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GRAVITON AND TOPOLOGY CONTRIBUTIONS TO SELF-CONSISTENT COSMOLOGY

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A B S T R A C T

We study the graviton contribution and the topology effects of antipodal identification in constant curvature solutions of semi-classical Einstein equations. We analyze the curvature R as a function of the cosmological constant Λ , of the topology (labelled here by a discrete parameter σ), and of the trace anomaly γ , the mass m and the coupling ξ of quantum matter fields. For $m = 0$, we find eight possible (some of them classically forbidden) configurations depending on the graviton-matter balance. Even if $\Lambda > 0$, R can be negative and even if $\Lambda \neq 0$, R goes to zero when N (the number of matter fields) goes to infinity. For $m \neq 0$ we find five characteristic types of behaviours depending on the values of ξ and σ . The "back-reaction" effects of the topology appear more important for small ξ and increasing R .

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1. - INTRODUCTION

Cosmological models based on self-consistent solutions of semi-classical Einstein equations have attracted much attention in the last years¹⁾⁻⁶⁾. Semi-classical in this context means that matter fields $\hat{\phi}$, including the graviton, are quantized to the one-loop level and coupled to (c-number) gravity through the equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G \langle T_{\mu\nu}(\hat{\phi}, g_{\mu\nu}) \rangle \quad (1)$$

The space-time metric $g_{\mu\nu}$ generates a non-zero vacuum expectation value of the energy-momentum tensor $\langle T_{\mu\nu} \rangle$, which in turn acts as a source of curvature. (Here $\langle T_{\mu\nu} \rangle$ stands for its renormalized value.) In general, this "back-reaction" problem is highly complicated. Besides the natural local dependence on $g_{\mu\nu}$, $\langle T_{\mu\nu} \rangle$ depends on the state $|\rangle$ and on the global space-time topology. Therefore, $\langle T_{\mu\nu} \rangle$ is a non-local geometrical object. In this paper, we study the back-reaction effects of non-trivial topology as well as the graviton contribution to constant curvature self-consistent solutions of Eq. (1). These solutions may describe the early Universe as being singularity-free and in the most symmetric (de Sitter) state. They have been much studied recently but only partial results have been obtained⁴⁾⁻⁶⁾. In particular, the effects considered here have not been taken into account.

For maximally symmetric (constant curvature) spaces, such as de Sitter (and anti-de Sitter), Eqs. (1) read

$$\left(\Lambda - \frac{R}{4} \right) g_{\mu\nu} = -8\pi \langle T_{\mu\nu} \rangle \quad (2)$$

which give the trace equation

$$(4\Lambda - R) = -8\pi \langle T_{\mu}^{\mu} \rangle \quad (3)$$

$\langle T_{\mu\nu} \rangle$ is proportional to $g_{\mu\nu}$ and therefore de Sitter (dS) [as well as anti-de Sitter (adS)] are self-consistent solutions with back-reaction included. In dS space, there is one real parameter family of dS $[0(1,4)]$ invariant vacua. On the other hand, these spaces are invariant under the discrete antipodal transformation

$$J: P(X) \rightarrow \bar{P}(\bar{X})$$

[the antipodal $\bar{P}(\bar{x})$ of a point $P(x)$ being defined as having its light-cone without intersecting points with the one of P]. J has a simple expression as an inversion ($x^a \rightarrow -x^a$) in $\mathbb{R}^5$ where the dS (adS) hyperboloid is embedded. For the trivial ordinary topology $R^1 \times S^3$, P and $\bar{P}$ are different points but if P and $\bar{P}$ are identified ("elliptic interpretation"⁷⁾⁻⁹⁾), we have a multiply-connected topology. In de Sitter space, both the dependence on the state and on the space-time topology can be expressed in terms of two parameters α , σ as

$$\langle T_{\mu\nu} \rangle_{\alpha} = e^{\alpha} \left[\langle T_{\mu\nu} \rangle + \sigma \langle \bar{T}_{\mu\nu} \rangle \right] \quad (4)$$

