a_1 a_5} + \\ & + \frac{3i}{8} \epsilon_{\alpha\beta} A^\alpha V_+^\beta \bar{\lambda} \Gamma^{a_1 a_2} \psi V^{a_1 a_2} - \frac{3i}{8} \epsilon_{\alpha\beta} A^\alpha V_-^\beta \bar{\psi} \Gamma^{a_1 a_2} \lambda V^{a_1 a_2} \end{aligned}$$

$$\mathcal{D}\lambda = i \rho_a \Gamma^a \psi^* - \frac{i}{8} \Gamma^{a_1 a_3} \psi \epsilon_{\alpha\beta} V_+^\alpha F_{a_1 a_3}^\beta$$

$$\mathcal{D}V_+^\alpha = V_-^\alpha \bar{\psi}^* \lambda + V_-^\alpha \rho_a V^a$$

$$\mathcal{D}V_-^\alpha = V_+^\alpha \bar{\psi} \lambda^* + V_+^\alpha \rho_a^* V^a$$

$$\begin{aligned} R^{ab} = & R^{ab}_{cd} V^{cd} - i \bar{\psi} \Gamma^a \rho_{bc} V^c \\ & + i \bar{\psi} \left[\frac{i}{96} \Gamma^a \Gamma^{b_1 b_5} \Gamma^b \psi F_{b_1 b_5} - \frac{1}{32} \Gamma^a \Gamma^{b a_2 a_4} \psi^* V_+^\alpha F_{a_2 a_4}^\beta \epsilon_{\alpha\beta} \right. \\ & \left. + \frac{9}{32} \Gamma^a \Gamma^{a_2 a_3} \psi^* V_+^\alpha F_{a_2 a_3}^\beta \epsilon_{\alpha\beta} - \frac{3i}{128} \Gamma^a \Gamma^{b a_2} \psi \bar{\lambda} \Gamma^{a_2} \lambda \right] \end{aligned}$$

TABLE 2 continued

$$\begin{aligned}
 & + \frac{2i}{128} \Gamma^a \psi \bar{\lambda} \Gamma^b \lambda - \frac{1}{8} \Gamma^a \Gamma^{b a_1 a_2} \psi \bar{\lambda} \Gamma^{a_1 a_2} \lambda \\
 & + \frac{5}{8} \Gamma^a \Gamma^{a_1 a_2} \psi \bar{\lambda} \Gamma^{a_1 a_2 b} \lambda - \frac{1}{960} \Gamma^a \Gamma^{a_1 a_2 a_3 b} \psi \bar{\lambda} \Gamma^{a_1 a_2 a_3} \lambda \\
 & + \frac{5}{960} \Gamma^a \Gamma^{a_1 a_2 a_3} \psi \bar{\lambda} \Gamma^{a_1 a_2 a_3 b} \lambda \Big] + \text{herm. conj.}
 \end{aligned}$$

REFERENCES

- 1) F. Gliozzi, D. Olive and J. Scherk, Nucl. Phys B122 (1977) 253.
- 2) For a review on superstrings see for ex. :
J. H. Schwarz, Phys. Rep. 89 (1982) 223;
M.B. Green, J.H. Schwarz and E. Witten, "Superstring Theory", Cambridge Univ. Press 1987.
- 3) A.H. Chamseddine, Nucl. Phys. B185 (1981) 403;
E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, Nucl. Phys. B195 (1982) 97;
G.F. Chapline and N. Manton, Phys. Lett. 120B (1983) 105.
- 4) I.C. Campbell and P.C. West, Nucl. Phys. B243 (1984) 112;
F. Giani and M. Pernici, Phys. Rev. D30 (1984) 325;
M. Huq and M.A. Namazie, Class. Quant. Grav. 2 (1985) 293.
- 5) E. Cremmer, B. Julia and J. Scherk, Phys. Lett. 76B (1978) 409.
- 6a) M.B. Green and J.H. Schwarz, Phys. Lett. 109B (1982) 444; 122B (1983) 43;
J.H. Schwarz and P. C. West, Phys. Lett. 126B (1983) 301;
J.H. Schwarz, Nucl. Phys. B266 (1983) 269.
- 6b) P. Howe and P.C. West, Nucl. Phys. B243 (1984) 181.
- 7) N. Marcus and J.H. Schwarz, Phys. Lett. 115B (1982) 111;
For a possible resolution using auxiliary fields (lagrange multipliers) see W. Siegel, Nucl. Phys. B238 (1984) 307.
- 8) L. Castellani, in preparation.
- 9) For reviews on the group manifold method see for ex.:
L. Castellani, R. D' Auria and P. Fré in "Supersymmetry and Supergravity 1983 ", Proceedings of the XIX-th Winter School and Workshop, Karpacz 1983 , ed. B. Milewski, World Scientific 1983;
L. Castellani, R. D' Auria and P. Fré, "Supergravity Theory: a Geometrical Perspective", World Scientific 1987, to appear.
- 10) R. D'Auria, P. Fré and T. Regge, Phys. Lett. B128 (1983) 44.

- 11) D. Sullivan, Bulletin de l' Institut des Hautes Etudes Scientifiques
Publication Mathématique n^o 47 (1981).
- 12) R. D' Auria and P. Fré, Nucl. Phys. B201 (1982) 101.
- 13) L. Castellani, P. Fré, G. Giani, K. Pilch and P. van Nieuwenhuizen,
Ann. of Phys. 146 (1983) 35.
- 14) C. Chevalley and S. Eilenberg, Trans. Amer. Math. Soc. 63 (1948) 85.
- 15) R. D' Auria, P. Fré and A.J. Da Silva, Nucl. Phys. B196 (1982) 205.
- 16) S. Ferrara, P. Fré and M. Porrati, "First order higher curvature
supergravities and effective theories of strings", preprint UCLA
86/TEP/20 (1986).