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ON A POSSIBLE WESS-ZUMINO TERM IN THE HETEROTIC STRING

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ABSTRACT

We examine the low-energy limits of the one-particle irreducible one-loop amplitudes in the heterotic superstring for five gravitons, for three gravitons and two gauge bosons, and for one graviton and four gauge bosons. In agreement with general arguments, explicit calculation shows that there is a Wess-Zumino term in the low-energy effective action for $D = 10$ supergravity. However, we argue that this term is not relevant to the effective low-energy theory in four dimensions after Calabi-Yau compactification.

1. INTRODUCTION

The present emergence of interest in the superstring as a possible Theory of Everything (TOE) was triggered by an observation of Green and Schwarz [1]. They pointed out that the one-loop gauge, gravitational, and mixed anomalies would vanish in a superstring theory with gauge group $SO(32)$ or $E_8 \times E_8$. This was shown by studying the anomalous behaviour of the topologically distinct parity-violating contributions to the six-point functions at the one-loop level of the type I superstring. The low-energy effective $D = 10$ supergravity theory, on the other hand, is anomalous. However, for gauge groups $E_8 \times E_8$ and $SO(32)$ a local counter term (commonly called a Wess–Zumino term) can be added to the action which would make it anomaly-free. This Wess–Zumino term [2] results from a local coupling in the field theory limit induced by the factorized parity-violating non-planar (fig. 1a) six-point function. It corresponds to an interaction of the antisymmetric tensor field with four gravitons or gauge bosons. For the heterotic string [3], modular invariance [when the gauge group is $E_8 \times E_8$ or $SO(32)$] was used to show [4] that there is again no anomaly in the parity-violating six-point functions. Once again a Wess–Zumino term should result in the field theory limit of the factorized amplitude (see fig. 1b).

In this paper we calculate the one-loop amplitudes in the heterotic string for five external bosons: for five gravitons, for three gravitons and two gauge bosons, and for one graviton and four gauge bosons. We perform this calculation in two stages. First, we calculate the amplitudes in the light-cone gauge, finding the parity-conserving parts of the amplitudes. We also identify the terms proportional to the 8-index ϵ -symbol carrying at least 4-powers of momenta which are candidates for corresponding to a Wess–Zumino term. Then, in order to make more explicit the covariantization of this term we calculate the parity-violating part of the amplitudes in the covariant formalism and find the Wess–Zumino term corresponding to the covariant form of the 8-index ϵ -symbol. These one-loop five-point amplitudes are important not only because they contain the Wess–Zumino term whose existence is essential to the consistency of the effective field theory, but also because the interactions which they induce can be studied to find other possible corrections to the effective $D = 10$ action. A subsequent question is then whether these terms have effects on the effective $d = 4$ action when the theory is dimensionally reduced.

The paper is organized as follows. In Section 2 we calculate the five-graviton amplitude and in Section 3 we present the other five-boson amplitudes in the light-cone gauge and identify the parity-violating term. In Section 4 we study the parity-violating part of the amplitudes in the covariant formalism and establish the existence of a Wess–Zumino term. Since the physical content of this term has been a subject of debate [5] we add in Section 6 a discussion of its possible physical implications. In particular, we point out that the tree-level Kähler potential [6] obtained after Calabi–Yau compactification does not include any interaction which could generate a one-loop term in the $d = 4$ theory of the form obtained by naïve reduction of the Wess–Zumino term. We argue that care should be exercised in this reduction, and that there is no term in the effective $d = 4$ action corresponding to the $D = 10$ Wess–Zumino term after Calabi–Yau compactification. Section 6 contains some concluding remarks.

2. THE FIVE-GRAVITON AMPLITUDE

The one-loop diagrams that we compute are shown in fig. 2. Each amplitude takes the form

$$A = \text{Tr} \left[\Delta_1 V(1) \Delta_2 V(2) \Delta_3 V(3) \Delta_4 V(4) \Delta_5 V(5) \right] , \quad (1)$$

where the trace runs over all zero and non-zero bosonic and fermionic modes, Δ_a ($a = 1, \dots, 5$) are string propagators between neighbouring vertices, and V_a are the appropriate vertex operators. The vertex operator for the emission of a graviton or its bosonic partners (dilaton and antisymmetric tensor field) can be written in the light-cone gauge as [3]

$$V_G = \frac{4g}{\pi} g_{ij}(k) \int_0^\pi d\tau B^i \tilde{P}^j e^{ik \cdot x} \quad (2)$$

with

$$k^2 = 0, \quad k_\mu g^{\mu\nu} = k_\nu g^{\mu\nu} = g^{\mu+} = g^{+\mu} = 0. \quad (3)$$

Here B^i is the supersymmetrized right-moving momentum operator, and $\tilde{P}^i$ is the left-moving momentum operator:

$$B^i = P^i + \frac{1}{2} k_j R^{ij} \quad (4a)$$

$$R^{ij} = \frac{1}{8} \bar{S} \gamma^{ij} S \quad (4b)$$

and

$$S^a = \sum_{n \neq 0} S_n^a e^{-2in(\tau-\sigma)} + S_0^a \quad (4c)$$

with S_n^a being the fermionic oscillators and γ^{ij-} is the totally antisymmetric product of γ^i, γ^j , and $\gamma^- = (\gamma^0 - \gamma^9)/\sqrt{2}$. The quantities P^i and $\tilde{P}^i$ can be expanded in terms of the right-moving and left-moving oscillators ($\alpha_n^i, \tilde{\alpha}_n^i$, respectively), similarly to the expansion of eq. (4c) (with $\tau + \sigma$ being the argument of the exponent in $\tilde{P}^i$).

The string propagator in eq. (1) has the form:

$$\Delta_a = \int \frac{d^2 z_a}{\pi} |\bar{z}_a|^{k_a^2 - 2} \bar{z}_a^N \bar{z}_a^{\tilde{N} - 1 + \frac{1}{2} Q_a^2} , \quad (5a)$$

where q_a ($a = 1, \dots, 5$) is the 10-momentum in external space and Q_a is the 16-momentum in the compactified internal space

$$q_a = \ell - \sum_{b=1}^{Q-1} k_b \quad (5b)$$

$$Q_a = L - \sum_{b=1}^{Q-1} k_b \quad (5c)$$

with ℓ and L the loop momenta in external and internal spaces respectively. We will integrate later on over $d^{10}\ell$ and sum over $L \in \Lambda$, where Λ is the self-dual lattice of $E_8 \times E_8$ or $SO(32)$. In eq. (5a) N and $\tilde{N}$ are the number operators in the right-moving and left-moving sectors respectively.

