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SUPERSPACE RENORMALIZATION OF $N = 1$, $d = 4$ SUPERSYMMETRIC GAUGE THEORIES:

I. - NON-LINEAR FIELD RENORMALIZATION

II. - CONFORMAL INVARIANCE

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A B S T R A C T

I. - The gauge superfield, being dimensionless, is subject to a non-linear field renormalization, parametrized by an infinite set of gauge parameters. These parameters are shown to be of gauge type, hence non-physical. The argument is also used in order to cure an off-shell infra-red problem.

II. - Simple criteria involving one-loop calculations are proved to be sufficient for assuring the vanishing of the β -functions of the theory at all orders of perturbation theory.

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Contents: I. Non-Linear Field Renormalization
 II. Conformal Invariance

These two somewhat unrelated talks deal with the renormalization of $N = 1$ supersymmetric gauge theories in four-dimensional space-time. We are working in the superfield formalism, i.e., in a linear realization of supersymmetry, with a supersymmetry-invariant gauge-fixing condition. This is to be contrasted with the Wess-Zumino gauge approach, where the non-linear realization of supersymmetry causes some difficulties which are still awaiting a complete solution¹⁾. However, although renormalization is made simpler by the superfield approach^{2),3)}, a substantial price has to be paid, due to the fact that the gauge superfield is dimensionless and massless.

Indeed, the consequences of this fact are, first, the occurrence of a non-linear renormalization of the gauge superfield^{3),4)}, a phenomenon which was also met later on in the study of two-dimensional σ -models⁵⁾⁻⁷⁾, and, second, off-shell infra-red singularities due to a propagator of the form $1/k^4$ for this same gauge superfield. This is the subject of the first talk, where we show that the infinite set of arbitrary parameters describing the non-linear field renormalization^{*)} are gauge parameters, and thus do not contribute to physical quantities like Green functions of gauge-invariant operators. The method of the proof consists of allowing these parameters to transform under BRS and of proving the corresponding Slavnov identity. This procedure is explained in Ref. 9) for the case of gauge parameters in ordinary Yang-Mills theories, and was in fact already advocated in Ref. 10). The application to the supersymmetric case we discuss here was given in Refs. 3) and 4). We also briefly describe in this first talk the use of this procedure for curing the infra-red singularity, by introducing an infra-red cut-off mass and showing that it is a gauge parameter¹¹⁾.

The second talk deals with the problem of finite theories. More precisely, we consider theories with vanishing Callan-Symanzik β -functions, namely conformal invariant theories, which can be interpreted as finite "on the mass-shell". For these, in particular, the Green functions of gauge-invariant operators without

*) This phenomenon was also discovered, independently, by explicit one-loop graph computations⁸⁾.

anomalous dimensions, e.g., conserved flavour currents, have no ultra-violet divergences. We shall show that $N = 1$ super-Yang-Mills theories coupled with matter indeed have vanishing β functions, if they satisfy three conditions which can be checked by simple one-loop computations^{12),13)}. These criteria may be expressed in the following way.

- (1) The gauge coupling β -function vanishes in the one-loop approximation.
- (2) The anomalies of the axial currents associated with the set of chiral invariances of the superpotential, i.e., of the action describing the self-interaction of the matter fields, vanish.
- (3) The coupling constants are completely reduced¹⁴⁾. In other words, all matter self-interaction coupling constants λ_I can be chosen in a consistent way as functions of the gauge coupling constant g^* , so that the theory depends only on one coupling constant.

These criteria will be shown to be sufficient for the vanishing of the β -functions. On the other hand, condition (1) is clearly necessary. Condition (3) is also necessary in view of the lower-order calculations of Ref. 15). Let us mention that in the latter reference, as well as in the remaining literature, the vanishing of the anomalous dimensions of the matter fields is required. This is indeed sufficient for the matter self-interaction β -functions to vanish, but in general not necessary. Our three criteria can be seen to be fulfilled¹²⁾ by the extended $N = 4$ super-Yang-Mills theory, as well as by a class of $N = 2$ theories, all written in terms of $N = 1$ superfields. This confirms the known results^{16),17)}. The criteria are also satisfied by $N = 1$ theories with complex representations for the matter fields¹³⁾.

I. - NON-LINEAR FIELD RENORMALIZATION^{3),4)}

I.1 Classical Theory

The field content of the theory is given by a set of real gauge superfields $\phi^i(x, \theta, \bar{\theta})$ (dimension 0), a set of Lagrange multiplier chiral superfields $B^i(x, \theta)$ (dimension 1) and a set of anticommuting ghost and antighost chiral superfields $c_+^i(x, \theta)$ and $c_-^i(x, \theta)$ (dimensions 0 and 1). No coupling with matter fields will be

*) We consider a simple gauge group; thus there is only one gauge coupling constant.

considered in this section. The superscript i is the Yang-Mills index, and we shall use the matrix notation

$$\phi = \phi^i \tau_i, \quad B = B^i \tau_i, \quad c_{\pm} = c_{\pm}^i \tau_i \quad (1.1)$$

where the matrices τ_i are the generators of the gauge group in the fundamental representation. Notations and conventions are those of Ref. 3). The gauge group is chosen to be simple.