Here $\sigma = 0, 1, -1$, depending on whether we consider the ordinary space-time topology ($\sigma = 0$) or the so-called "elliptic" one [$\sigma = 1(-1)$ for fields which are symmetric (antisymmetric) under J]. α is a real parameter labelling the one-parameter family of dS $O(1,4)$ invariant vacua $|\rangle_{\alpha}$. $\langle T_{\mu\nu} \rangle$ is the expectation value in the Bunch-Davies¹⁰⁾ ("Euclidean" or "inflationary") vacuum obtained for $\alpha = 0$ and given by

$$\langle T_{\mu\nu} \rangle = \frac{g_{\mu\nu}}{64\pi^2} f(m, R, \xi)$$

$$f(m, R, \xi) = m^2 \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] \left[\psi\left(\frac{3}{2} + \nu\right) + \psi\left(\frac{3}{2} - \nu\right) + \ln \frac{R}{12m^2} \right] - \left(\xi - \frac{1}{6} \right) m^2 R - \frac{1}{18} m^2 R - \frac{1}{2} \left(\xi - \frac{1}{6} \right)^2 R^2 + \frac{R^2}{2160} \quad (5)$$

where m is the mass of the fields, R the scalar curvature and ξ the coupling parameter; the coefficient γ takes into account the degrees of freedom of the fields. Here,

$$\nu \equiv \left(\frac{9}{4} - \frac{12 M^2}{R} \right)^{1/2}, \quad M^2 \equiv m^2 + \xi R, \quad \psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

The term $\langle \bar{T}_{\mu\nu} \rangle$ taking into account the antipodal contribution is given by⁹⁾

$$\langle \bar{T}_{\mu\nu} \rangle = \frac{-m^2}{64\pi \cos \pi \nu} \left[m^2 + \left(\xi - \frac{1}{6} \right) R \right] g_{\mu\nu} \quad (6)$$

This term vanishes for massless fields ($m^2 = 0$), i.e.,

$$\langle T_{\mu\nu} \rangle_\sigma = \langle T_{\mu\nu} \rangle \quad \text{if} \quad M^2 = \frac{2}{3} R$$

But in the inflationary case, it is $\langle T_{\mu\nu} \rangle \propto \langle \bar{T}_{\mu\nu} \rangle$ and so we have

$$\langle T_{\mu\nu} \rangle_\sigma = \begin{cases} \frac{61}{151} \langle T_{\mu\nu} \rangle, & \sigma = 1 \\ -\frac{29}{61} \langle T_{\mu\nu} \rangle, & \sigma = -1 \end{cases} \quad \text{if} \quad M^2 \ll R$$

2. - MASSLESS FIELDS AND GRAVITON CONTRIBUTION

In this section we deal with the massless case ($m^2 = 0$) so the effect of the topology will not appear. $\langle T_\mu^\mu \rangle$ is given by^{11),12)}

$$\langle T_\mu^\mu \rangle = \frac{1}{2880\pi^2} \left(d - \frac{b}{12} \right) R^2 \quad (7)$$

where d and b are constants depending on the spin (S) of the fields. Their values are given in the Table. For a collection of N_0 scalars, $N_{\frac{1}{2}}$ neutrinos, N_1 photons, $N_{3/2}$ gravitinos and the graviton, we have

$$\langle T_\mu^\mu \rangle = \frac{\gamma}{34560\pi^2} R^2 \quad (8)$$

where

$$\gamma = N_0 + \frac{11}{2} N_{\frac{1}{2}} + 62 N_1 + \frac{183}{2} N_{3/2} - 2151$$

The trace equation (3) reads

$$R - 4\Lambda = \frac{\gamma}{4320\pi} R^2,$$

and the classical constant curvature relation $R = 4\Lambda$ becomes

$$R_\pm = 4\Lambda_\pm(\Lambda, \gamma) \quad (9)$$

here

$$\Lambda_{\pm}(\Lambda, \gamma) = \frac{540\pi}{\gamma} A_{\pm}, \quad A_{\pm} = 1 \pm \sqrt{1 - \frac{\Lambda\gamma}{270\pi}}$$

The character of the solutions depends on the sign of $\Lambda\gamma$. The contribution to γ of matter fields is positive and that of gravitons is negative (a feature which also holds for the trace anomaly in two dimensions¹³). Therefore:

(I) If $\text{sg}\Lambda \neq \text{sg}\gamma$, then $A_+ > 0$ and $A_- < 0$ ($A_{\pm} = 1 \pm \sqrt{1 + |\Lambda\gamma|/270\pi}$)