To carry out the trace in eq. (1) we notice that the momentum operators in the vertex (2), spoil the factorization property of the trace into a product of traces over the individual oscillators. To circumvent this complication we use the generating functional of the momentum operators

$$p^{ia} = \frac{d}{dt_{ia}} \exp(t_b \cdot p_b) \Big|_{t_a=0} \quad (6a)$$

$$\tilde{p}^{ia} = \frac{d}{d\tilde{t}_{ia}} \exp(\tilde{t}_b \cdot \tilde{p}_b) \Big|_{\tilde{t}_a=0} \quad (6b)$$

thereby recovering the factorization property [7]. Thus the traces can easily be carried out and the differentiations with respect to t_a and $\tilde{t}_a$ are performed at the end. In calculating the amplitude with 5 external gravitons we encounter a factor $(B\tilde{P})^5$ in the trace (1). It is well known that only the traces including at least four powers of $R_0^{ij} = (1/8)\tilde{S}_0\gamma^{ij} - S_0$ are non-zero [8]. Thus we have generically two contributions resulting from $\text{tr}(R_0^5)$ and $\text{tr}(R_0^4)$ respectively:

$$I_1 = \frac{g^5}{32} \text{Tr} \left[: \frac{5}{\pi} \Delta_a R^{iaja} \tilde{p}_{ra} e^{ik \cdot x} k_{ja} g_{ia} : \right] \quad (7a)$$

and

$$I_2 = \frac{g^5}{16} \text{Tr} \left[: \Delta_a p^{ia} \tilde{p}_{ra} e^{ik \cdot x} \frac{5}{\pi} \Delta_a R^{iaja} \tilde{p}_{ra} e^{ik \cdot x} k_{ja} g_{ia} : \right] \\ + \text{permutation of 4 out of 5.} \quad (7b)$$

To calculate I_1 we use the result of ref. [9]. In that paper, the one-loop pentagon diagram in the $SO(32)$ open superstring theory was computed. Using this, we can extract I_1 . Here we have two possible contributions: the first has typically the form $\text{Tr}(R_0^5)$ and the second is $\text{Tr}[(S_n S_0)^2 R_0^3]$, and their sum can be extracted from ref. [9]. For the rest we use eq. (6b) and the traces over the individual oscillators as explained in refs. [7] and [8] (see also ref. [10] for a similar calculation of 4-point amplitudes). Thus eq. (7a) can be written as

$$I_1 = \left(\frac{g}{2\pi}\right)^5 \int \frac{5}{\pi} d^2 z_a \frac{[f(\bar{\omega})]^{-24}}{\bar{\omega} |\omega|^2} \prod_{b < c} [\chi(c_{cb}, \omega)]^{k_c \cdot k_b} \left(\frac{-4\pi}{\ell_n(\omega)}\right)^5 \\ \cdot \sum_{\ell \in \mathbb{N}} \exp\left(\frac{\ell^2 \ell_n \bar{\omega}}{2}\right) \sum_{e \in \mathbb{Z}} u_{ef}^{r_1 r_2 r_3 r_4 r_5} B_{ef} T_{r_1 r_2 r_3 r_4 r_5} \quad (8)$$

Here we use the variables $q_a = \prod_{b=1}^a z_b = \bar{q}_a^*$, $q_5 = \omega$, $c_{ab} = (q_a/q_b)$; and the functions

$$B_{ab} = \frac{1}{2} + \sum_{n=1}^{\infty} (c_{ba}^n - \omega^n c_{ab}^n) \frac{1}{1 - \omega^n} \quad (9)$$

and

$$T_{r_1 r_2 r_3 r_4 r_5} = \frac{5}{\pi} \frac{d}{d\tilde{t}_{r_a}} \exp \left(\tilde{t}_a \cdot \tilde{A}_a + \sum_{a < b} \tilde{t}_a \cdot R_{ab} \tilde{t}_b \right) \Big|_{\tilde{t}_a = 0} \quad (10)$$

where

$$\tilde{A}_a = \frac{1}{2} \sum_{b=1}^5 \left(\tilde{B}_{ab} + \frac{\ell_n |c_{4b}|}{\ell_n |w|} \right) k_b \quad (11a)$$

$$\tilde{B}_{ab} = B_{ab}^* \quad (11b)$$

$$R_{ab} = D_{ab} - \frac{1}{2 \ell_n |w|} \quad (11c)$$

$$D_{ab} = \sum_{n=1}^{\infty} \left(\bar{c}_{ba}^n + \bar{\omega}^n \bar{c}_{ab}^n \right) \frac{n}{1 - \bar{\omega}^n} \quad (11d)$$

The tensor $U_{ab}^{r_1 r_2 r_3 r_4 r_5}$ in eq. (8) is defined as [9]

$$U_{ab}^{r_1 r_2 r_3 r_4 r_5} = S_{ab}^{i_1 j_1 i_2 j_2 i_3 j_3 i_4 j_4 i_5 j_5} \frac{5}{\pi} k_{j_a} p_{i_a r_a} \quad (12)$$

and $S_{12}^{ij \dots}$ is given by [9]

$$S_{12}^{i_1 j_1 i_2 j_2 i_3 j_3 i_4 j_4 i_5 j_5} = \text{Tr} \left(\gamma^{i_1 j_1} \gamma^{i_2 j_2} \gamma^{i_3 j_3} \right) \text{Tr} \left(R_0^{i_4 j_4} R_0^{i_5 j_5} R_0^{i_1 j_1} R_0^{i_2 j_2} \right) \quad (13)$$

such that for general a, b we use $\text{Tr} (\gamma^{i_a j_a} \gamma^{i_b j_b} \gamma^{ij})$ and in $\text{Tr} R_0^4$ we include the indices $i_c j_c, i_d j_d, i_f j_f$ with $c, d, f = 1, 2, \dots, 5$ and $c, d, f \neq a, b$. The differentiations in eq. (10) with respect to $\tilde{t}_a$ yields

$$\begin{aligned} T_{r_1 r_2 r_3 r_4 r_5} = & \tilde{A}_{r_1} \delta_{r_2 r_3} \delta_{r_4 r_5} R_{23} R_{45} + \tilde{A}_{r_1} \delta_{r_2 r_5} \delta_{r_3 r_4} R_{25} R_{34} + \text{permutations} \\ & + \tilde{A}_{r_1} \tilde{A}_{r_2} \tilde{A}_{r_3} \delta_{r_4 r_5} R_{45} + \text{permutations} \\ & + \tilde{A}_{r_1} \tilde{A}_{r_2} \tilde{A}_{r_3} \tilde{A}_{r_4} \tilde{A}_{r_5} \end{aligned} \quad (14)$$

By studying eqs. (8) to (14) we find that there is a polynomial of k_a ($a = 1, \dots, 5$), whose lowest order is k^6 , in addition to the exponential dependence on $k_b \cdot k_c$.