The BRS transformations may be written as

$$\begin{aligned} s e^{\phi} &= e^{\phi} c_{+} - \bar{c}_{+} e^{\phi} \\ s \phi &= c_{+} - \bar{c}_{+} + \frac{1}{2} [\phi, c_{+} + \bar{c}_{+}] + \dots \equiv Q_s(\phi, c_{+}) \end{aligned} \quad (1.2)$$

$$s c_{+} = -c_{+}^2$$

$$s c_{-} = B, \quad s B = 0$$

and are nilpotent:

$$s^2 = 0. \quad (1.3)$$

Introducing external superfields ρ^i and σ^i coupled to the BRS variations of ϕ^i and c_{+}^i respectively, we can write an action invariant under (1.2) as: [matrix notation (1.1) is used]:

$$\begin{aligned} \Gamma_s(\phi, c_{\pm}, B, \rho, \sigma) &= -\frac{1}{128g^2} \text{Tr} \int dS F^{\alpha} F_{\alpha} \\ &+ \text{Tr} \int dV \left\{ -\frac{1}{8} (D D c_{-} + \bar{D} \bar{D} \bar{c}_{-}) Q_s(\phi, c_{+}) + \rho Q_s(\phi, c_{+}) \right\} \\ &- \text{Tr} \int dS \sigma c_{+}^2 - \text{Tr} \int d\bar{S} \bar{\sigma} \bar{c}_{+}^2 \\ &+ \text{Tr} \int dV \{ (D D B + \bar{D} \bar{D} \bar{B}) \phi + \alpha B \bar{B} \} \end{aligned} \quad (1.4)$$

where

$$F_{\alpha} = \bar{D}\bar{D}e_{\alpha}, \quad e_{\alpha} = e^{-\phi} D_{\alpha} e^{\phi} \quad (1.5)$$

$$dV = d^4x DD \bar{D}\bar{D}, \quad dS = d^4x DD, \quad d\bar{S} = d^4x \bar{D}\bar{D} \quad (1.6)$$

$D_{\alpha}, \bar{D}_{\alpha}$ are the superspace covariant derivatives.

The BRS invariance of (1.4) may be expressed by the Slavnov identity¹⁸⁾

$$\begin{aligned} S(\Gamma) = & : \text{Tr} \int dV \frac{\delta\Gamma}{\delta\rho} \frac{\delta\Gamma}{\delta\phi} \\ & + \text{Tr} \int dS \left[\frac{\delta\Gamma}{\delta\sigma} \frac{\delta\Gamma}{\delta c_{+}} + B \frac{\delta\Gamma}{\delta c_{-}} \right] - \text{c.c.} = 0 \end{aligned} \quad (1.7)$$

and the gauge fixing [last terms in (1.4)] by the (linear) gauge-fixing condition

$$\frac{\delta\Gamma}{\delta B} = \frac{1}{8} \bar{D}\bar{D} DD\phi + \alpha \bar{D}\bar{D} \bar{B}. \quad (1.8)$$

The theory is further specified by supersymmetry and rigid invariance

$$\begin{aligned} \delta_{\alpha}\phi &= \left(\frac{\partial}{\partial\theta\alpha} + i\sigma_{\alpha\dot{\alpha}}^{\mu} \bar{\theta}^{\dot{\alpha}}\partial_{\mu} \right) \phi \\ \delta_{\text{rig}}\phi &= i [\phi, \omega], \quad \omega = \omega^i \tau_i, \quad \omega^i = \text{const.} \end{aligned} \quad (1.9)$$

$$\phi = \phi, \quad c_{\pm}, \quad B, \quad \rho, \quad \sigma$$

expressed through the Ward identities¹⁸⁾

$$\begin{aligned} W_{\alpha}\Gamma &= -i \sum_{\phi} \int \delta_{\alpha}\phi \frac{\delta\Gamma}{\delta\phi} = 0 \\ W_{\text{rig}}\Gamma &= -i \sum_{\phi} \int \delta_{\text{rig}}\phi \frac{\delta\Gamma}{\delta\phi} = 0 \end{aligned} \quad (1.10)$$

From now on, we define the theory through the functional identities (1.7), (1.8) and (1.10). This is the appropriate way for the extension to the quantized theory.

Whereas the requirements (1.8) and (1.10) are straightforward, we shall see that the action (1.4) is not the most general classical solution of the Slavnov identity (1.7). In order to investigate this, let us consider the following stability problem: the special solution Γ_s (1.4) being given, find the most general form of the perturbed action (ε small)

$$\Gamma = \Gamma_g + \epsilon \Delta(\phi, c_+, c_-, B, \rho, \sigma) \quad (1.11)$$

fulfilling all of our requirements, and having its dimension bounded by four in order to preserve power-counting renormalizability. From supersymmetry and rigid invariance (1.10), we know that Δ is a linear combination of superspace integrals of rigid-invariant superfield monomials - there is an infinite set of them, since ϕ is dimensionless.