There are four possible real solutions: two dS and two adS

This occurs for

(i) $\Lambda > 0$ and $\gamma < 0$ (graviton-dominated Universe). Here $R_{(-)}$ is dS and $R_{(+)}$ is adS

or

(ii) $\Lambda < 0$ and $\gamma > 0$ (matter-dominated Universe). Here $R_{(+)}$ is dS and $R_{(-)}$ is adS

(II) If $\text{sg}\Lambda = \text{sg}\gamma$, then $A_{\pm} > 0$, ($A_{\pm} = 1 \pm \sqrt{1 - |\Lambda\gamma|/270\pi}$), there are two families of solutions depending on $|\Lambda\gamma| > 270\pi$:

If $|\Lambda\gamma| < 270\pi$ there are four real solutions

(i) $\Lambda > 0$ and $\gamma > 0$ (matter-dominated Universe); here $R_{(\pm)}$ are dS

(ii) $\Lambda < 0$ and $\gamma < 0$ (graviton-dominated Universe); here $R_{(\pm)}$ are adS

If $|\Lambda\gamma| > 270\pi$ there are two pairs of complex conjugate solutions

$R_{(+)} = R_{(-)}^*$

$R_{(\pm)} = (2160\pi/\gamma) [1 \pm i\sqrt{|\Lambda\gamma|/270\pi - 1}]$

If $|\Lambda\gamma| = 270\pi$:

$R_{(+)} = R_{(-)} = 2160\pi/\gamma$

(i) $\Lambda > 0$ and $\gamma > 0$ (dS)

(ii) $\Lambda < 0$ and $\gamma < 0$ (adS)

Note that even if $\Lambda > 0$, R could be negative. This means that for a given sign of Λ , the character of the solutions depends on the number of matter fields in the Universe. This contrasts sharply with the standard classical situation in which R only depends on Λ . In Figs. 1 and 2, we show $R = R(\Lambda)$ and $R = R(\gamma)$ for different matter-graviton balances. For $sg\Lambda \neq sg\gamma$, the configurations corresponding to the positive branch $R_{(+)}$ [dS in I(i) and adS in I(ii)] do not occur classically. On the contrary, the negative branch $R_{(-)}$ always represents a classically allowed configuration. Classical space-time ($R_{(-)} = 4\Lambda$) is obtained from the $\gamma \rightarrow 0$ limit of this branch. For $\Lambda = 0$, the negative branch gives flat space-time ($R_{(-)} = 0$) but $R_{(+)}$ allows for dS and adS configurations. In the interval $-270/\pi < |\Lambda\gamma| < 270/\pi$ both branches are classically allowed. We have

$$\lim_{\gamma \rightarrow 0} \begin{cases} \Lambda_+ = 1/\gamma \\ \Lambda_- = \Lambda \end{cases} \quad (\text{classical space})$$

$$\lim_{\Lambda \rightarrow 0} \begin{cases} \Lambda_+ = 1080/\pi\gamma \\ \Lambda_- = 0 \end{cases} \quad (\text{flat space})$$

$$\lim_{\gamma \rightarrow \infty} \Lambda_{\pm} = 0 \quad (\text{flat space})$$

The dS semi-classical geometries ($R_{\pm} > 0$) have an effective surface gravity ($K_{\pm}$) and Hawking temperature ($T_{\pm}$) which are modified with respect to those of the classical geometry, i.e.,

$$K_{\pm} = \frac{\Lambda_{\pm}}{3}, \quad T_{\pm} = \frac{K_{\pm}}{2\pi} = \sqrt{\frac{45\pi}{\gamma} \left(1 \pm \sqrt{1 - \frac{\Lambda\gamma}{270\pi}} \right)}$$

If N (the number of matter fields) $\rightarrow \infty$, then $T_{\pm} \rightarrow 0$ and the semi-classical geometry is flat even if $\Lambda \neq 0$.

The interpretation of complex solutions for $|\gamma\Lambda| > 270\pi$ is not clear and needs further investigation. They could be interpreted as possible unstable configurations, the imaginary part being related to the decay probability. In the path integral of gravity and matter fields, these solutions will give a real contribution for the partition function and the wave function of the Universe.