To calculate I_2 we once again use eqs. (6a) and (6b), whereas for $\text{Tr} (R^4)$ we need to take only its zero-mode contribution $\text{Tr} (R_0^4)$ so as to get a non-vanishing result. Defining

$$\text{Tr} (R_0^{ij} R_0^{kl} R_0^{mn} R_0^{pq}) = \epsilon^{ijklmnpq} \quad (15)$$

then the first term in eq. (7b) has the form

$$\begin{aligned} I_2^{(1)} = & 2 \left(\frac{g}{2\pi} \right)^5 \int \frac{5}{\pi} d^2 z_a \frac{[\zeta(\bar{w})]^{-24}}{\bar{\omega} |w|^2} \prod_{b < c} [\chi(c_b, w)]^{k_2 k_c \cdot k_b} \left(\frac{-4\pi}{\ell_n |w|} \right)^5 \\ & \cdot \sum_{i \in A} \exp \left(\frac{k^2 \ell_n \bar{w}}{2} \right) \epsilon^{i_1 j_1 i_2 j_2 i_3 j_3 i_4 j_4 i_5 j_5} \frac{5}{\pi} k_{j_e} p_{i_e r_e} p_{i_i r_i} T_{r_1 r_2 r_3 r_4 r_5}^{i_i} \end{aligned} \quad (16)$$

Here

$$T_{r_1 r_2 r_3 r_4 r_5}^{i_1} = \frac{d}{dt_{i_1}} \frac{5}{\pi} \frac{d}{d\tilde{t}_{r_a}} \exp \left(t_i \cdot A_i + \tilde{t}_a \cdot \tilde{A}_a + \sum_{a < b} \tilde{t}_a \cdot R_{ab} \tilde{t}_b - \frac{1}{2 \ell_n |w|} \sum_{b=1}^5 t_i \cdot \tilde{t}_b \right) \Big|_{\tilde{t}_i = \tilde{t}_a = 0} \quad (17)$$

and A_1 is defined as in eq. (11a) with B_{ab} replacing $\tilde{B}_{ab}$. Performing now the differentiations in eq. (17) we find

$$T_{r_1 r_2 r_3 r_4 r_5}^{i_1} = A_{i_1} T_{r_1 r_2 r_3 r_4 r_5} - \frac{1}{2\epsilon_{n1\omega}} (\delta_{r_1 i_1} T_{r_2 r_3 r_4 r_5} + \text{all permutations}), \quad (18)$$

where $T_{r_1 r_2 r_3 r_4 r_5}$ is defined in eq. (14) and

$$\begin{aligned} T_{r_2 r_3 r_4 r_5} = & \delta_{r_2 r_3} \delta_{r_4 r_5} R_{23} R_{45} + \text{permutations} \\ & + \delta_{r_2 r_3} R_{23} \tilde{A}_4 \tilde{A}_5 + \text{permutations} + \tilde{A}_{r_2} \tilde{A}_{r_3} \tilde{A}_{r_4} \tilde{A}_{r_5}. \end{aligned} \quad (19)$$

We note that the first term in eq. (18) is a polynomial in k whose lowest order is k^2 , whereas in the second term we have contributions independent of k of the type $\delta_{r_1 i_1} \delta_{r_2 r_3} \delta_{r_4 r_5} R_{23} R_{45}$. If, in addition, the four powers of k appearing in eq. (16) are taken into account we find a polynomial whose lowest order of k is k^4 . By contracting the indices in eq. (16) we find the kinematical factor

$$\begin{aligned} K_{r_2 r_3 r_4 r_5} = & -\frac{1}{2} \epsilon^{i_2 i_3 i_4 i_5} \epsilon^{j_2 j_3 j_4 j_5} \prod_{a=2}^5 k_{j_a} f_{i_a} \tau_a \\ & + \left\{ -\frac{1}{4} [st p^{(2)} p^{(4)} p^{(5)} p^{(3)} + su p^{(3)} p^{(4)} p^{(2)} p^{(5)} + tu p^{(3)} p^{(5)} p^{(4)} p^{(2)}] \right. \\ & + \frac{s}{2} [p^{(2)} k_5 p^{(4)} k_3 p^{(3)} p^{(5)} + p^{(3)} k_4 p^{(5)} k_2 p^{(2)} p^{(4)} \\ & \quad + p^{(2)} k_4 p^{(5)} k_3 p^{(3)} p^{(4)} + p^{(3)} k_5 p^{(4)} k_2 p^{(2)} p^{(5)}] \\ & + \frac{t}{2} [p^{(3)} k_2 p^{(5)} k_4 p^{(4)} p^{(2)} + p^{(4)} k_5 p^{(2)} k_3 p^{(3)} p^{(5)} \\ & \quad + p^{(3)} k_5 p^{(2)} k_4 p^{(4)} p^{(5)} + p^{(4)} k_2 p^{(5)} k_3 p^{(2)} p^{(3)}] \\ & \left. + \frac{u}{2} [p^{(2)} k_3 p^{(5)} k_2 p^{(4)} p^{(3)} + p^{(4)} k_5 p^{(3)} k_2 p^{(2)} p^{(5)} \right. \\ & \quad \left. + p^{(3)} k_5 p^{(2)} k_4 p^{(4)} p^{(5)} + p^{(4)} k_3 p^{(5)} k_2 p^{(2)} p^{(3)}] \right\} r_2 r_3 r_4 r_5 \end{aligned} \quad (20)$$

Here $q^{(a)}(k_a)$ is the polarization tensor of particle a , and the contraction of indices should be understood in the following way:

$$\{p^{(2)} p^{(3)} p^{(4)} p^{(5)}\}_{r_2 r_3 r_4 r_5} = p_{i r_2}^{(2)} p_{i r_3}^{(3)} p_{j r_4}^{(4)} p_{j r_5}^{(5)} \quad (21)$$

etc.; while

$$s = (k_2 + k_3)^2 ; \quad t = (k_3 + k_4)^2 ; \quad u = (k_2 + k_4)^2. \quad (22)$$

Note the first term in eq. (20) which involves an 8-index ϵ -symbol. When covariantized this term would yield the Wess–Zumino term. As this can be seen more directly in the covariant formalism, we leave the discussion of this term to Section 4, where the parity-violating part of the amplitude is calculated in the covariant formalism.

3. OTHER FIVE-BOSON AMPLITUDES

We now proceed to calculate the amplitudes for three gravitons and two gauge bosons, and one graviton and four gauge bosons which are also candidates to contribute to the Wess–Zumino term. As in the previous case, we will identify the term involving the 8-index ϵ -symbol and discuss it in more detail in Section 4 where we calculate the parity-violating amplitudes in the covariant formalism.