The gauge-fixing condition (1.8) implies

$$\begin{aligned} \Gamma(\phi, c_+, c_-, B, \rho, \sigma) &= \bar{\Gamma}(\phi, c_+, \eta, \sigma) \\ &+ \int \left\{ \frac{1}{8} (DDB + \bar{D}\bar{D}\bar{B})\phi + \alpha B\bar{B} \right\} \end{aligned} \quad (1.12)$$

with

$$\eta = \rho - \frac{1}{8} (DDc_- + \bar{D}\bar{D}\bar{c}_-). \quad (1.13)$$

That the dependence on c_- and ρ occurs through the combination η (1.13) is a consequence of the ghost equation

$$G\Gamma = : \left[\frac{\delta}{\delta c_-} + \frac{1}{8} \bar{D}\bar{D} DD \frac{\delta}{\delta \rho} \right] \Gamma = 0 \quad (1.14)$$

which in turn follows from the gauge-fixing condition and the Slavnov identity.

The ansatz (1.12) allows us to write the Slavnov identity (1.7) in the form

$$S(\Gamma) = \frac{1}{2} B_{\bar{\Gamma}} \bar{\Gamma} = 0 \quad (1.15)$$

with

$$B_{\bar{\Gamma}} = : \int \left\{ \frac{\delta \bar{\Gamma}}{\delta \eta} \frac{\delta}{\delta \phi} + \frac{\delta \bar{\Gamma}}{\delta \phi} \frac{\delta}{\delta \eta} + \frac{\delta \bar{\Gamma}}{\delta \sigma} \frac{\delta}{\delta c_+} + \frac{\delta \bar{\Gamma}}{\delta c_+} \frac{\delta}{\delta \sigma} \right\} \quad (1.16)$$

(We drop all trace and integration measure symbols.) The functional-dependent linear operator (1.16) obeys the identities

$$B_{\gamma} B_{\gamma} \gamma = 0, \quad \forall \gamma \quad (1.17)$$

$$B_{\gamma}^2 = 0, \text{ if } B_{\gamma} \gamma = 0. \quad (1.18)$$

Γ and Γ_s being each decomposed according to (1.12), (1.11) reads

$$\bar{\Gamma} = \bar{\Gamma}_s + \varepsilon \Delta(\phi, c_+, \eta, \sigma). \quad (1.19)$$

Substituting this in the Slavnov identity (1.15) and retaining the terms of first order in ε , we obtain for Δ the equation

$$b\Delta = 0 \quad (1.20)$$

$$\text{with } b = : B_{\bar{\Gamma}_s}, \quad b^2 = 0 \quad (1.21)$$

[The nilpotency of b follows from (1.18) since Γ_s is a solution of the Slavnov identity.] Note that b , when acting on ϕ and c_+ , coincides with the BRS operator s (1.2). But it acts non-trivially on the external fields:

$$b\eta = \frac{\delta \bar{\Gamma}_s}{\delta \phi}, \quad b\sigma = \frac{\delta \bar{\Gamma}_s}{\delta c_+} \quad (1.22)$$

To solve (1.20) is a cohomology problem, with the coboundary operator given by (1.21). The most general solution Δ having ghost number 0^{*} and dimension 4 has the form⁴⁾

$$\Delta = z \Gamma_{\text{SYM}}(\phi) + b \hat{\Delta}(\phi, c_+, \eta, \sigma) \quad (1.23)$$

where

$$\Gamma_{\text{SYM}} = - \frac{1}{128} \text{Tr} \int dS F^{\alpha} F_{\alpha} \quad (1.24)$$

is the super-Yang-Mills gauge-invariant action occurring in (1.4), and $\hat{\Delta}$ is an arbitrary local functional of dimension 4 and ghost number -1:

$$\hat{\Delta} = \text{Tr} \int dV f(\phi) \eta - [x \text{Tr} \int dS c_+ \sigma + \text{c.c.}] \quad (1.25)$$

with

$$f(\phi) = \sum_{k=1}^{\infty} x_k (\phi)^k \quad (1.26)$$

*) The ghost numbers of ϕ , c_+ , c_- , ρ , σ are 0, 1, -1, -1, -2 respectively.

or, more precisely:

$$f_i(\phi) = \sum_{k=1}^{\infty} \sum_{\omega=1}^{\Omega_k} x_{k,\omega} t_i^{\omega}(i_1 \dots i_k) \phi^{i_1} \dots \phi^{i_k} \quad (1.27)$$

where $t_i^{\omega}(i_1 \dots i_k)$ are the Ω_k invariant tensors of rank $k+1$, symmetric in their k last indices (rigid invariance is taken into account). z , x and $x_{k,\omega}$ are arbitrary parameters.

Computing $b\hat{\Delta}$ according to the definitions (1.21) and (1.16) we find (integration measures and trace symbols omitted)

$$b\Delta = \int \left\{ f_i \frac{\delta \bar{\Gamma}s}{\delta \phi_i} - \eta_i \partial_j f_i \frac{\delta \bar{\Gamma}s}{\delta \eta_j} + x \left(c_+ \frac{\delta \bar{\Gamma}s}{\delta c_+} - \sigma \frac{\delta \bar{\Gamma}s}{\delta \sigma} \right) \right\} \quad (1.28)$$

with $\partial_j f_i = (\partial/\partial \phi^j) f_i(\phi)$.