3. - MASSIVE FIELDS AND TOPOLOGY EFFECTS

We study now the "back-reaction" effects of massive fields and non-trivial topology. It is sufficient to consider here scalar fields. $\langle T_{\mu\nu} \rangle$ is given by Eqs (4)-(6). In this case, the trace Eq. (3) establishes a relation $R = R(m, \Lambda)$ which will be different for different values of σ and ξ . We study this relation in detail. We analyze numerically $R = R(m)$ for $\Lambda = 0$ and $R = R(\Lambda)$ for fixed m , both in the cases $\sigma = 0, 1, -1$ and for different ξ values ($0, 1/6$ and $\xi \neq 1/6$). The results of this study allow us to characterize five different types of behaviours depending on the values of ξ and σ :

$$\begin{array}{llll}
 \text{Type (I):} & \xi_- < \xi < \xi_+ & , & \sigma = 0, 1 \\
 \text{(II):} & \xi_- < \xi < \xi_+ & , & \sigma = -1 \\
 \text{(III):} & \xi = 0 & , & \sigma = 0, -1 \\
 \text{(IV):} & \xi = 0 & , & \sigma = 1 \\
 & \text{or} & & \\
 & 0 < \xi < \xi_0 & , & \sigma = 0, 1, -1 \\
 \text{(V):} & \xi_0 < \xi < \xi_- \quad \text{and} \quad \xi > \xi_+ & , & \sigma = 0, 1, -1
 \end{array}$$

with

$$\xi_{\pm} = \frac{1}{6} \left(1 \pm \frac{1}{\sqrt{30}} \right) \quad \text{and} \quad \xi_0 \simeq 0.102$$

We show these different types of solutions in Figs. 4 and 5 for $\Lambda = 0$ and $\Lambda \neq 0$, respectively. We analyze them now.

3.1. - $\Lambda = 0$ case

Classically, the only solution is $R = 0$ but semi-classically, Eq. (3) gives:

$$R = (12 \pi^2) \frac{2890}{F(x)} \quad (10)$$

where

$$F(x) = 15x \left[x + 12 \left(\xi - \frac{1}{6} \right) \right] \left[\Psi \left(\frac{3}{2} + \gamma \right) + \Psi \left(\frac{3}{2} - \gamma \right) - \ln x + \frac{\sigma \pi}{\cos \pi \gamma} \right] -$$

$$- 10x + 1 - 180x \left(\xi - \frac{1}{6} \right) - 1080 \left(\xi - \frac{1}{6} \right)^2,$$

$$x = 12 \frac{m^2}{R}$$

In order to study $R = R(m)$ we proceed as follows. Given x , calculate $F(x)$ and then from Eq. (10) we obtain R . Known x and R , we get the corresponding value of m^2 . We have in this way pairs (m^2, R) . We illustrate this procedure with the following examples.

Conformal coupling ($\xi = 1/6$)

$F(x)$ has the following asymptotic behaviours:

$$F(x) \underset{\substack{x \rightarrow 0 \\ \xi = 1/6}}{=} 1 - (10 - 15\sigma)x + 15(1 - 2\gamma - \sigma)x^2 - 15x^2 \ln x + O(x^3) \quad (11)$$

$$F(x) \underset{\substack{x \rightarrow \infty \\ \forall \xi}}{=} \left[-\frac{8}{24} + 12 \left(\xi - \frac{1}{6} \right) - 4320 \left(\xi - \frac{1}{6} \right)^3 \right] \frac{1}{x} + O\left(\frac{1}{x^2}\right) \quad (12)$$

where γ is the Euler constant.

In the three cases $\sigma = 0, 1, -1$, $F(x)$ is bounded, which implies that $R(m)$ is always larger than a certain value $R_0 \equiv R(m=0)$ [$R(m)$ never vanishes]. We have

$$F(x) \leq 1 \quad \text{and} \quad R(m) \geq R(m=0) \quad \text{for} \quad \sigma = 0, -1$$

and

$$F(x) \lesssim 6 \quad \text{and} \quad R(m) \gtrsim \frac{1}{6} R(m=0) \quad \text{for} \quad \sigma = 1$$

The behaviour of $R(m)$ for $m \rightarrow \infty$ is known computing the values x_σ such that $F(x_\sigma) = 0$. $R \rightarrow \infty$ for $m \rightarrow \infty$ with $x = x_\sigma = \text{const.}$ Numerically, we obtain

$$R_{m \rightarrow \infty} = \frac{12 m^2}{x_\sigma}, \quad x_\sigma \approx \begin{cases} 0.2 & , \sigma = 0 \\ 0.7 & , \sigma = 1 \\ 0.04 & , \sigma = -1 \end{cases}$$

Because $\lim_{x \rightarrow 0} F(x) = 1$ for the three values of σ , the curves $R = R(m)$ start at the same point $m = 0$, $R = R_0$ [see Fig. 4 (I-II)].