As to the gauge bosons we need to distinguish between the charged and neutral gauge bosons. The vertex operator for the charged gauge bosons in the light cone gauge is [3]

$$V_K = \frac{4g}{\pi} g_i(k) \int_0^\pi d\sigma \theta^i e^{ik \cdot x} : e^{2ik^\perp x^\perp} : C(K) , \quad (23)$$

where K^I is a vector in the self-dual lattice of $E_8 \times E_8$ or $SO(32)$ satisfying $K^2 = 2$, x^I ($I = 1, \dots, 16$) are the internal coordinates of the string, and $C(K)$ generates the projective representation of the translation group in internal space defining the 2-cocycles $\epsilon(K, L)$ [3]. The vertex operator for the neutral gauge bosons is [3]

$$V_\perp = \frac{4g}{\pi} g_i^\perp(k) \int_0^\pi d\sigma \theta^i \tilde{P}^\perp e^{ik \cdot x} \quad (24)$$

where $\tilde{P}^\perp$ is the left-moving momentum operator in internal space. It has a Fourier expansion in terms of the left-moving internal oscillators $\tilde{\alpha}_n^I$ similar to eq. (4c) (with an argument $\tau + \sigma$ in the exponent). In both eqs. (23) and (24) $k^2 = 0$ and $k_i q_i = 0$.

3.1 Three-graviton/two-gauge boson amplitudes

To calculate the amplitude for the emission of three gravitons (particles 1, 2, 3) and two gauge bosons (particles 4, 5), we need to insert eqs. (2) and (23) or (24) in eq. (1) and take the trace over all the oscillators. As in eq. (7) we have two types of contributions: the first contains R^5 as in eq. (7a) and the second contains R^4 as in eq. (7b). The new features in eq. (23) are the cocycles $C(K)$ and $:e^{2ik^\perp x^\perp}:$. The product of the cocycles $C(K_3)C(K_4)$ when summed over the internal loop momentum $L \in \Lambda$ yields the 2-cocycle $\epsilon(K_3, K_4) = -1$ ($K_3 + K_4 = 0$), and the trace over the internal oscillators $\tilde{\alpha}_n^I$ appearing in $x^\perp$ yields the function $\psi(\tilde{c}_{54}, \tilde{\omega})$. Thus in the amplitude for the emission of three gravitons and two charged gauge bosons we have

$$A_{3g}^{2K} = I_1 + I_2 \quad (25)$$

where

$$I_1 = -\left(\frac{g}{2\pi}\right)^5 \int \frac{d^2 z_a}{\pi} \frac{[f(\tilde{\omega})]^{-24}}{\tilde{\omega} |\omega|^2} \prod_{a,b} [\chi(c_{ab}, \omega)]^{k_a \cdot k_b} \left(-\frac{4\pi}{\ell_n |\omega|}\right)^5 \\ \cdot \sum_{L \in \Lambda} \exp\left[\frac{1}{2} \ell_n \tilde{\omega} \left(L - \frac{\ell_n \tilde{\omega}}{\ell_n \tilde{\omega}} K_4\right)^2\right] [\psi(\tilde{c}_{54}, \tilde{\omega})]^{-2} \sum_{e \in \mathcal{E}} U_{ef}^{r_1 r_2 r_3} \theta_{ef} T_{r_1 r_2 r_3} \quad (26)$$

and $U_{ef}^{r_1 r_2 r_3}$ is defined as in eq. (12), remembering that only the three gravitons wave functions $\varrho^{(a)}$ ($a = 1, 2, 3$) carry a second space-time index. Similarly,

$$T_{r_1 r_2 r_3} = \frac{3}{\pi} \frac{d}{d\tilde{t}_a} \exp(\tilde{t}_a \tilde{A}_a + \tilde{t}_a R_{ab} \tilde{t}_b) \Big|_{\tilde{t}_a=0} \\ = \tilde{A}_{r_1} R_{r_2 r_3} \delta_{r_1 r_3} + \text{permutations} + \tilde{A}_{r_1} \tilde{A}_{r_2} \tilde{A}_{r_3} \quad (27)$$

In I_2 in eq. (25) we include all the terms containing R_0^4 . Here we have the two possibilities R_0^3 for gravitons and R_0 for one of the gauge bosons, or R_0^2 for the gravitons and R_0^2 for the gauge bosons with all the possible permutations. Thus the first term in I_2 has the form

$$I_2^{(4)} = -2 \left(\frac{g}{2\pi} \right)^5 \int \frac{\pi}{\pi} d^2 z_a \frac{[f(\bar{\omega})]^{-24}}{\bar{\omega} |\omega|^2} \prod_{b \in c} [\chi(c_b, \omega)]^{k_b \cdot k_b} \left(\frac{-4\pi}{2\pi i \omega} \right)^5 \\ \times \sum_{L \in n} \exp \left[\frac{L \bar{\omega}}{2} \left(L - \frac{L \bar{\omega}}{2\pi i} K_4 \right)^2 \right] [\psi(\bar{c}_{54}, \bar{\omega})]^{-2} t^{i_1 j_1 i_2 j_2 i_3 j_3 i_4 j_4} \\ \times \frac{3}{\pi} k_{j_a} g_{i_a r_a} k_{j_4} g_{i_4 r_4} T_{r_1 r_2 r_3}^{i_5} \quad (28)$$

where $T_{r_1 r_2 r_3}^{i_5}$ is defined as in eq. (17) with three differentiations with respect to $\tilde{t}_{r_a}$ ($a = 1, 2, 3$) and one with respect to t_{i_5} :

$$T_{r_1 r_2 r_3}^{i_5} = A_{i_5} T_{r_1 r_2 r_3} - \frac{1}{2\pi i \omega} \left[\delta_{r_1 i_5} (\delta_{r_2 r_3} R_{23} + \tilde{A}_{r_2} \tilde{A}_{r_3}) + \text{permutations} \right]. \quad (29)$$

The second term of I_2 containing R_0^2 for the gravitons and R_0^2 for the gauge bosons can be written similarly to eq. (28) with, say, $i_3 j_3 \rightarrow i_5 j_5$ and $i_5 \rightarrow i_3$. The important point to be noticed about eq. (25) is that, apart from the exponential dependence on $k_b \cdot k_c$, I_1 in eq. (26) is a polynomial in k whose lowest order is k^6 [see eqs. (11a), (12), and (27)], whereas I_2 in eq. (28) has a polynomial dependence on k of the order of k^4 at least [see the second term of eq. (29) and eq. (11c)]. The kinematical factor (of the order of k^4) in eq. (28) can be written as in eq. (20):

$$k_{r_1 r_2 r_3} = \frac{3}{\pi} k_{i_a} g_{i_a r_a} k_{i_4} g_{i_4 r_4} t^{i_1 j_1 i_2 j_2 i_3 j_3 i_4 j_4} \quad (30)$$