Substituting (1.23) into (1.19) yields, at the first order in ϵ ,

$$\bar{\Gamma}(\phi, c_+, \eta, \sigma) = \bar{\Gamma}_s(\hat{\phi}, \hat{c}_+, \hat{\eta}, \hat{\sigma})|_{g^2 \rightarrow g^2 - \epsilon z} \quad (1.29)$$

with

$$\begin{aligned} \hat{\phi}_i &= \phi_i + \epsilon f_i(\phi), \quad \hat{\eta}_i = \eta_i - \epsilon \eta_j \partial_i f_j(\phi) \\ \hat{c}_+ &= (1 + \epsilon x) c_+, \quad \hat{\sigma} = (1 - \epsilon x) \sigma. \end{aligned} \quad (1.30)$$

This means that the general solution in the neighbourhood of the special solution $\bar{\Gamma}_s$ is obtained by a coupling constant renormalization $g^2 \rightarrow g^2 - \epsilon z$ and the field substitutions (1.30). For c_+ this is just a usual field amplitude renormalization, but for ϕ we have a generalized, non-linear field amplitude renormalization.

We notice that the sources η and σ for the BRS transformations of ϕ and c_+ are redefined, too: this amounts to a redefinition of the BRS transformation laws (1.2) $s \rightarrow \hat{s}$, such that

$$\hat{s} e^{\hat{\phi}} = e^{\hat{\phi}} \hat{c}_+ - \hat{c}_+ e^{\hat{\phi}}, \quad \hat{s} \hat{c}_+ = -\hat{c}_+^2. \quad (1.31)$$

This is in fact³⁾ the most general change of s keeping its nilpotency - which is implicitly contained in the definition of the Slavnov operator [see (1.17) and (1.18)].

The relevance of studying the general solution in the infinitesimal form (1.11) lies in the fact that it yields the general structure of the counterterms of the quantized theory in the perturbative framework. The occurrence of the non-linear renormalization (1.30) was indeed confirmed by explicit one-loop computations⁸⁾ which showed the presence of infinities, absorbable only through a non-linear redefinition of ϕ .

One has, however, to look for the general classical solution of the Slavnov identity in finite form, since this is the starting point for the perturbative construction of the quantum theory. It turns out⁴⁾ that this general solution is again obtained from a special solution $\bar{\Gamma}_s$ by a substitution exactly as in (1.29), but now with the finite field redefinitions

$$\hat{\phi}_i = F_i(\phi), \quad \hat{\eta}_i = \eta_j \left[\frac{\partial}{\partial \hat{\phi}_i} F_j^{-1}(\hat{\phi}) \right] \hat{\phi} = F(\phi) \quad (1.32)$$

$$\hat{c}_+ = Z_{c_+} c_+, \quad \hat{\sigma} = Z_{c_+}^{-1} \sigma$$

where [in the short-hand notation (1.26) instead of (1.27)]

$$F(\phi) = Z_\phi \phi + \sum_{k \geq 2} a_k(\phi)^k \quad (1.33)$$

is an arbitrary invertible, dimension 0, function of ϕ . Z_{c_+} , Z_ϕ , a_k are arbitrary constants.

We may conclude that the theory depends on infinitely numerous parameters and hence is non-renormalizable! The following formal argument suggests that the parameters a_k are in fact gauge parameters, hence non-physical. (A rigorous proof will be given in the next subsection for the quantized theory.) We first observe that the substitutions (1.32) for ϕ and η (we now take $\hat{c}_+ = c_+$, $\hat{\sigma} = \sigma$) take place in $\bar{\Gamma}$ as defined by (1.12), and not in the whole action Γ_s . But if, after this, we perform the inverse transformation $\phi \rightarrow F^{-1}(\phi)$, and similarly for η , in the whole action Γ (this defines a canonical transformation) we arrive at the equivalent action

$$\Gamma'(\phi, c_+, c_-, B, \rho, \sigma) = \bar{\Gamma}_s(\phi, c_+, \eta, \sigma) + \int \left\{ \frac{1}{8} (DDB + \overline{DD} \bar{B}) F^{-1}(\phi) + \alpha B \bar{B} \right\} \quad (1.34)$$

We see in this new formulation that the numbers a_k now parametrize the non-linear gauge fixing condition

$$\frac{\delta \Gamma'}{\delta B} = \frac{1}{8} \overline{DD} DD F^{-1}(\phi) + \alpha \overline{DD} \bar{B} \quad (1.35)$$

which replaces the linear condition (1.8): they are indeed gauge parameters.