Minimal coupling ($\xi = 0$)

In this case, the behaviour of $F(x)$ for $x \rightarrow 0$ is different from Eq. (11). One should carefully take into account the existence of poles in $\phi(3/2-\nu)$ and $\sigma\pi/\cos\pi\nu$. We have

$$F(x)_{\substack{x \rightarrow 0 \\ \xi = 0}} = 90(1-\sigma) - 29 + [30\sigma - 25 - 45(1-\sigma)]x - [15x^2 - 30x] \log x + O(x^2) \quad (13)$$

Note that

$$F'(0) < 0 \quad \text{for} \quad \sigma = 0, 1, -1$$

$$F(0) \begin{cases} > 0 \\ < 0 \end{cases} \quad , \quad \begin{matrix} \sigma = 0, -1 \\ \sigma = 1 \end{matrix}$$

and

$$F(x)_{\substack{x \rightarrow \infty \\ \xi = 0}} = \left(\frac{370}{21} \right) \frac{1}{x} + O\left(\frac{1}{x^2} \right) > 0 \quad \text{for} \quad \sigma = 0, 1, -1$$

As a consequence, there always exist self-consistent solutions of dS type because there is a range of values of $x > 0$ such that $F(x) > 0$. For $\sigma = +1$, $R(m; \xi=0)$ is a two-valued function [Fig. 4 (IV)]. There exists a value x_A such that $F(x) < F(x_A)$ for all x and so $R > R_A = (2880\pi^2/F(x_A))(12)$. Straight lines through the origin intersect twice the curve $F(x)$ for $\sigma = +1$. For $\sigma = 0, -1$, $R(m; \xi=0)$ is not very different from $R(m; \xi=1/6)$ for $\sigma = 0, 1$. This is due to the infra-red behaviours of these functions. For $\xi = 1/6$, the terms in $\phi(3/2-\nu)$ and $\sigma\pi/\cos\pi\nu$ have poles at $\nu_n = (\frac{1}{2}+n)$ with residues $(1-\sigma n)$ and $\sigma(-1)^n$ respectively. [In the ordinary case ($\sigma=0$), only one residue appears.] For $\xi = 1/6$ and $x = 0$ (i.e., $\nu=\frac{1}{2}$) the residues are 1 and $-\sigma$ respectively (and so $\sigma=-1$ reinforces the total value). For $\xi = 0$ and $x = 0$ (i.e., $\nu=3/2$) the residues are 1 and σ (and so $\sigma=+1$ reinforces the total residue).

Other values of ξ ($\xi > 0$)

In this case, there are no divergences from the terms $\phi(3/2-\nu)$ and $\sigma\pi/\cos\pi\nu$:

$$F(x=0) = 1 - 1080 \left(\xi - \frac{1}{6} \right)^2 \quad (14)$$

$\xi \neq 0$
 $\xi \neq 1/6$

As a consequence:

$$0 < F(x=0) < 1 \quad \text{for} \quad \xi_- < \xi < \xi_+$$

and

$$F(x=0) < 0 \quad \text{for} \quad \xi < \xi_- \quad \xi > \xi_+$$

where

$$\xi_{\pm} = \frac{1}{6} \left(1 \pm \frac{1}{\sqrt{30}} \right)$$

Therefore, there exist de Sitter self-consistent solutions with $m = 0$, for $\xi \in (\xi_-, \xi_+)$, but for $\xi < \xi_-$ and $\xi > \xi_+$, there are no such solutions. For $\xi_- < \xi < \xi_+$, the solutions do not differ essentially from those with $\xi = 1/6$. In addition we have

$$F(x \rightarrow \infty) \rightarrow \frac{c}{x}$$

and by analyzing Eq. (12) we see that

$$\begin{aligned} c > 0 & \quad \text{if} \quad \xi < \xi_0, \\ c < 0 & \quad \text{if} \quad \xi > \xi_0 \end{aligned} \quad , \quad \xi_0 = 0.102, \quad \sigma = 0, 1, -1$$

Therefore, for $\xi_0 < \xi < \xi_-$ and $\xi > \xi_+$, there are no dS solutions because $F(x) < 0$ for all x . For $\xi < \xi_0$, $R(m)$ is two-valued, as it is for $\xi = 0$. That is, there exists x_1, x_2 , such that $F(x_1)/x_1 = F(x_2)/x_2 = a/m^2$, i.e., the mass value given by $x_1/F(x_1)$ has two possible values of R [Fig. 4 (IV)].