In the amplitude for the emission of three gravitons and two neutral gauge bosons we need to insert the vertex operator (24) for the gauge bosons (particles 4, 5). Here instead of: $e^{2iK^I x^I} : C(K)$ we have the internal momenta $\tilde{P}^I$ which need to be represented by their generating functional (6) with external sources $\tilde{t}^I$ ($I = 1, \dots, 16$). Once again we have the analogues of I_1 and I_2 in eqs. (7) or (25). Here we have expressions similar to eqs. (26) up to (30) but with

$$- \sum_{L \in n} \exp \left[\frac{L \bar{\omega}}{2} \left(L - \frac{L \bar{\omega}}{2\pi i} K_4 \right)^2 \right] [\psi(\bar{c}_{54}, \bar{\omega})]^{-2} \\ \rightarrow \sum_{L \in n} \exp \left(\frac{L^2 \bar{\omega}}{2} \right) (D_{45} \delta_{I_4 I_5} + L \cdot I_4 L \cdot I_5) \quad (31)$$

The difference lies in the fact that the external sources $\tilde{t}_a^I$ ($a = 4, 5$) induce the term

$$\frac{d}{d\tilde{t}_{I_4}} \frac{d}{d\tilde{t}_{I_5}} \exp(\tilde{t}_{I_4} D_{45} \tilde{t}_{I_5} + L \cdot \sum_{a=4}^5 \tilde{t}_a) \Big|_{\tilde{t}_a=0}$$

whose differentiation yields the right-hand side of eq. (31), whereas the left-hand side of eq. (31) disappears because the internal momenta $K^I = 0$ for the neutral gauge bosons. As before, the

dependence on the space-time momenta k_a ($a = 1, \dots, 5$) is polynomial (up to the exponential factor $k_b \cdot k_c$) of order k^4 at least and the Wess–Zumino term would result from the covariantization of the 8-index ϵ -symbol as will be seen in Section 5.

3.2 One-graviton/four-gauge-boson amplitudes

We continue to list now the rest of the amplitudes for the emission of one graviton (particle 1) and four gauge bosons (particles 2, ..., 5). The graviton's contribution to I_1 , the term containing R^5 [see eq. (7a)], is $\tilde{P}^{r_1}$, which induces the external source $\tilde{t}^{r_1}$. When we differentiate $\tilde{t}^{r_1}$ we get $\tilde{A}_{r_1}$ [eq. (11a)], which replaces $T_{r_1 r_2 r_3 r_4 r_5}$ in eq. (8), and the trace over R^5 yields $\Sigma_{e < f} U_{ef}^{r_1} B_{ef}$, where $U_{ef}^{r_1}$ is defined as in eqs. (12) and (13), carrying only one space-time index belonging to the graviton's wave function $q_{i_1 r_1}$ [i_1 is contracted as in eq. (12)]. We also have the usual term

$$\frac{[f(\bar{\omega})]^{-24}}{\bar{\omega} |\omega|^2} \prod_{b < c} [\chi(c_b, \omega)]^{\frac{1}{2} k_c \cdot k_b} \left(\frac{-4\pi}{\epsilon_n |\omega|} \right)^5$$

which is common to all the amplitudes. For the rest we need to specify the particular processes. Before doing so we analyse the term I_2 , containing R_0^4 . Here we need to remember the different permutations where there is a factor R_0 for the graviton and R_0^3 for the gauge bosons or R_0^4 for the gauge bosons. In both cases we have the kinematical factor K similar to eq. (30) [and eq. (20)] which would carry one space-time index r_1 belonging to the graviton in the first case. In addition we have two external sources: t^{ia} ($a = 1, \dots, 5$) resulting from P^{ia} in the vertices (2), (23) or (24) and $\tilde{t}^{r_1}$ resulting from $\tilde{P}^{r_1}$ in the graviton vertex (2). These sources are coupled as in the exponential of eq. (17) (without the term R_{ab} because we have only one left-moving external source), and when differentiated we get $A^{ia} \tilde{A}^{r_1} = \delta_{ia r_1} / 2 \ln |\omega|$ replacing $T_{r_1 r_2 r_3 r_4 r_5}^i$ in eq. (16).

We now specify the different amplitudes.

i) In the case of four charged gauge bosons we need to replace

$$\sum_{L \in n} \exp\left(\frac{L^2 \ell_n \bar{\omega}}{2}\right) \rightarrow \bar{E} \sum_{L \in n} \exp\left[\frac{\ell_n \bar{\omega}}{2} \left(L - \frac{\ell_n \bar{z}_a}{\ell_n \bar{\omega}} S_a\right)^2\right] \prod_{b < c} [\psi(\bar{c}_b, \bar{\omega})]^{k_c \cdot k_b} \quad (32)$$

in eq. (8) and in eq. (16) for I_1 and I_2 , respectively, where

$$S_a = \sum_{b=2}^{a-1} k_b \quad \text{and} \quad \bar{E} = E(k_2, k_3) E(k_4, k_5) (-1)^{k_4 \cdot k_5}. \quad (33)$$

The right-hand side of eq. (32) is the contribution typical of the four-charged-gauge-bosons amplitude. (For a derivation see ref. [3].)

ii) In the amplitude containing four neutral gauge bosons we need to replace

$$\sum_{L \in n} \exp\left(\frac{L^2 \ell_n \bar{\omega}}{2}\right) \longrightarrow \sum_{L \in n} \exp\left(\frac{L^2 \ell_n \bar{\omega}}{2}\right) T_{I_2 I_3 I_4 I_5} \quad (34)$$

where

$$\begin{aligned}
T_{I_2 I_3 I_4 I_5} &= \pi \frac{d}{d\tilde{t}_{I_a}} \exp \left(\sum_{b < c} \tilde{t}_b \cdot D_{bc} \tilde{t}_c + L \cdot \sum_{b=2}^5 \tilde{t}_b \right) \Big|_{\tilde{t}_a=0} \\
&= \delta_{I_2 I_3} \delta_{I_4 I_5} D_{23} D_{45} + \text{permutations} \\
&+ \delta_{I_2 I_3} D_{23} L_{I_4} L_{I_5} + \text{permutations} + L_{I_2} L_{I_3} L_{I_4} L_{I_5} .
\end{aligned} \tag{35}$$

As is clear from expressions (34) and (35), this replacement results from the coupling of the external source $\tilde{t}_a^I$ ($a = 2, \dots, 5$) to the momentum in internal space of the neutral gauge bosons [see eq. (24)]. We should also note that the kinematical factor $K_{I_1}^{I_2 \dots I_5}$ in eq. (30) depends on the indices I_a ($a = 2, \dots, 5$) via the neutral gauge bosons' wave functions $q_i^{I_a}$. Finally, as in expression (31), we have the functions D_{ab} , instead of R_{ab} [eq. (11c)]. This is due to the fact that the external sources $\tilde{t}_a^I$ (or $\tilde{t}_a$) couple to the loop momentum ℓ , which when integrated yields the second term of eq. (11c), whereas $\tilde{t}_a^I$ ($I = 1, \dots, 16$) couple to the loop internal momentum $L \in \Lambda$ only, so the integration over ℓ has no influence on the coupling $\tilde{t}_a^I \cdot D_{ab} \tilde{t}_b^I$ in eq. (35).