I.2 Renormalization

The quantum theory is described by the generating functional $Z(J_\phi, J_{c_+}, J_{c_-}, J_B, \rho, \sigma)$ of the Green functions, or by

$$Z_c = \frac{\hbar}{i} \log Z \quad (1.36)$$

which generates the connected Green functions, or by the vertex functional

$$\Gamma(\phi, c_+, c_-, B, \rho, \sigma) = Z_c(J_\phi, J_{c_+}, J_{c_-}, J_B, \rho, \sigma) - \int \{J_\phi \phi + J_{c_+} c_+ + J_{c_-} c_- + J_B B\} \quad (1.37)$$

which generates the one-particle irreducible graphs and coincides with the classical action at $\hbar = 0$, $\hbar$ being taken as the perturbation expansion parameter (loop expansion). The theory is defined by requiring the supersymmetry and rigid invariance Ward identities (1.10), the linear gauge fixing condition (1.8) and the Slavnov identity (1.7). The latter reads, for the Green functional Z , in short-hand notation ^{*}:

$$SZ =: \int \left\{ -J_\phi \frac{\delta}{\delta \rho} + J_{c_+} \frac{\delta}{\delta \sigma} + J_{c_-} \frac{\delta}{\delta J_B} \right\} Z \sim 0 \quad (1.38)$$

with $S^2 = 0$.

As we have seen, the theory depends on the infinite set of parameters a_k describing the general non-linear ϕ -field renormalization. We will now show that this dependence has the peculiar form

$$\frac{\partial}{\partial a_k} Z \sim S(\Delta_k, Z) \quad (1.39)$$

where Δ_k are some insertions with ghost number -1, or, equivalently for Γ :

$$\frac{\partial}{\partial a_k} \Gamma \sim B_{\bar{\Gamma}}(\Delta_k, \Gamma) \quad (1.40)$$

with $B_{\bar{\Gamma}}$ defined by (1.16). The consequence of (1.39) is that the S-matrix - if it can be defined - is independent of the a_k 's. More generally one can define the Green

^{*}) In order to avoid here any infra-red problem caused by the presence of dimensionless fields we introduce masses which preserve supersymmetry but break BRS invariance. Hence the Slavnov identity can only hold up to soft terms: this is expressed by the sign $\sim$ in all subsequent identities.

functions of gauge invariant operators Q_a through the introduction of BRS invariant external fields q_a . Their generating functional

$$Z_{\text{inv}}(q) = Z(J, \rho, \sigma, q) \Big|_{J=\rho=\sigma=0} \quad (1.41)$$

is then a_k -independent:

$$\frac{\partial}{\partial a_k} Z_{\text{inv}}(q) = [S(\Delta_k \cdot Z)]_{J=\rho=\sigma=0} = 0. \quad (1.42)$$

The validity of (1.39) or (1.40) is easily checked in the classical approximation. [In the infinitesimal form (1.11) this directly follows from the fact that the a_k - (or x_k -) dependent part of the perturbation Δ is of the form $B_{\bar{\Gamma}}(\phi)^k \eta$ as can be seen from Eqs. (1.23)-(1.26).]

In order to extend the property (1.40) to the quantum theory, we require the latter to obey the new Slavnov identity

$$S(\Gamma) = S_{\text{old}}(\Gamma) + \sum_k x_k \frac{\partial \Gamma}{\partial a_k} \sim 0 \quad (1.43)$$

where we have introduced an infinite set of anticommuting parameters x_k . This amounts to considering the a_k 's as transforming under BRS (in a way respecting the nilpotency):

$$s a_k = x_k, \quad s x_k = 0 \quad (1.44)$$

It is checked that this works by differentiating (1.43) with respect to x_k , which yields

$$\frac{\partial \Gamma}{\partial a_k} - B_{\bar{\Gamma}} \frac{\partial \Gamma}{\partial x_k} \sim 0 \quad (1.45)$$

with

$$B_{\bar{\Gamma}} = B_{\bar{\Gamma}}^{\text{old}} + \sum_k x_k \frac{\partial}{\partial a_k}. \quad (1.46)$$

Equation (1.45) indeed reproduces (1.40) with the identification

$$\Delta_k \cdot \Gamma = \frac{\partial \Gamma}{\partial x_k}. \quad (1.47)$$

The construction of an x_k -dependent classical action fulfilling the new Slavnov identity is straightforward. By standard arguments¹⁸⁾ we know that the construction is then feasible at all orders of perturbation theory if the cohomology equation

$$b\Delta = 0 \quad (1.48)$$

admits only trivial solutions

$$\Delta = b\hat{\Delta}. \quad (1.49)$$

Here, Δ being a local functional $\Delta(\phi, c_+, \eta, \sigma, a_k, x_k)$ of dimension four and ghost number one, the solution $\hat{\Delta}$ must be local, too, with dimension four and ghost number zero. The coboundary operator is

$$b = B \bar{\Gamma}_{\text{classical}}, \quad b^2 = 0. \quad (1.50)$$

In particular

$$ba_k = x_k, \quad bx_k = 0. \quad (1.51)$$

Concerning the x_k and a_k dependence of Δ , the cohomology is that of polynomials^{*}), and is thus trivial:

$$\Delta = b\hat{\Delta}_1 + \Delta_2 \quad (1.52)$$

with

$$b\Delta_2 = 0, \quad \frac{\partial \Delta_2}{\partial a_k} = \frac{\partial \Delta_2}{\partial x_k} = 0. \quad (1.53)$$

The remaining cohomology problem (1.53) is well known^{3), 19)}: the only non-trivial solution is the chiral anomaly, which we assume to be absent.