3.2. - $\Lambda \neq 0$ case

In this case, the trace Eq. (3) can be written as

$$R - R^2 F(x) = 4\Lambda \quad (15)$$

for convenience we have absorbed here the numerical factors in R, Λ [$R \rightarrow (1/12\pi^2) R/2880, \Lambda \rightarrow (1/12\pi^2) \Lambda/2880$]

We study the function $R = R(\Lambda)$ for each value of m and ξ . For fixed m and a given R , we obtain $F(x)$ and from Eq. (15) we calculate the corresponding Λ . The zeros of $F(x)$ determine the values of R for which $R(\Lambda)$ intersects the classical solution $R = 4\Lambda$. For fixed m , $F(x)$ has the asymptotic behaviours

$$F(x) \xrightarrow{x \rightarrow 0} \alpha + \beta x$$

$$F(x) \xrightarrow{x \rightarrow \infty} \frac{c}{x}$$

The values of α , β and c depend on ξ and on σ . For instance, for $\xi = 1/6$: $\alpha = 1$ and $c < 0$ for all $\sigma = 0, 1, -1$; $\beta < 0$ if $\sigma = 0, 1$ and $\beta > 0$ if $\sigma = -1$.

For $R \rightarrow \infty$, Eq. (15) gives

$$R - \alpha R^2 = 4\Lambda$$

and for $R \rightarrow 0$

$$R - \frac{c}{12m^2} R^3 = 4\Lambda$$

Obviously, for $R \rightarrow \infty$, the curves $R(\Lambda)$ lie very far from the classical line $R = 4\Lambda$. For $R \rightarrow 0$, $R(\Lambda)$ fall below (above) $R = 4\Lambda$ depending on $c < 0$ (or $c > 0$). When m increases R approaches the classical value 4Λ in both cases.

The effect of the antipodal topology does not manifests itself for small R [it is determined by the behaviour of $F(x)$ for $x \rightarrow \infty$ which is independent of σ]. For increasing R [determined by $F(x)$ for $x \rightarrow 0$] the effect introduced by the antipodal identification is more important (the signs of α and β depend strongly on σ).

4. - CONCLUSIONS AND DISCUSSION

We have investigated constant curvature self-consistent solutions of semi-classical Einstein equations taking into account two types of effects: (i) the graviton contribution to the trace anomaly at the one-loop level and (ii) the "back-reaction" effects of a non-trivial topology produced by antipodal identification. For massless fields coupled to gravity, we have analyzed the relation $R = R(\Lambda, \gamma)$, γ being the trace anomaly. The graviton contribution allows for 12 possible configurations (8 real and 4 complex conjugate). For a given sign of the cosmological constant Λ , the behaviour (dS or adS) depends on the quantum matter content of the Universe. If N (the number of matter fields) goes to ∞ ,

the Hawking temperature vanishes and the geometry is flat even if $\Lambda \neq 0$. The two pairs of complex conjugate solutions appear for values of $|\Lambda\gamma| > 270\pi$. For massless fields, the effects of the antipodal identification do not appear because their contribution to $\langle T_{\mu\nu\sigma} \rangle$ vanishes in that case [$\sigma=0$ for fields in the ordinary topology, $\sigma=1(-1)$ for fields symmetric (anti-symmetric) under the antipodal transformation]. For massive fields, we have studied in detail the relation $R = R(m, \Lambda)$ for different values of ξ and σ . The combined results give rise to five characteristic types of solutions, depending on the values of ξ and σ . The effects of the antipodal identification are more relevant for small ξ . When $\Lambda = 0$, a clear difference appears for $\xi = 0$ between the ordinary ($\sigma=0$) theory and the antipodal symmetric ($\sigma=1$) one. These differences are more relevant for small masses. For increasing ξ , the differences turn out to be less important. For small R , the effects of $\sigma \neq 0$ do not appear but these effects become important for growing R ($R \gg \lambda_c^{-2}$), $\lambda_c = h/m_c$ being the Compton length of the fields (although still $R \ll \ell_p^{-2}$ in the semi-classical regime).