iii) The two-neutral- and two-charged-gauge-boson amplitude is a hybrid of both types of particles [3, 10]. Here we need to replace

$$\sum_{L \in \Lambda} \exp \left(\frac{L^2 \ell_n \bar{\omega}}{2} \right) \rightarrow - \sum_{L \in \Lambda} \exp \left[\frac{\ell_n \bar{\omega}}{2} (L - \kappa_n \frac{\ell_n \bar{\omega}}{\ell_n \bar{\omega}})^2 \right] T_{I_2 I_3} [\Psi(\bar{E}_{I_4}, \bar{\omega})]^{-2} \tag{36}$$

where

$$\begin{aligned}
T_{I_2 I_3} &= \pi \frac{d}{d\tilde{t}_{I_a}} \exp \left[\tilde{t}_2 \cdot D_{23} \tilde{t}_3 + \sum_{a=2}^3 (L + \bar{A}_a) \cdot \tilde{t}_a \right] \Big|_{\tilde{t}_a=0} \\
&= \delta_{I_2 I_3} D_{23} + (L_{I_2} + \bar{A}_{I_2})(L_{I_3} + \bar{A}_{I_3}) .
\end{aligned} \tag{37}$$

Here we take particles 2, 3 to be neutral and particles 4, 5 to be charged gauge bosons. The term $\bar{A}_I$ in eq. (37) is characteristic of this amplitude [10], and it results from the coupling of the external source to K_a^I ($a = 4, 5$). Such a coupling is induced by the trace over the oscillators $\tilde{\alpha}_n^I$, and it is missing in the expressions (31) and (35) because when there are no charged gauge bosons all the internal momenta are zero. The quantity $\bar{A}_{I_a}$ is defined as in eq. (11a) without the second term in the bracket (which results from momentum integration)

$$\bar{A}_a = \frac{1}{2} \sum_{b=4}^5 \tilde{\Theta}_{ab} K_b \quad ; \quad K_b \in \Lambda . \tag{38}$$

iv) Finally, the amplitude with one graviton, three charged and one neutral gauge bosons can be similarly constructed. We have a replacement similar to expression (36) with $T_{I_2} = L_{I_2} + \bar{A}_{I_2}$ (particle 2 is the neutral gauge boson) and for the rest we need to take an expression similar to (32) where the indices $a, b, c = 3, 4, 5$ (for the charged bosons).

To summarize, all the amplitudes we have calculated contain a term proportional to an 8-index ϵ -symbol [see eqs. (20) and (30)]. When covariantized it would yield the Wess–Zumino term. We proceed now to calculate this term in the covariant formalism.

4. PARITY-VIOLATING AMPLITUDES IN THE COVARIANT FORMALISM

The covariantization of the terms involving the 8-index ϵ -symbols is not transparent. We therefore calculate these parity-violating amplitudes in the covariant formalism to confirm the existence of the couplings typical of the Wess–Zumino terms. Parity-violating amplitudes with N ($N \geq 6$) external massless particles for various models were calculated in Refs. [1, 4, 11] with particular emphasis on the evaluation of the anomaly. Here we make a similar computation for the parity-violating 5-point functions in the heterotic string.

The parity-violating amplitude in the covariant formalism can be written as

$$A_{PV} = -\frac{1}{2} \text{Tr} \left[\Gamma_{11} \prod_{a=1}^5 F_0(a) \Delta_R(a) V_R^{(2)}(a, 1) \right], \quad (39)$$

where $V_R^{(2)}(a, z)$ ($z = e^{2i(\tau-\sigma)}$) are the vertex operators in the Ramond sector in the R_2 formalism [12], $\Delta_R(a)$ is the string propagator in the Ramond sector, and $F_0(a)$ is the string analogue of the Dirac operator for point particles. The Ramond sector is characterized by the periodic boundary conditions which the fermionic fields satisfy. Thus the right-moving fermionic fields are

$$\Gamma_\mu(z) = \gamma_\mu + \sqrt{2} \sum_{n \neq 0} b_n^\mu z^{-n} \quad (40)$$

and

$$F_0 = \frac{1}{i\sqrt{2}} \gamma \cdot P + \gamma_{11} \sum_{n=1}^{\infty} (\alpha_{-n}^\mu b_n^\mu + b_{-n}^\mu \alpha_n^\mu) \quad (41)$$

$$\Gamma_{11} = \gamma_{11} (-1)^{\sum_{n=1}^{\infty} b_{-n}^\mu b_n^\mu} \quad (42)$$

In eqs. (40)–(42), the b_n^μ are the fermionic oscillators in the Ramond sector, the α_n^μ are the bosonic ones in the covariant formalism ($\mu = 0, \dots, 9$), and γ_{11} is defined to anticommute with all the γ^μ 's. The vertex operator in the R_2 formalism can be written as

$$V_R^{(2)} = \frac{ig}{\sqrt{2}} g_\mu^\lambda \Gamma^\mu(z) e^{ik_\nu x^\nu} T^\lambda(\bar{z}) \quad (43)$$

where $q^{(a)}$ are the appropriate wave functions and $T^\lambda(\bar{z})$ is the contribution of the left-moving modes to the vertex operator. Thus $T^\lambda(\bar{z})$ would be $\tilde{P}^\mu(\bar{z})$ ($\mu = 0, \dots, 9$) for the graviton, $\tilde{P}^I(\bar{z})$ ($I = 1, \dots, 16$) for the neutral gauge bosons, and $:e^{2ik^1 x^1}: C(K)$ for the charged gauge bosons. [Compare with eqs. (2), (23), and (24).] The string propagator in the Ramond sector is defined as in eq. (5) but with N_R (the number operator in the Ramond sector) replacing N , and an extra factor i multiplying the integral.