We have proved in this way the possibility of constructing a theory obeying the new Slavnov identity (1.43). In other words there always exists a set of insertions Δ_k such that the physical a_k independence condition (1.39) holds.

*) The a_k 's play the role of coupling constants. Since they couple with increasing powers of ϕ as k increases, any term of a given order in $\hbar$ (i.e., given number of loops) and of a given degree in ϕ can only depend on a finite number of a_k .

I.3 The off-shell infra-red problem^{3),11)}

Since the $\theta = 0$ component of the gauge superfield ϕ is of dimension zero, and massless in the case of strict gauge invariance, it has a propagator of the form $1/k^4$, which causes infra-red (IR) divergent Green functions.

In order to cure this disease we take advantage of the freedom of doing an arbitrary field redefinition (1.32). We choose a θ -dependent redefinition

$$\phi \rightarrow F(\phi) = (1 + \frac{1}{2} \mu^2 \theta^2 \bar{\theta}^2) \phi \quad (1.54)$$

where μ has the dimension of a mass (η must be changed accordingly). The substitution of (1.54) in $\bar{\Gamma}$ [according to (1.29)] has the effect of changing in particular the above IR-singular propagator into $1/(k^2 - \mu^2)^2$. Thus μ^2 is an IR-cut-off.

Moreover, this IR-cut-off appearing as a parameter of the field redefinition, is a gauge parameter, in the same sense as the parameters a_k previously discussed. The proof is the same, too, although the presence of fields staying massless (but with non-singular propagators) complicates considerably the technical task of solving the BRS-cohomology.

Thus the physical quantities do not depend on the IR-cut-off. In other words, the IR-singularities cancel when computing these quantities.

Of course, the ansatz (1.54) breaks supersymmetry explicitly. This soft breaking is, however, controllable and can be shown not to affect the physical quantities.

II. CONFORMAL INVARIANCE^{12),13)}

Let us consider the super-Yang-Mills model of Section I, with a simple gauge group G , and couple it with chiral matter fields A^R . Here R labels both the field and the irreducible representation of G where it lives. Its BRS transformation is

$$sA^R = -c_{+i} T^i_R A^R \quad (2.1)$$

$$s\bar{A}_R = \bar{A}_R T^i_R c_{+},$$

the Hermitian matrices T^i_R representing the generators of G in the representation R . The Slavnov identity (1.7) has a corresponding piece:

$$S(\Gamma) = \dots + \sum_l \int dS \frac{\delta \Gamma}{\delta Y_R} \frac{\delta \Gamma}{\delta A^R} - \text{c.c.} \sim 0 \quad (2.2)$$

where Y_R is the external field coupled to the BRS transformation of A^R . The matter field contribution to the action (1.4) is

$$\Gamma_{\text{matter}} = 1/16 \int dV \sum_R \bar{A}_R e^{\phi_i T^i} A^R + \int dS \sum_I \lambda_I W^I(A) + Y_R s A^R + \text{c.c.} \quad (2.3)$$

where $W^I(A)$ is a basis of invariant cubic polynomials of A , and the λ_I 's are the "Yukawa" coupling constants. Mass terms, not necessarily gauge invariant but supersymmetric, are supposed to be present in order to avoid the off-shell infra-red problem discussed in Section I.3. The Slavnov identity is thus softly broken, as well as all other following equations [this is expressed by the sign $\sim$ in (2.2)].

Our strategy for studying the properties of the Callan-Symanzik β functions and for finding conditions under which they vanish is based on the existence of a BRS-invariant supercurrent³⁾ V_μ obeying the Ward identity

$$\bar{D}^{\dot{\alpha}} V_{\alpha\dot{\alpha}} \sim -2w_\alpha \Gamma - 4/3 D_\alpha (S+S_0) \quad (2.4)$$

with

$$V_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^\mu V_\mu, \quad \bar{D}_\alpha S = \bar{D}_{\dot{\alpha}} S_0 = 0.$$

w_α is a functional differential operator expressing the different symmetries (superconformal group) involved. The letters V , S and S_0 stand for insertions¹⁸⁾. V and S are BRS-invariant, i.e.,

$$B_{\bar{\Gamma}} V_\mu \sim 0, \quad B_{\bar{\Gamma}} S \sim 0 \quad (2.5)$$

[see (1.16) and Ref. 18) for the definition of $B_{\bar{\Gamma}}$].

Let us forget S_0 , an effect of the gauge fixing, irrelevant for the present discussion. The supercurrent V_μ contains²⁰⁾ among its components an axial current ($\theta = 0$ component) associated with R-invariance^{21), 3)} and the conserved symmetric energy-momentum tensor ($\theta\sigma\bar{\theta}$ -component). The BRS-invariant chiral insertion S , of dimension 3, and of order $\hbar$, describes in particular the anomalies of the axial current and of the trace of the energy-momentum tensor, the latter being related to the dilatation anomaly, hence to the Callan-Symanzik equation. The precise relation is the following. We expand S in a basis of BRS-invariant insertions L_n of dimension