Classical Einstein equations fix the local geometry but not the global space-time topology. However, each possible choice of the topology reacts back (through $\langle T_{\mu\nu} \rangle$) in the semi-classical Einstein equations and give a different semi-classical geometry (with a topology fixed by that choice). We have implemented these "back-reaction" topological effects in dS space with antipodal identification. This identification yields a modification of the full space-time topology (J reverses time) and not only of the spatial one.

The antipodal identification in $\langle T_{\mu\nu\sigma} \rangle$ is implemented at the level of the field operators in the space configuration. The vacuum is chosen to be the ordinary $O(1,4)$ state^{8),9)}. The antipodal symmetry cannot be implemented without problems at the level of the Fock space^{8),14)} [this is analogous to the problem of the $m^2 = \xi = 0$ limit of the ordinary ($\sigma=0$) theory in which the usual Fock space construction breaks down rigorously, but $\langle T_{\mu\nu} \rangle$ can be defined in that limit]. In any case, $\langle T_{\mu\nu} \rangle$ in Eq. (1) is considered as an "effective source".

For massless fields, $\langle T_{\mu\nu} \rangle$ in (dS) and (adS) spaces have the same expressions. For massive fields, they are different. However, $\langle T_{\mu\nu\sigma} \rangle$ for $\sigma = \pm 1$ in (dS) and (adS) can be related. The choice of the symmetric (antisymmetric) boundary conditions required by the formulation of a consistent quantum field theory in (adS) space¹⁵⁾ (that is also motivated by supersymmetry) corresponds just to the $\sigma = +1(-1)$ antipodal theory in dS space.

The qualitative behaviour of the semi-classical solutions found here appears to hold in lower as well as in higher dimensions. Two-dimensional semi-classical gravity also exhibits these features¹⁶⁾. On the other hand, the slope expansion in the parameter α' of string models admits solutions of the effective field equations with constant curvature given by¹⁷⁾

$$\Lambda_{\pm} = -\frac{n(n+1)}{2\tilde{\alpha}K^2} \left[1 \pm \sqrt{1 - \frac{4\tilde{\alpha}K^2\Lambda}{n(n+1)}} \right]$$

here n is the space-time dimension and K the area of a unit- n -sphere (this corresponds to quadratic curvature corrections to the Einstein action in terms of the Gauss-Bonnet term). The same behaviour as in Eq. (9) appears with the string parameter α' playing the role of the trace anomaly. [If sign α' is dictated by string theory ($\alpha' > 0$), only (adS) space is obtained and the branch $\Lambda_{(-)}$ singled out by stability.] At the semi-classical level, the corrections to the curvature introduced by the effective equations are qualitatively the same in both cases (from semi-classical Einstein equations in four dimensions as from string theory in D dimensions).

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S	b	d	(12d-b)
0	-1	0	1
$\frac{1}{2}$	-11/2	0	11/2
1	-62	0	62
3/2	0	61/8	183/2
2	0	-717/4	-2151

- TABLE -

Trace anomaly coefficients for different values of the spin (S) [$\langle T^{\mu}_{\mu} \rangle = (R^2/34560\pi)^2(12d-b)$].

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FIGURE CAPTIONS

- Fig. 1 The curvature as a function of the cosmological constant Λ for massless quantum fields and different graviton-matter balances.
- Fig. 2 The curvature as a function of the trace anomaly coefficient γ for different values of Λ .
- Fig. 3 Characteristic behaviours of the function $f(x=12m^2/R)$ for $\Lambda = 0$ and different values of ξ and σ (labelling the field couplings and the different choices of space-time topology).
- Fig. 4 The curvature as a function of the mass m of quantum fields, for $\Lambda = 0$ and the four characteristic types of solutions.
- Fig. 5 The curvature as a function of Λ for fixed m and the five types of solutions.

- Figure 1 -

- Figure 2 -

$$\Lambda = 0$$

$$\xi_+ = 1/6 \left(1 \pm 1/\sqrt{30} \right) \quad \xi_0 = 0.102$$

- Figure 3 -

Type V ($\xi_{00} < \xi < \xi_-$ and $\xi > \xi_+$, $\sigma = 0, 1, -1$)
 There are no self-consistent solutions

- Figure 4 -

$$\Lambda = 0$$

- Figure 5 -