The parity-violating amplitude is the term in eq. (39) proportional to a 10-index ϵ -symbol. Using the properties of the γ -matrices we find

$$\text{tr} \left(\gamma_{11} \prod_{a=1}^{10} \gamma^{\mu_a} \right) = -32i \epsilon^{\mu_1 \dots \mu_{10}} \quad (44)$$

Thus only the factor γ^μ from each Γ^μ and each F_0 [eqs. (40) and (41) respectively] would contribute to the parity-violating amplitude. For one graviton (particle 5) and four bosons (gauged or gravitational) it can be written as:

$$A_{PV} = -\frac{1}{2} \left(\frac{g}{2\pi} \right)^5 \epsilon^{\mu_1 \mu_2 \dots \mu_5 \lambda_5} \prod_{a=1}^4 k_{\mu_a} \rho_{\lambda_a}^{A_a} \rho_{\lambda_5}^{A_5} \int d^4 \ell \ell_{\mu_5} \text{Tr} \left[(-1)^{\sum_{n=1}^5 b_n^\dagger b_n} \prod_{c=1}^5 \Delta_R(c) e^{ik_c \cdot x} T^{A_c} \right] \quad (45)$$

The traces over the fermionic and bosonic oscillators in eq. (45) can now be evaluated in a way similar to the one used in the previous sections. We only need to take care of the term $e^{i\pi \sum_{n=1}^{\infty} b_n^\dagger b_n}$ which introduces a minus sign for each occupied fermionic state. Taking into account the ghost contributions we find that the trace over the fermionic oscillators yields a factor $[f(\omega)]^2 = \prod_{n=1}^{\infty} (1 - \omega^n)^8$. The traces over the rest of oscillators are as before (for the different processes). The integration over ℓ in eq. (45) yields two possible contributions: (i) a term proportional to k_a ($a = 1, \dots, 5$) or (ii) $[1/2 \ln |\omega|] \delta_{\mu_5 \lambda_5}$. The first would vanish when contracted with the ϵ symbol. (Recall that the k_a , $a = 1, \dots, 5$, are linearly dependent owing to momentum conservation.) The second term yields the Wess-Zumino term for the string model. In comparing this result with the coefficients of the 8-index ϵ -symbol found in the light-cone gauge, we find that for the 5-graviton amplitude the coefficient is the second term of eq. (18), for 3 gravitons and 2 gauge bosons it is the second term of eq. (29), and for 1 graviton and 4 gauge bosons it is simply $1/2 \ln |\omega|$. One needs only to add the various dynamical factors appearing in the amplitudes for the different processes [see eqs. (16), (28), (32), (34)–(38)]. These extra factors result from the traces over bosonic oscillators for the right- and left-moving modes. One last remark is that there are other terms involving the 8-index ϵ -symbol in the light-cone gauge such as the first term of eq. (18) or (29). However, these would cancel when the contributions from $\text{Tr } R_0^5$ in I_1 [eq. (7a)] are added up [9].

Thus we have shown that the parity-violating amplitude in the covariant formalism is the term appropriate to the covariantization of the 8-index ϵ -symbol of the light-cone gauge. It has typically the form (for 1 graviton and 4 charged gauge bosons)

$$A_{PV} = -\left(\frac{g}{2\pi} \right)^5 \epsilon^{\mu_1 \mu_2 \mu_3 \mu_4 \mu_5 \lambda_5} \rho_{\mu_5 \lambda_5} \prod_{a=1}^4 k_{\mu_a} \rho_{\lambda_a} \int \prod_{a=1}^5 d^2 z_a \frac{[f(\bar{\omega})]^{-24}}{\bar{\omega} |\omega|^2} \prod_{b < c} [\chi(c_{cb}, \omega)]^{\frac{1}{2} k_c \cdot k_b} \left(-\frac{4\pi}{\ln |\omega|} \right)^5 \frac{1}{2 \ln |\omega|} \cdot \bar{E} \sum_{L \in \mathcal{N}} \exp \left[\frac{\ell_n \bar{\omega}}{2} \left(L - S_d \frac{\ell_n \bar{z}_d}{\ell_n \bar{\omega}} \right)^2 \right] \prod_{e < f} [\psi(\bar{c}_{fe}, \bar{\omega})]^{k_e \cdot k_f} \quad (46)$$

This would clearly yield the Wess-Zumino term in the field-theory limit, and has the typical form $\epsilon^{\mu_1 \nu_1 \dots \mu_5 \nu_5} B_{\mu_1 \nu_1} \prod_{a=2}^4 F_{\mu_a \nu_a}$. For the other amplitudes the integrand in eq. (46) would change according to the specific process as explained above.

5. INTERPRETATION OF THE WESS-ZUMINO TERM

We have confirmed in the previous sections that the effective action for the massless states in the 10-dimensional heterotic superstring does indeed contain a Wess-Zumino term at the one-loop level in the topological expansion for string configurations. In this section we address the question whether this term has any importance for four-dimensional phenomenology: the answer will be 'not necessarily'. We express our arguments in naïve physical terms, for which we apologize to our mathematically more sophisticated colleagues.

Symmetries of field theories lead, in general, to Ward identities interrelating different Green functions, which take the generic form

$$k^\mu \Gamma_\mu(k, q_i) = \Gamma(k, q_i) \quad , \quad (47)$$

where k is the momentum associated with the symmetry current and the q_i are the other field momenta. In general, there will be several different Green functions $\Gamma(k, q_i)$ on the right-hand side. If the symmetry is anomalous, an additional term $A(k, q_i)$ appears in $O(\hbar)$ on the right-hand side of eq. (47):

$$k^\mu \Gamma_\mu(k, q_i) = \Gamma(k, q_i) + \hbar A(k, q_i) \quad . \quad (48)$$

The anomaly $A(k, q_i)$ in general depends on the momenta (k, q_i) in a non-trivial way. For example, the classic chiral triangle anomaly contains a non-trivial kinematic factor $K(k, q_i)$ including contractions with external wave functions, which multiplies a reduced form factor $\bar{A}(k, q_i)$:

$$A(k, q_i) = K(k, q_i) \bar{A}(k, q_i) \quad . \quad (49)$$

If the theory only contains massless particles and no other mass-scale, $\bar{A}(k, q_i)$ will be determined by dimensional analysis, scaling like a simple power of the momenta: a simple constant in the case of the chiral triangle anomaly. However, in general, the form factor $\bar{A}(k, q_i)$ may depend on (k, q_i) in a non-trivial way, and may also depend on particle masses and other mass-scales in the theory. An example would be quantum electrodynamics (QED) with one massless fermion f_0 and one massive fermion f_m . In this case

$$\begin{aligned} \lim_{k, q_i \ll m} \bar{A}(k, q_i) &= c Q_0^2 \\ \lim_{k, q_i \gg m} \bar{A}(k, q_i) &= c (Q_0^2 + Q_m^2) \quad , \end{aligned} \quad (50)$$

with $\bar{A}(k, q_i)$ smoothly interpolating between these two limits when $(k, q_i) = O(m)$. Normally, one considers physics in the range of (k, q_i) between the masses of some light fermions (e.g. the u and d quarks in the case of $\pi^0 \rightarrow \gamma\gamma$ decay) which one can legitimately approximate as massless, and some other massive fermions. It is well known that only the massless fermions contribute to the effective anomaly $A(k, q_i)$ in this momentum range, as is shown explicitly by a simple kinematical argument. The heterotic superstring is, however, an explicit example where $A(k, q_i)$ has a complicated integral representation with both *massive* and massless particles running around the loop, and which add up

to zero when summed up [4]. In the zero-slope limit $[\alpha'(k \cdot q) \ll 1]$ only the contribution of the massless modes remains and the irreducible diagram reduces to the conventional field-theoretical expression for the anomaly derived from loops of massless particles. The reducible part of the diagram, on the other hand, reduces to the conventional Wess–Zumino term, thereby making the field theoretic model anomaly free. However, had there been no anomaly in the field theory in the first place, there would be no Wess–Zumino term in the $D = 10$ model to cancel it.