3 defined through the action principle¹⁸⁾ by

$$\nabla_n \Gamma \sim \int dS L_n + \int d\bar{S} \bar{L}_n \quad (2.6)$$

where the ∇_n 's are a basis of BRS-invariant differential operators:

$$\begin{aligned} \nabla_g &= \partial_g, \quad \nabla_{\lambda_I} = \partial_{\lambda_I}, \quad \nabla_k = \partial_{a_k}, \\ \nabla_\phi &= \tilde{N}_\phi =: N_\phi - N_\rho - N_{c_-} - N_{\bar{c}_-} - N_B - N_{\bar{B}} + 2\alpha\partial_\alpha \\ \nabla_S^R &= \tilde{N}_S^R =: N_{AS}^R - N_{YS}^R \\ \nabla_+ &= \tilde{N}_+ =: N_{c_+} - N_\sigma + c.c. \end{aligned} \quad (2.7)$$

the N 's being the counting operators

$$N_\phi = \int dV \phi \frac{\delta}{\delta\phi}, \quad N_S^R = \int dS A^R \frac{\delta}{\delta A^S}, \quad \text{etc.} \quad (2.8)$$

One sees in particular that

$$\begin{aligned} L_S^R &= (A^R \frac{\delta}{\delta A^S} - Y_S \frac{\delta}{\delta Y_R}) \Gamma \\ L_+ &= \text{Tr}(c_+ \frac{\delta}{\delta c_+} - \sigma \frac{\delta}{\delta \sigma}) \Gamma \end{aligned} \quad (2.9)$$

and one can show that

$$L_\phi = \overline{DD} \, \ell_\phi, \quad L_k = \overline{DD} \, \ell_k \quad (2.10)$$

where ℓ_ϕ and ℓ_k are BRS-invariant.

We thus write

$$S = \beta_g L_g + \sum_I \beta_{\lambda_I} L_{\lambda_I} - \gamma_\phi L_\phi - \sum_{R,S} \gamma_R^S L_S^R - \gamma_+ L_+ - \sum_k \beta_k L_k. \quad (2.11)$$

The connection with the Callan-Symanzik equation

$$\begin{aligned}
 C\Gamma =: & [\sum_a m_a \partial_{m_a} + \beta_g \partial_g + \beta_{\lambda_I} \partial_{\lambda_I} + \beta_{\bar{\lambda}_I} \partial_{\bar{\lambda}_I} - \gamma_\phi \tilde{N}_\phi - \gamma_R^S \tilde{N}_S^R - \gamma_{\tilde{N}_+} - \\
 & - \gamma_k \partial_{a_k}] \Gamma \sim 0
 \end{aligned}
 \tag{2.12}$$

follows from the identity³⁾

$$\sum_a m_a \partial_{m_a} \Gamma + \int dS S + c.c. = 0
 \tag{2.13}$$

(summation is over all mass parameters of the theory).

Let us perform a change of basis for the insertions L_S^R , the new basis $\{L_{0a}, L_{1A}\}$ being defined according to (2.6) through the counting operators

$$\begin{aligned}
 \tilde{N}_{0a} &= \sum_{R,S} e_{aR}^S \tilde{N}_S^R \\
 \tilde{N}_{1A} &= \sum_{R,S} f_{AR}^S \tilde{N}_S^R
 \end{aligned}
 \tag{2.14}$$

where the operators $\tilde{N}_{0a}$ form a basis of counting operators annihilating the superpotential terms W^I of the action (2.3):

$$\tilde{N}_{0a} W^I(A) = 0, \quad \forall I
 \tag{2.15}$$

The $\tilde{N}_{1A}$ complete the basis of matter field counting operators.

The expansion of S in this new basis reads [with (2.10) taken into account]

$$S = \beta_g L_g + \sum_I \beta_{\lambda_I} L_{\lambda_I} - \gamma_\phi \overline{DD} \ell_\phi - \sum_a \gamma_{0a} L_{0a} - \sum_A \gamma_{1A} L_{1A} - \gamma_{\tilde{N}_+} - \sum_k \gamma_k \overline{DD} \ell_k.
 \tag{2.16}$$

A key remark, now, is that the conditions (2.15) defining the insertion L_{0a} express the invariance of the theory (at the classical approximation) under the chiral transformations

$$\delta_a^R = i e_{aR}^S \delta_a^S, \quad \delta_a^S = -i e_{aR}^S \delta_a^R
 \tag{2.17}$$

(which obviously leave invariant the rest of the action). This invariance can be extended to the quantum theory, but the associated axial currents become anomalous. A relation between the coefficients β, γ of (2.16) and the coefficients of these

anomalies, as well as with the coefficient of the anomaly for the axial R-current will follow from the fact¹²⁾ that the BRS-invariant chiral insertion $T = S, L_g, \dots$, can be written in the form

$$T \sim \overline{DD} [rK^\circ + J^{\text{inv}}] + T^c \quad (2.18)$$

where T^c is genuinely chiral [i.e., it cannot locally be written as $\overline{DD}(\dots)$] and J^{inv} is BRS-invariant. The insertion K° is not invariant, although $\overline{DD}K^\circ$ is, and is defined^{*)} through the supersymmetric descent equations

$$\begin{aligned} B_{\overline{I}} K^\circ &\sim \overline{D}_{\dot{\alpha}} K^{1\dot{\alpha}} & B_{\overline{I}} K^{1\dot{\alpha}} &\sim (\overline{DD} + 2\overline{DD})^{\dot{\alpha}\alpha} K_{\alpha}^2 \\ B_{\overline{I}} K_{\alpha}^2 &\sim D_{\alpha} K^3 & B_{\overline{I}} K^3 &\sim 0, \quad \overline{D}_{\dot{\alpha}} K^3 = 0 \end{aligned} \quad (2.19)$$

(the superscript denotes the ghost number).

The dimensionless insertion K^3 is proportional to c^3_+ and can be shown to be finite, hence uniquely defined up to a numerical factor, chosen to be 1/3 by convention. It turns out that K° is then uniquely defined up to an invariant. The coefficient r in (2.18) is thus defined unambiguously and is moreover gauge independent.

The genuinely chiral insertion T^c being expanded in terms of the chiral insertions L_{1A}, L_+ , we can write (2.18), for $T = S, L_g, \dots$, as

$$\begin{aligned} S &\sim \overline{DD}[rK^\circ + J^{\text{inv}}] + S^c & L_g &\sim \overline{DD}\left[\left(\frac{1}{128 g^3} + r_g\right)K^\circ + J_g^{\text{inv}}\right] + L_g^c \\ L_{\lambda_I} &\sim \overline{DD}[r_{\lambda_I}K^\circ + J_I^{\text{inv}}] + L_I^c & L_{0a} &\sim \overline{DD}[r_{0a}K^\circ + J_{0a}^{\text{inv}}] + L_{0a}^c. \end{aligned} \quad (2.20)$$

The corresponding expressions for L_ϕ, L_k, L_{1A}, L_+ have a coefficient $r = 0$ due to our choice of basis. The coefficients r, r_{λ_I}, r_g and r_{0a} are of order $\hbar$ at least. The zeroth order coefficient in L_g comes from the fact that

$$\overline{DD}K^\circ = \text{Tr } F^{\alpha F}_{\alpha} + O(\hbar) \quad (2.21)$$

is the integrand of the Yang-Mills action (1.4).

*) Up to an invariant.

Moreover, the coefficients r and r_{0a} in (2.20), which can be interpreted as the anomalies of the axial currents associated with R-invariance and the chiral invariances (2.17), respectively, can be proved^{*} to be non-renormalized: they have only one-loop contributions, which may be computed, with the result

$$\begin{aligned} r &= \frac{1}{512(4\pi)^2} (-3C_2(G) + \sum_R T(R)) \\ r_{0a} &= - \frac{1}{256(4\pi)^2} \sum_R e_{aR}^R T(R). \end{aligned} \quad (2.22)$$

$T(R)$ is defined for the irreducible representation R by

$$T(R) \delta^{ij} = \text{Tr } T_R^i T_R^j \quad (2.23)$$

and

$$C_2(G) = T(\text{ad}) \quad (2.24)$$

is the quadratic Casimir operator of the group. The numbers e_{aR}^R were defined by (2.14) and (2.15).

The substitution of the expressions (2.20) in (2.16) and the identification of the coefficients of K° yield the equation

$$r = \beta_g \left(\frac{1}{128g^3} + r_g \right) + \sum_I \beta_{\lambda_I} r_{\lambda_I} - \sum_a r_{0a} \gamma_{0a}. \quad (2.25)$$

This is the announced relation between the Callan-Symanzik functions and the anomaly coefficients r , r_{0a} . One sees in particular that r is proportional to the one-loop β_g function.

If the representations R of the matter fields are chosen such that the coefficients (2.22) vanish,

$$r = r_{0a} = 0 \quad (2.26)$$

then Eq. (2.25) becomes homogeneous in β_g , β_{λ_I} .

^{*}) The proof¹²⁾ is based on: (i) the finiteness of $K^3 = 1/3 \text{Tr } c_+^3$; (ii) the Callan-Symanzik equation, obeyed by the superspace integrals of the insertion S and L_{0a} without any of their own anomalous dimensions.

If, moreover, the theory can be completely reduced¹⁴⁾, i.e., that all Yukawa coupling constants λ_I can be chosen as power series in the gauge coupling constant g , then these functions $\lambda_I(g)$ must be solutions of the reduction equations¹⁴⁾

$$\beta_{\lambda_I} = \beta_g \partial_g \lambda_I \quad (2.27)$$

and Eq. (2.25) becomes

$$0 = \beta_g \left(\frac{1}{128g^3} + r_g + r_{\lambda_I} \partial_g \lambda_I \right) \quad (2.28)$$

whose solution is, in perturbation theory,

$$\beta_g = 0. \quad (2.29)$$

This also implies the vanishing of all β_{λ_I} , due to (2.27).

We have thus proved that the model is asymptotically scale invariant, as announced in the Introduction, if, first, the representations are such that the vanishing of the quantities (2.22) holds and, second, that the reduction equations (2.27) admit non-trivial solutions. The latter is, as a rule, true if it is verified in the one-loop approximation.

A closer look at the superconformal group reveals that the physical quantities are also superconformally covariant.

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