The physical problem we are concerned with actually has two large energy scales, equivalent to two small distance scales: the string length $L \sim \sqrt{\alpha'}$ and the radius R_c of compactification from ten to four dimensions. String excitations have masses $m_s \sim L^{-1}$ and Kaluza–Klein excitations have masses $m_K \sim R_c^{-1}$. Considerations of the ten-dimensional $N = 1$ supergravity theory as the low-energy limit of the superstring are based on the working hypothesis that $R_c \gg L$ so that $m_K \ll m_s$. The Green–Schwarz form of the anomaly is applicable when $m_K \ll (k, q_i) \ll m_s$. For larger values of $(k, q_i) \gtrsim O(m_s)$ one must use the string integral representation of the anomaly, which includes the effects of particles with masses $O(m_s)$ going around the loop. If we want to go to $(k, q_i) \lesssim O(m_K)$ we must remove the effects of the Kaluza–Klein excitations with masses $O(m_K)$ going around the loop. It is not sufficient just to replace the external fields appearing in anomalous Green functions by their classical expectation values in some compactification scheme; one must take into account the Kaluza–Klein splitting $\delta m = O(m_K)$ of the particles going around the loop. In the case of Calabi–Yau compactification, it is difficult to integrate the Kaluza–Klein modes out explicitly, as was done for the string modes in the reduction from the string-theoretical $D = 10$ anomaly to the field-theoretical $D = 10$ anomaly. This is because even the Calabi–Yau metric is not known explicitly, let alone the associated Kaluza–Klein spectrum. However, we can evaluate the residue of the anomaly explicitly in the low-energy effective $d = 4$ field theory obtained after compactification, since topology specifies for us the spectrum of fields with $m \ll m_K$.

We are interested in a possible anomaly in the non-compact global $U(1)$ symmetry: $T \rightarrow T + i\beta$, where T is a component of the ten-dimensional antisymmetric two-index tensor field $B_{\mu\nu}$ with indices tangential to the Calabi–Yau manifold of compactification: $T = \epsilon_{IJ} B^{IJ}$, where (I, J) are three-dimensional complex indices. The ten-dimensional field $B_{\mu\nu}$ has a gauge invariance

$$\delta B_{\mu\nu} = i \partial_{[\mu} \Lambda_{\nu]} , \quad (51)$$

which reduces to $\delta T = i\beta : \beta = \epsilon_{IJ} \partial_{[I} \Lambda_{J]}$ when we compactify à la Calabi–Yau. There would be a coupling $\hbar \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} (\text{Im } T)$ in the low-energy effective theory if and only if the symmetry $\delta T = i\beta$ were anomalous, in much the same way as the $\pi^0 \rightarrow 2\gamma$ coupling exists in $O(\hbar)$ because the global $U_A(1)$ current is anomalous.

To probe the existence of such a coupling in $O(\hbar)$, we need to know the tree-level vertices of the theory. These are determined by a Kähler potential K , which is a real function of the light chiral fields, and a holomorphic function f_{ab} of the chiral fields, which characterizes the couplings of the gauge fields:

$$f_{ab} \rightarrow \text{Re } f_{ab} F_{\mu\nu}^a F^{b\mu\nu} , \quad \text{Im } f_{ab} F_{\mu\nu}^a F_{\rho\sigma}^b \epsilon^{\mu\nu\rho\sigma} . \quad (52)$$

To obtain a coupling $\hbar(\text{Im } T) F_{\mu\nu}^a F_{\rho\sigma}^a \epsilon^{\mu\nu\rho\sigma}$ we need to find that $f_{ab} \ni \hbar T \delta_{ab}$, which would require a tree-level coupling of $\text{Im } T$ to the light fermions. However, the tree-level Kähler potential does not depend on $\text{Im } T$:

$$G \ni -3 \ln (\tau + \tau^* - 2 \phi^i \phi_i^*) + \dots \quad (53)$$

when one approximates Calabi–Yau compactification by a simple truncation. It is easy to check that the effective $d = 4$ tree-level action given by expression (53) depends only on $\partial(\text{Re } T)$, $\partial(\text{Im } T)$ and $\text{Re } T$, so that it has $\delta T = i\beta$ as a trivial classical symmetry. The Ward identity in which we must seek an anomaly is $k^\mu \Gamma_\mu(k, q_1, q_2) = 0$, where Γ_μ here represents the Green function for one current J_μ generating the shift $\delta T = i\beta$, and two gauge fields A_μ . The only particles coupling to A_μ are the (almost) massless gauge non-singlet particles, which do not appear in J_μ . Hence there is no one-loop diagram in the effective low-energy theory which can contribute to Γ_μ . Since $\Gamma_\mu = 0$, it is *a fortiori* true that $k^\mu \Gamma_\mu = 0$ and hence there is no anomaly.

There is an independent argument which tells us not to expect a coupling $\hbar \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} (\text{Im } T)$ in the effective low-energy theory. Elementary topological considerations tell us that the effective action must be independent of $\text{Im } T$ (apart from derivative terms) if world-sheet instanton effects are not included, and the one-loop calculation does not include any instanton effects. In fact, it has also been pointed out that world-sheet instanton effects in Calabi–Yau compactification also do not renormalize the gauge kinetic terms [13], and hence do not generate a coupling $\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} (\text{Im } T)$ either.

6. CONCLUSIONS

We have shown by explicit calculation that the low-energy effective action obtained from the heterotic superstring at the one-loop level of five-boson amplitudes contains a Wess–Zumino term. However, there is no corresponding one-loop renormalization of the gauge kinetic term in four dimensions, at least when one compactifies on a Calabi–Yau manifold.

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Figure captions

Fig. 1 Factorized one-loop parity-violating six-point functions.

a) Non planar diagram in type II superstring.

b) One-loop diagram in the heterotic string.

Fig. 2 One-loop diagram contributing to the Wess–Zumino term in the heterotic string. The external legs are gravitons or gauge bosons.

a)

b)

Fig. 1

Fig. 2